University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 1 – Solutions

Due 12:30pm Monday, January 11, 2020

- Homework will be handed in electronically. Please do not hand in photographs of hand-written work. The preferred option is to typeset your homework. It is easy to do and you need to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples is provided here. If you need help getting set up or have any other questions please email me. I am happy to give TeX advice.
- To hand in your homework, please submit a pdf file through the course's Canvas website, under the assignment labelled hw01.

Thanks in advance for following these guidelines. Please ask me by email if you have any trouble.

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1. **Groundstate degeneracy and 1-form symmetry algebra.**

   (a) Suppose we have a system with Hamiltonian H with string operators W_C and $V_{\check{C}}$ supported on closed curves, and commuting with H, and satisfying $W^N = V^N = 1$. Suppose $[W_C, W_{C'}] = 0, [V_{\check{C}}, \check{C}']$ for all curves but

   $$W_C V_{\check{C}} = \omega^{\#\left(C \cap \check{C}\right)} V_{\check{C}} W_C$$

   where $\omega \equiv e^{\frac{2\pi \mathbf{i}}{N}}$ and $\#\left(C \cap \check{C}\right)$ is the number of intersection points of the curves. How many groundstates does such a system have on the two-torus (that is, with periodic boundary conditions on both spatial directions)?

   This is what happens in the $\mathbb{Z}_N$ toric code.

   For each non-contractible cycle $C_{x,y}$ of the torus, we get a pair of string operators $W_C, V_{\check{C}}$, with

   $$W_{C_x} V_{\check{C}_y} = \omega V_{\check{C}_y} W_{C_x}, \quad W_{C_y} V_{\check{C}_x} = \omega^{-1} V_{\check{C}_x} W_{C_y}$$

   – note that the orientation of the intersection matters now. Let's diagonalize W_{C_x} and W_{C_y}. Their eigenvalues are roots of unity. Starting from a state $|(1,1)\rangle$ with eigenvalues $(1,1)$, the action of $V^n_{\check{C}_y} V^m_{\check{C}_x}$ generates

   $$|(n,-m)\rangle = V^n_{\check{C}_y} V^m_{\check{C}_x} |(1,1)\rangle$$

with the eigenvalues ω^n and ω^{-m} under W_{C_x} and W_{C_y}. Since they have different eigenvalues they are linearly independent. This gives N^2 groundstates as the minimal representation of this algebra. On a genus g Riemann surface, with g conjugate pairs of cycles, we would find N^{2g} groundstates.

(b) Now suppose in a different system we have just one set of string operators W_C satisfying

$$W_C W_{C'} = \omega^{\#(C \cap C')} W_{C'} W_C,$$

with the same definitions as above. How many groundstates does this system have on the two-torus?

This is what happens in the Laughlin fractional quantum Hall state with filling fraction $\frac{1}{N}$.

Now we get just one conjugate pair of operators on the torus:

$$W_{C_x} W_{C_y} = \omega W_{C_y} W_{C_x}.$$

Acting on an eigenstate $|1\rangle$ of W_{C_x} with eigenvalue 1, $W_{C_y}^n$ generates

$$|n\rangle = W_{C_y}^n |1\rangle$$

with W_{C_x}-eigenvalue ω^n. Therefore the minimal representation has N states. On a genus-g Riemann surface, there would be N^g states.

(c) [Bonus problem] Redo the previous problems for a genus g Riemann surface, *i.e.* the surface of a donut with g handles.

In all parts of this problem you should make the assumption that the string operators are *deformable*: W_C acts in the same way as $W_{C+\partial p}$ on groundstates.

2. **Anyons in the toric code.**

(a) Show that when acting on a toric code groundstate the operator

$$W_C = \prod_{\ell \in C} X_\ell$$

creates a state which violates only the star operators at the sites in the boundary of C, ∂C, a pair of e-particles.

Clearly W_C commutes with B_p since they are both made of just Xs. If a site j touches C in the middle somewhere, A_j shares two edges with W_C and therefore $[A_j, W_C] = 0$. At the end of the curve is a site which shares only one edge with A_j and therefore they anticommute. Therefore acting with W_C changes the A_j-eigenvalue of the state from 1 to -1.

(b) Show that when acting on a toric code groundstate the operator

$$V_{\check{C}} = \prod_{\ell \perp \check{C}} Z_\ell$$

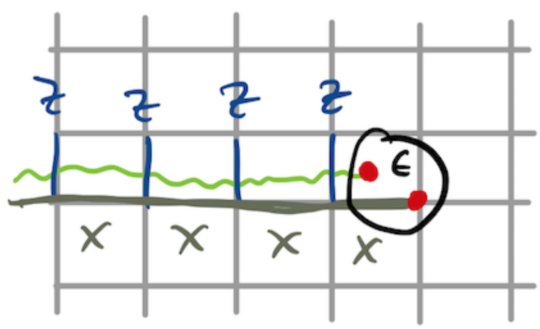
creates a state which violates only the plaquette operators in the boundary of $\check{C}$, $\partial \check{C}$.

This question is related to the previous by the duality map which takes the lattice to the dual lattice and interchanges $X \leftrightarrow Z$.

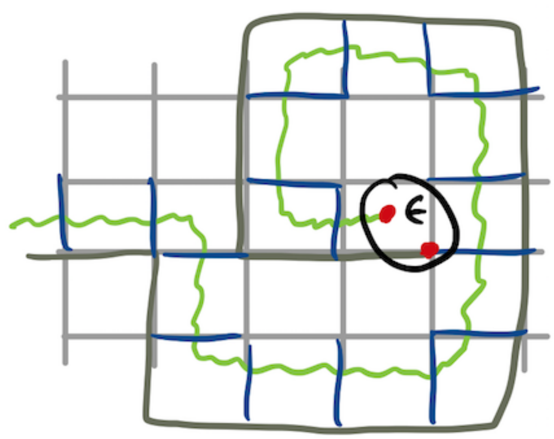
(c) Show that a boundstate of an e particle and an m particle in the 2d toric code must be a fermion.

Recall that a particle is a fermion if after rotating it by 2π its state picks up a minus sign:

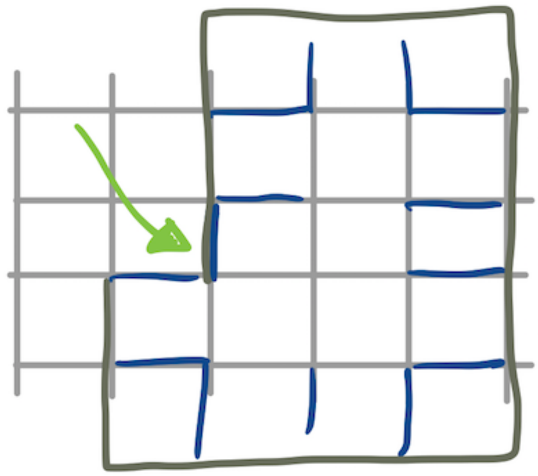
The operator which creates the epsilon particle looks like this, call it $\mathcal{O}_1$:



The operator which creates an epsilon particle and rotates it by 2π looks like this, call it $\mathcal{O}_2$:



The product of the two is this:



(1)

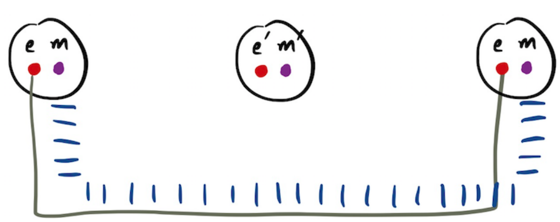
except I need to tell you the order in which the X and Z act on the indicated link. Therefore

$$\left\langle \quad \middle| \quad \right\rangle = \langle \mathrm{gs}| \, \mathcal{O}_2 \mathcal{O}_1 \, |\mathrm{gs}\rangle \, .$$

If we can separate the two loops of Xs and Zs in (1) they will give $W_C V_{\check{C}}$ for contractible curves, which gives 1 when acting on the groundstate. But: the X and Z on the indicated link need to be moved past each other first. This costs a minus sign and therefore

$$\left\langle \quad \middle| \quad \right\rangle = \langle \mathrm{gs}| \, \mathcal{O}_2 \mathcal{O}_1 \, |\mathrm{gs}\rangle = -1.$$

Alternatively, we can think about *exchange*. Exchanging two particles can be accomplished by first rotating one around the other by a π rotation, and then translating both of them by their separation. As you can see in this figure:



the first step requires the string creating the e particle to cross that creating the m particle on an odd number of links. (The second step is innocuous.)