University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2020
Assignment 6

Due 12:30pm Wednesday, February 26, 2020

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1. **Brainwarmer: Fundamental theorem of functional integrals.**

   (a) Show that
   $$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$
   [Hint: square the integral and go to polar coordinates.]

   (b) Consider a collection of variables $x_i, i = 1..N$ and a hermitian matrix a_{ij}. Show that
   $$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a^{-1}_{ij} J^j}.$$
   (Summation convention in effect, as always.)

   [Hint: change variables to diagonalize a. Recall that $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

   (c) Consider a Gaussian field Q, governed by the (quadratic) euclidean action in one dimension:
   $$S[x] = \int dt \frac{1}{2}\left(\dot{Q}^2 + \Omega^2 Q^2\right).$$
   Show that
   $$\left\langle e^{-\int ds J(s) Q(s)} \right\rangle_Q = \mathcal{N} e^{+\frac{1}{2}\int ds dt J(s) G(s,t) J(t)}$$
   where G is the (Feynman) Green's function for Q, satisfying:
   $$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t).$$
   Here $\mathcal{N}$ is a normalization factor which is independent of J. Note the similarity with the previous problem, under the replacement
   $$a = -\partial_s^2 + \Omega^2, \quad a^{-1} = G.$$

   (d) Consider a Gaussian field ϕ, governed by the (quadratic) euclidean action in D dimensions
   $$S[x] = \int dt \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\right).$$

Show that

$$\left\langle e^{-\int d^D x J(x)\phi(x)} \right\rangle_\phi = \mathcal{N} e^{+\frac{1}{2}\int d^D x d^D y J(x) G(x,y) J(y)}$$

where G is the (Feynman) Green's function for ϕ, satisfying:

$$\left(-\partial_\mu \partial^\mu + m^2\right) G(x,y) = \delta^D(x-y).$$

2. **Another consequence of unitarity of the S matrix.**

   (a) Show that unitarity of S, $S^\dagger S = \mathbb{1} = S S^\dagger$, implies that the transition matrix is *normal*:
   $$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{T} \; . \tag{1}$$

   (b) What does this mean for the amplitudes $\mathcal{M}_{\alpha\beta}$ (defined as usual by $\mathcal{T}_{\alpha\beta} = \not{\delta}(p_\alpha - p_\beta)\mathcal{M}_{\alpha\beta}$)?

   (c) The probability of a transition from α to β is
   $$P_{\alpha\to\beta} = |S_{\beta\alpha}|^2 = VT\not{\delta}(p_\alpha - p_\beta)|\mathcal{M}_{\alpha\beta}|^2$$
   which is IR divergent. More useful is the transition rate per unit time per unit volume:
   $$\Gamma_{\alpha\to\beta} \equiv \frac{P_{\alpha\to\beta}}{VT}.$$
   Show that the the total decay rate of the state α is
   $$\Gamma_\alpha \equiv \int d\beta \Gamma_{\alpha\to\beta} = 2\text{Im}\,\mathcal{M}_{\alpha\alpha}.$$

   (d) Consider an ensemble of states p_α evolving according to the evolution rule
   $$\partial_t p_\alpha = -p_\alpha \Gamma_\alpha + \int d\beta p_\beta \Gamma_{\beta\to\alpha}.$$
   $S[p] \equiv -\int d\alpha p_\alpha \ln p_\alpha$ is the Shannon entropy of the distribution. Show that
   $$\frac{dS}{dt} \geq 0$$
   as a consequence of (1). This is a version of the Boltzmann H-theorem.

   (e) [Bonus] Notice that we are doing something weird in the previous part by using classical probabilities. This is a special case; more generally, we should describe such an ensemble by a density matrix $\rho_{\alpha\beta}$. Generalize the result of the previous part appropriately.

3. **Schwinger-Dyson equations.**

Consider the path integral

$$\int [D\phi] e^{\mathbf{i}S[\phi]}.$$

Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \text{(anything)}$$

(as long as 'anything' doesn't grow at large ϕ). So this equation says that we can integrate by parts in the functional integral.

(Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int [D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial\phi_x} \text{(anything)}$$

with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

This trivial-seeming set of equations (we get to pick the 'anything') can be quite useful and are called Schwinger-Dyson equations. (Be warned that these equations are sometimes also called Ward identities.) Unlike many of the other things we've discussed, they are true non-perturbatively, *i.e.* are really true, even at finite coupling. They provide a quantum implementation of the equations of motion.

(a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \left(\phi(y) e^{\mathbf{i}S[\phi]}\right)$$

to conclude that

$$\left\langle \mathcal{T} \frac{\delta S}{\delta\phi(x)} \phi(y) \right\rangle = +\mathbf{i}\delta(x-y). \tag{2}$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (2) for the case of a free massive scalar field to show that the (two-point) time-ordered correlation functions of ϕ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\left(+\Box_x + m^2\right) \langle \mathcal{T}\phi(x)\phi(y)\rangle = -\mathbf{i}\delta(x-y), \tag{3}$$

with $\Box_x \equiv \partial_{x^\mu}\partial^{x^\mu}$.

(c) Find the generalization of (3) satisfied by (time-ordered) three-point functions of the free field ϕ.

(d) Remind yourself that last quarter you (probably) derived the equation (2) (for a free theory) more arduously, from a more canonical (*i.e.* Hamiltonian) point of view, by considering what happens when you act with the wave operator $+\Box_x + m^2$ on the time-ordered two-point function.

[Hints: Use the canonical equal-time commutation relations:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\partial_{x^0}\phi(\vec{x}), \phi(\vec{y})] = -\mathbf{i}\delta^{D-1}(\vec{x}-\vec{y}).$$

Do not neglect the fact that $\partial_t\theta(t) = \delta(t)$: the time derivatives act on the time-ordering symbol!]