University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2020
Assignment 2

Due 12:30pm Monday, January 27, 2020

Note that this is two weeks' worth of homework.

---

1. **An example of renormalization in classical physics.**

   Consider a classical field in $D+2$ spacetime dimensions coupled to an *impurity* (or defect or brane) in D dimensions, located at $X = (x^\mu, 0, 0)$. Suppose the field has a self-interaction which is localized on the defect. For definiteness and calculability, we'll consider the simple (quadratic) action

   $$S[\phi] = \int d^{D+2}X \left(\frac{1}{2}\partial_\mu \phi(X)\partial^\mu \phi(X) + g\delta^2(\vec{x}_\perp)\phi^2(X) \right).$$

   (a) What is the mass dimension of the coupling g? This is why I picked a codimension[1]-two defect.

   (b) Find the equation of motion for ϕ. Where have you seen an equation like this before?

   (c) We will study the propagator for the field in a mixed representation:

   $$G_k(x,y) \equiv \langle \phi(k,x)\phi(-k,y)\rangle = \int d^D z\ e^{\mathbf{i}k_\mu z^\mu} \langle \phi(z,x)\phi(0,y)\rangle$$

   – *i.e.* we go to momentum space in the directions in which translation symmetry is preserved by the defect. Find and evaluate the diagrams contributing to $G_k(x,y)$ in terms of the free propagator $D_k(x,y) \equiv \langle \phi(k,x)\phi(-k,y)\rangle_{g=0}$. (We will not need the full form of $D_k(x,y)$.) Sum the series.

   (d) You should find that your answer to part 1c depends on $D_k(0,0)$, which is divergent. This divergence arises from the fact that we are treating the defect as infinitely thin, as a pointlike object – the δ^2-function in the interaction involves arbitrarily short wavelengths. In general, as usual, we must really be agnostic about the short-distance structure of things. To reflect this, we introduce a regulator. For example, we can replace the fourier representation of $D_k(0,0)$ with the cutoff version

   $$D_k(0,0;\Lambda) = \int_0^\Lambda đ^2 q \frac{e^{\mathbf{i}q\cdot 0}}{k^2+q^2}. \tag{1}$$

   Do the integral.

[1] An object whose position requires specification of p coordinates has codimension p.

(e) Now we renormalize. We will let the *bare coupling* g (the one which appears in the Lagrangian, and in the series from part 1c) depend on the cutoff $g = g(\Lambda)$. We wish to eliminate $g(\Lambda)$ in our expressions in favor of some measurable quantity. To do this, we impose a renormalization condition: choose some reference scale μ, and demand that

$$G_\mu(x,y) \stackrel{!}{=} D_\mu(x,y) - g(\mu) D_\mu(x,0) D_\mu(0,y). \tag{2}$$

This equation defines $g(\mu)$, which we regard as a physical quantity. Show that (2) is satisfied if we let the bare coupling be $g(\Lambda) = g(\mu) Z$, with

$$Z = \frac{1}{1 - \frac{g(\mu)}{4\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right)}.$$

(f) Find the beta function for g,

$$\beta_g(g) \equiv \mu \frac{dg(\mu)}{d\mu},$$

and solve the resulting RG equation for $g(\mu)$ in terms of some initial condition $g(\mu_0)$. Does the coupling get weaker or stronger in the UV?

2. **Vacuum energy from the propagator.**

Consider a free scalar field with

$$S = \int d^{d+1}x \, \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right).$$

(a) (Brain-warmer) Find the Hamiltonian.

(b) Reproduce the formal expression for the vacuum energy

$$\langle 0 | \mathbf{H} | 0 \rangle = V \int đ^d k \frac{1}{2} \hbar \omega_{\vec{k}}$$

using the two point function

$$\langle 0 | \phi(x)^2 | 0 \rangle = \langle 0 | \phi(0)\phi(0) | 0 \rangle = \lim_{\vec{x},t\to 0} \langle 0 | \phi(x)\phi(0) | 0 \rangle$$

and its derivatives. (V is the volume of space.)

Thus, the vacuum energy can be described as a loop diagram of the form  , where the $\times$ represents the insertion of the operator $\mathbf{H}$.

3. **Propagator corrections in a solvable field theory.**

Consider a theory of a scalar field in D dimensions with action

$$S = S_0 + S_1$$

where

$$S_0 = \int d^D x \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2 \right)$$

and

$$S_1 = - \int d^D x \frac{1}{2} \delta m^2 \phi^2 \ .$$

We have artificially decomposed the mass term into two parts. We will do perturbation theory in small δm^2, treating S_1 as an 'interaction' term. We wish to show that the organization of perturbation theory that we've seen lecture will correctly reassemble the mass term.

(a) Write down all the Feynman rules for this perturbation theory.

(b) Determine the 1PI two-point function in this model, defined by

$$\mathbf{i}\Sigma \equiv \sum (\text{all 1PI diagrams with two nubbins}) \, .$$

(c) Show that the (geometric) summation of the propagator corrections correctly produces the propagator that you would have used had we not split up $m_0^2 + \delta m^2$.

4. **Meson scattering.** Now consider the Yukawa theory with fermions, with

$$\mathcal{L} = \bar{\Psi} \left(\mathbf{i} \partial\!\!\!/ - m \right) \Psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \mathcal{L}_{\text{int}}$$

and $\mathcal{L}_{\text{int}} = g \bar{\Psi} \Psi \phi$.

(a) Draw the Feynman diagram(s) which give(s) the leading contribution to the process $\phi\phi \to \phi\phi$.

(b) Derive the correct sign of the amplitude by considering the relevant matrix elements of powers of the interaction hamiltonian. Compare with the Feynman rules for fermions.

(c) Evaluate the diagram in terms of a spinor trace and a momentum integral. Do not do the momentum integral. Suppose that the integral is cutoff at large k by some cutoff Λ. Estimate the dependence on Λ, in particular in $D = 4$.

(d) What counterterm is required to renormalize this interaction?

(e) Do you need a counterterm of the form $\delta_3\phi^3$ in this theory?

5. **Electron-photon scattering at low energy.** [This is an optional bonus problem for those of you who wish to experience some of the glory of tree-level QED.]

   Consider the process $e\gamma \to e\gamma$ in QED at leading order.

   (a) Draw and evaluate the two diagrams.

   (b) Find $\frac{1}{4}\sum_{\text{spins/polarizations}} |\mathcal{M}|^2$.

   (c) Construct the two-body final-state phase space measure in the limit where the photon frequency is $\omega \ll m$ (the electron mass), in the rest frame of the electron. I suggest the following kinematical variables:

   $$p_1 = (\omega, 0, 0, \omega), p_2 = (m, 0, 0, 0), p_4 = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta), p_3 = p_1 + p_2 - p_4 = (E', p')$$

   for the incoming photon, incoming electron, outgoing photon and outgoing electron respectively.

   (d) Find the differential cross section $\frac{d\sigma}{d\cos\theta}$ as a function of ω, θ, m. (The expression can be prettified by using the on-shell condition $p_3^2 = m^2$ to relate ω' to ω, θ. It is named after Klein and Nishina.) Compare to experiment.

   (e) Show that the limit $E \ll m$ gives the (Thomson) scattering cross section for classical electromagnetic radiation from a free electron.

6. **Brain-cooler.**

   Show that we did the right thing in the numerator of the electron self-energy: use the Clifford algebra to show that

   $$\gamma^\mu \left(x\not{p} + m_0 \right) \gamma_\mu = -2x\not{p} + 4m_0.$$