University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2020
Assignment 9 ("Final Exam")

Due 12:30pm Wednesday, March 18, 2020

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1. **An example of the power of the RG logic.**

   Consider quantum mechanics of a single particle in d dimensions, with Hamiltonian

   $$H = \frac{p^2}{2m} + V(q), \quad [q, p] = i.$$

   Consider the (say, euclidean) path integral for this problem,

   $$Z = \int [dq] e^{-S[q]}, \quad S[q] = \int dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right).$$

   To be more precise, with periodic boundary conditions, $Z(\beta) = \int_{q(t+\beta)=q(t)} [dq] e^{-S[q]} = \mathrm{tr} e^{-\beta H}$ is the thermal partition function. Alternatively, instead of Z, we could consider the Green's function $G(q_1, t_1; q_2, t_2) = \int_{q(t_1)=q_1}^{q(t_2)=q_2} [dq] e^{-S[q]}$.

   Working by analogy with our treatment of field theory, show that any smooth[1] potential V is a *relevant* perturbation of the free particle, *i.e.* the Gaussian fixed point with $H = \frac{p^2}{2m}$.

   Hint: change variables to $\phi(t) \equiv \sqrt{m} q(t)$.

   Use this to explain in words why the high energy asymptotics of the density of states

   $$N(E) \equiv \{\# \text{ of eigenvalues of } H \text{ less than } E\}$$

   is given by the *Weyl formula* (even for $V(q) \neq 0$):

   $$N(E) = E^{d/2} K_d L^d + ...$$

   where $K_d = \frac{\Omega_{d-1}}{(2\pi)^d}$ as usual, and L is the linear size of the box in which we put the particle (an IR cutoff).

   Hint: we can represent the density of states by a path integral using an inverse Laplace transform:

   $$\mathrm{tr} \frac{1}{\omega - H} = \int d\beta \; e^{\beta\omega} Z(\beta)$$

   and the relation

   $$\mathrm{Im} \frac{1}{\omega + \mathbf{i}\epsilon - H} = \pi \delta(\omega - H).$$

[1] Some singular potentials are also relevant perturbations. If $V(q) \sim q^{-\alpha}$, how big can α be for my statement to remain true? Thanks to Brian Vermilyea for reminding me that a singular enough potential will cause trouble.

2. **An application of effective field theory in quantum mechanics.**

Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model studied in lecture with two QM variables where both natural frequencies are taken to zero.)

(a) Based on a semiclassical analysis, would you think that the spectrum is discrete?

(b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

(c) The result is not analytic in x at $x = 0$. Why?

(d) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime which matters for the semi-classical analysis?

[Bonus: determine the spectrum of $\mathbf{H}_{\text{eff}}$.]

3. **RG analysis of less-symmetric spin systems.**

Suppose that we break the rotation symmetry of the $\mathsf{O}(n)$ model to the subgroup of $\pi/2$ rotations, *i.e.* the cubic symmetry, (for example, for $n = 2$, $(s_1, s_2) \to (s_2, -s_1)$.) If the spins live on the cubic lattice, a spin-orbit coupling could do this.

(a) Don't look at the next part of the problem yet! What functions of an n-vector ϕ_a that are invariant under $\pi/2$ rotations, but not general rotations?

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⋮

⋮

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(b) Show that in addition to the usual $\mathsf{O}(n)$-symmetric interaction

$$\int d^d x \sum_{a,b=1}^{n} u\phi_a^2\phi_b^2,$$

the LG free energy should include a term of the form

$$\int d^d x \sum_{a=1}^{n} v\phi_a^4.$$

Argue that this is the only new term (preserving cubic symmetry but not the full $\mathsf{O}(n)$ symmetry) which can be a relevant perturbation of the Gaussian fixed point near $d = 4$.

(c) Treating $\mathcal{O}(u) = \mathcal{O}(v) = \mathcal{O}(\epsilon)$, redo the analysis of the running couplings in $d = 4 - \epsilon$ dimensions to derive beta functions for u and v up to corrections of order $\mathcal{O}(u^3) = \mathcal{O}(v^2 u) = ... = \mathcal{O}(\epsilon^3)$.

(d) Your answer to the previous part will be of the form

$$\begin{aligned} -b\partial_b u &= -\epsilon u + A_1 u^2 + A_2 uv + A_3 v^2 + \mathcal{O}(u^3) \\ -b\partial_b v &= -\epsilon v + B_1 u^2 + B_2 uv + B_3 v^2 + \mathcal{O}(u^3). \end{aligned} \tag{1}$$

You should find that $A_3 = B_1 = 0$. Find four fixed points:

- The gaussian fixed point.
- A fixed point where only $u \neq 0$.
- A fixed point where only $v \neq 0$. Describe the physics of this fixed point. (Hint: the action is a sum of n terms.)
- A fixed point where both (u, v) are nonzero.

(In every case, the assumption of $u \sim v \sim \epsilon$ is self-consistent.)

(e) Analyze the stability of these fixed points (by computing the matrix of derivatives of the beta functions at each fixed point). Draw the phase diagram. Which fixed point dominates the critical behavior? You will want to consider different cases depending on whether $n > 4$ or $n < 4$.

(f) When $n > 4$ you may find that v wants to become negative. This means that the effective potential for m becomes unbounded, within our approximation. What have we left out that will restore sanity? What does this mean for the order of the phase transition? (Notice that mean field theory predicts a continuous transition, so any change in this conclusion is a dramatic effect of the fluctuations, more dramatic than just changing the values of critical exponents by a little.)

4. **$\mathsf{O}(N)$ model at large N.** In lecture we studied the $\mathsf{O}(N)$ model in an expansion in $\epsilon = 4 - D$. When N is large, there is another small parameter in which to expand. We'll see that the results are consistent with the ϵ expansion. It is also an example which illustrates the manipulations we did in describing the BCS phenomenon.

   Consider the (Euclidean) partition function for an N-vector of scalar fields in D dimensions:

$$Z = \int [d\phi] e^{\mathbf{i}S[\phi]}, \quad S[\vec{\phi}] = \int \mathrm{d}^D x \left(\partial_\mu \phi^a \partial^\mu \phi^a - r\phi^a \phi^a - \frac{g}{N} \left(\phi^a \phi^a\right)^2 \right).$$

   (a) At the free fixed point, what is the dimension of the coupling g as a function of the number of spacetime dimensions D? Show that it is classically marginal in $D = 4$, so that this action is (classically) scale invariant.

   (b) [optional] Show that the definition above of $u = g/N$ is a good idea if we want to take $N \to \infty$, at fixed g. Do this by considering the N-dependence of diagrams which contribute to, say, the free energy, and demanding that in the large-N limit the interaction terms contribute with the same power of N as the leading term.

   (c) Analyze, at large N, the critical behavior of the model as r is varied. You'll need to consider separately the regimes $D > 4, D = 4, 2 < D < 4, D \leq 2$. Here are the steps: first use the Hubbard-Stratonovich trick to replace ϕ^4 by $\sigma\phi^2 + \sigma^2$ (up to factors) in the action, where σ is a new scalar field[2]. Then integrate out the ϕ fields. Find the saddle point equation for σ; argue that the saddle point dominates the integral for large N. Regulate the integrals in a convenient way. Find a translation invariant saddle point (*i.e.* where σ is constant). Plug the saddle point configuration of σ back into the action for ϕ and describe the resulting dynamics.

   (d) [bonus] Compute the correlation-length critical exponent ν at leading order in large N. Compare with the epsilon expansion results.

5. **Gross-Neveu model.** [optional] This problem uses *very* similar steps to the previous one, but leads to very different physical conclusions. I include it here to emphasize the many applications of this method.

[2] To be more explicit, use (a path-integral version of) the identity

$$e^{u\phi^4} = \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} d\sigma \; e^{-\sigma^2/u - 2\phi^2\sigma}.$$

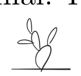
Consider the partition function for an N-vector of fermionic spinor fields in D dimensions:

$$Z = \int [d\psi d\bar{\psi}] e^{\mathbf{i}S[\psi]}, \quad S[\vec{\psi}] = \int \mathrm{d}^D x \left(\bar{\psi}^a \mathbf{i} \partial\!\!\!/ \psi^a - \frac{g}{N} \left(\bar{\psi}^a \psi^a \right)^2 \right).$$

(a) At the free fixed point, what is the dimension of the coupling g as a function of the number of spacetime dimensions D? Show that it is classically marginal in $D = 2$, so that this action is (classically) scale invariant.

(b) We will show that this model in $D = 2$ exhibits dimensional transmutation in the form of a dynamically generated mass gap. Here are the steps: first use the Hubbard-Stratonovich trick to replace ψ^4 by $\sigma\psi^2 + \sigma^2$ in the action, where σ is a scalar field. Then integrate out the ψ fields. Find the saddle point equation for σ; argue that the saddle point dominates the integral for large N. Find a translation invariant saddle point. Plug the saddle point configuration of σ back into the action for ψ and describe the resulting dynamics.

6. **Diagrammatic understanding of BCS instability of Fermi liquid theory.** [optional]

(a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.

(b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

(c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

$$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi^\dagger_{\vec{k},\omega_3,\downarrow} \psi^\dagger_{-\vec{k},\omega_4,\uparrow} \psi_{\vec{p},\omega_1,\downarrow} \psi_{-\vec{p},\omega_2,\uparrow} \right\rangle$$

as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta} \psi^\dagger_\alpha \psi_\alpha$ at zero momentum.

Sum the geometric series in terms of a (one-loop) integral kernel.

(d) Do the integrals. In the loops, restrict the range of energies to $|\omega| < E_D$ (or $|\epsilon(k)| < E_D$), the Debye energy, since it is electrons with these energies which experience attractive interactions.

Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, make the approximation $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^d k \simeq k_F^{d-1} \frac{d\xi}{v_F} d\Omega_{d-1}$.

(e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.

(f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi^\dagger_{k,\uparrow}\psi^\dagger_{-k,\downarrow}|F\rangle$$

where $|F\rangle = \prod_{k<k_F} \psi^\dagger_{k,\uparrow}\psi^\dagger_{k,\downarrow}|0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi^\dagger_{k,\sigma}\psi_{k,\sigma} + \sum_{k,k'} V_{k,k'}\psi^\dagger_{k,\sigma}\psi_{k,\sigma}\psi^\dagger_{k',\sigma'}\psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k)a_k = \sum_{k'} V_{k,k'}a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w^\star_{k'} w_k, \qquad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega^\star_k a_k$, show that the Schrödinger equation requires

$$1 = V\sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \tag{2}$$

Assuming V is attractive, find a bound state. Compare (26) to the condition for a pole found from the bubble chains above.

7. **Abrikosov-Nielsen-Oleson vortex string.** [optional]

Consider the Abelian Higgs model in $D = 3 + 1$:

$$\mathcal{L}_h \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - V(|\phi|)$$

where ϕ is a scalar field of charge e whose covariant derivative is $D_\mu\phi = (\partial_\mu - \mathbf{i}qA_\mu)\,\phi$, and let's take

$$V(|\phi|) = \frac{\kappa}{2}(|\phi|^2 - v^2)^2$$

for some couplings κ, v. Here we are going to do some interesting classical field theory, to show that magnetic flux lines in a superconductor collimate into a string. Set $q = 1$ for a bit.

(a) Consider a configuration which is independent of x^3, one of the spatial coordinates, and static (independent of time). Show that its energy density (energy per unit length in x^3) is

$$U = \int d^2x \, \left(\frac{1}{2} F_{12}^2 + \frac{1}{2} |D_i \phi|^2 + V(|\phi|) \right).$$

(b) Consider the special case where $\kappa = 1$. Assuming that the integrand falls off sufficiently quickly at large $x^{1,2}$, show that

$$U_{\kappa=1} = \int d^2x \, \left(\frac{1}{2} \left(F_{12} + |\phi|^2 - v^2 \right)^2 + \frac{1}{4} |D_i \phi + \mathbf{i} \epsilon_{ij} D_j \phi|^2 + v^2 F_{12} - \frac{1}{2} \mathbf{i} \epsilon_{k\ell} \partial_k \left(\phi^\star D_\ell \phi \right) \right).$$

(c) The first two terms in the energy density of the previous part are squares and hence manifestly positive, so setting to zero the things being squared will minimize the energy density. Show that the resulting first-order equations (they are called BPS equations after people with those initials, Bogolmonyi, Prasad, Sommerfeld)

$$0 = \left(D_i + \mathbf{i} \epsilon_{ij} D_j \right) \phi, \quad F_{12} = -|\phi|^2 + v^2$$

are solved by ($x^1 + \mathbf{i} x^2 \equiv r e^{\mathbf{i}\varphi}$)

$$\phi = e^{\mathbf{i} n \varphi} f(r), \quad A_1 + \mathbf{i} A_2 = -\mathbf{i} e^{\mathbf{i}\varphi} \frac{a(r) - n}{r}$$

if

$$f' = \frac{a}{r} f, \; a' = r(f^2 - v^2)$$

with boundary conditions

$$a \to 0, f \to v + \mathcal{O}\left(e^{-mr}\right), \quad \text{at } r \to \infty \tag{3}$$

$$a \to n + \mathcal{O}(r^2), f \to r^n(1 + \mathcal{O}(r^2)), \quad \text{at } r \to 0.$$

(For other values of κ, the story is not as simple, but there is a solution with the same qualitative properties. See for example §3.3 of E. Weinberg, *Classical solutions in Quantum Field Theory.*)

(d) The second BPS equation and (27) imply that the region where field configuration differs from its vacuum behavior (in particular F_{12}) is localized near $r = 0$. Evaluate the magnetic flux through the $x^1 - x^2$ plane, $\Phi \equiv \int B \cdot da$ in the vortex configuration labelled by n. Show that the energy density is $U = \frac{v^2}{2} \cdot \Phi$.

(e) Show that the previous result for the flux follows from demanding that the two terms in $D_i\phi$ cancel at large r so that

$$D_i\phi \stackrel{r\to\infty}{\to} 0 \tag{4}$$

faster than $1/r$. Solve (30) for A_i in terms of ϕ and integrate $\int d^2x F_{12}$.

(f) Argue that a single vortex (string) in the *ungauged* theory (with global $\mathsf{U}(1)$ symmetry)

$$\mathcal{L} = |\partial\phi|^2 + V(|\phi|)$$

does not have finite energy per unit length. By a vortex, I mean a configuration where $\phi \stackrel{r\to\infty}{\to} ve^{\mathbf{i}\varphi}$, where $x^1 + \mathbf{i}x^2 = re^{\mathbf{i}\varphi}$.

(g) Consider now the case where the scalar field has charge q. (Recall that in a superconductor made by BCS pairing of electrons, the charged field which condenses has electric charge two.) Show that the magnetic flux in the core of the minimal ($n = 1$) vortex is now (restoring units) $\frac{hc}{qe}$.

8. **BPS conditions from supersymmetry.** [bonus!] What's special about $\kappa = 1$? For one thing, it is the boundary between type I and type II superconductors (which are distinguished by the size of the vortex core). More sharply, it means the mass of the scalar equals the mass of the vector, at least classically. Moreover, in the presence of some extra fermionic fields, the model with this coupling has an additional symmetry mixing bosons and fermions, namely supersymmetry. This symmetry underlies the special features we found above. Here is an outline (you can do some parts without doing others) of how this works. The logic in part (c) underlies a lot of the progress in string theory since the mid-1990s. Please do not trust my numerical factors.

(a) Add to $\mathcal{L}_h$ a charged fermion Ψ (partner of ϕ) and a neutral Majorana fermion λ (partner of A_μ):

$$\mathcal{L}_f = \frac{1}{2}\mathbf{i}\bar{\Psi}\not{D}\Psi + \mathbf{i}\bar{\lambda}\not{D}\lambda + \bar{\lambda}\Psi\phi + h.c..$$

Consider the transformation rules

$$\delta_\epsilon A_\mu = \mathbf{i}\bar{\epsilon}\gamma_\mu\lambda, \delta_\epsilon\Psi = D_\mu\phi\gamma^\mu\epsilon, \quad \delta_\epsilon\phi = -\mathbf{i}\bar{\epsilon}\Psi, \delta_\epsilon\lambda = -\frac{1}{2}\mathbf{i}\sigma^{\mu\nu}F_{\mu\nu}\epsilon + \mathbf{i}(|\phi|^2 - v)\epsilon$$

where the transformation parameter ϵ is a Majorana spinor (and a grassmann variable). Show that (something like this) is a symmetry of $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_f$. This is $\mathcal{N} = 1$ supersymmetry in $D = 4$.

(b) Show that the conserved charges associated with these transformations Q_α (they are grassmann objects and spinors, since they generate the transformations, via $\delta_\epsilon \text{fields} = [\epsilon_\alpha Q_\alpha + h.c., \text{fields}]$), satisfy the algebra

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu + 2\gamma^\mu \Sigma_\mu \tag{5}$$

where P_μ is the usual generator of spacetime translations and Σ_μ is the *vortex string charge*, which is nonzero in a state with a vortex string stretching in the μ direction. $\bar{Q} \equiv Q^\dagger \gamma^0$ as usual.

(c) If we multiply (31) on the right by γ^0, we get the positive operator $\{Q_\alpha, Q^\dagger_\beta\}$. This operator annihilates states which satisfy $Q\,|BPS\rangle = 0$ for some components of Q. Such a state is therefore invariant under some subgroup of the superymmetry, and is called a BPS state. Now look at the right hand side of (31)$\times\gamma^0$ in a configuration where $\Sigma_3 = \pi n v^2$ and show that its energy density is $E \geq \pi|n|v^2$, with the inequality saturated only for BPS states.

(d) To find BPS configurations, we can simply set to zero the relevant supersymmetry variations of the fields. Since we are going to get rid of the fermion fields anyway, we can set them to zero and consider just the (bosonic) variations of the fermionic fields. Show that this reproduces the BPS equations.