University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 9

Due 12:30pm Wednesday, March 13, 2019

1. **Brain-warmer: Spectral representation at finite temperature.**

   In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

   $$\mathbf{i}\mathcal{D}(x) = \langle 0| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |0\rangle$$

   Derive a spectral representation for the two-point function of a scalar operator at a nonzero temperature:

   $$\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr}\frac{e^{-\beta\mathbf{H}}}{Z_\beta}\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n}\langle n| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |n\rangle .$$

   Here $Z_\beta \equiv \mathrm{tr}e^{-\beta\mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result. Assume that $\mathcal{O} = \mathcal{O}^\dagger$ if you wish.

2. **Another consequence of the optical theorem.**

   A general statement of the optical theorem is:

   $$-\mathbf{i}\left(\mathcal{M}(a \to b) - \mathcal{M}(b \to a)\right) = \sum_f \int d\Phi_f \mathcal{M}^\star(b \to f)\mathcal{M}(a \to f) .$$

   Consider QED with electrons and muons.

   (a) Consider scattering of an electron (e^-) and a positron (e^+) into e^-e^+ (so $a = b$ in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to $Q_e^2 Q_\mu^2$ where Q_e and Q_μ are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude $\mathrm{Im}\,\Pi_\mu(q^2)$ (just the $Q_e^2 Q_\mu^2$ bit) as a function of $s \equiv (k_1 + k_2)^2$ where $k_{1,2}$ are the momenta of the incoming e^+ and e^-.

   Check that the imaginary part is independent of the cutoff.

(b) Use the optical theorem and the fact that the total cross section for $e^+e^- \to \mu^+\mu^-$ must be positive

$$\sigma(e^+e^- \to \mu^+\mu^-) \geq 0$$

to show that a Feynman diagram with a fermion loop must come with a minus sign. Check that with the correct sign, the optical theorem is verified.

3. **Bubble-chain approximation for bound states.**

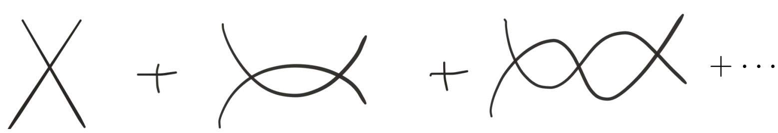
In discussing the form of the spectral density for an operator which creates a massive particle, I mentioned that in addition to the single-particle delta function at $s = m^2$, and the continuum above $s = (2m)^2$, there could be delta functions associated with bound states at $m^2 < s < (2m)^2$. Here we'll get an idea how we might discover such a thing theoretically.

For this problem, we're going to work in $D = 2 + 1$ dimensions, so that we can avoid the problem of UV divergences. Consider the theory of a single real scalar with action

$$S[\phi] = \int d^3x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \right)$$

where m, g are real. In this problem we will consider both signs of g, without worrying about questions of the stability of the vacuum.

(a) Consider the amplitude $\mathcal{M}(s)$ for elastic scattering $\phi\phi \to \phi\phi$, with $s = E_T^2$, the square of the total center of mass energy. Compute $\mathcal{M}(s)$ in the bubble-chain approximation, defined as the following infinite sum of Feynman diagrams:



Do not worry about justifying the validity of the approximation (it is not justified in this theory, though it is in a large-n version of the theory), and do not worry about convergence of the series. You can leave your answer as a Feynman parameter integral.

(b) Show, by explicit calculation, that the bubble chain approximation to the scattering amplitude obeys the optical theorem. [In elastic scattering in the center of mass frame in 3d, the element of solid angle $d\Omega$ is just an element of ordinary angle $d\theta$, and $d\sigma/d\theta = \frac{|\mathcal{M}|^2}{32\pi p E_T^2}$ where p is the magnitude of the spatial momentum of either particle.]

(c) The interaction between the ϕ quanta could result in two of them formal a bound state of mass M_B. A signal of such a bound state is the appearance of a pole in $\mathcal{M}(s)$ at $s = M_B^2$ on the real axis, but below threshold ($0 < M_B^2 < 4m^2$). Find the values of g for which the bubble-chain approximation predicts bound states. [You are not asked to give an analytic expression for M_B.]

4. **Soft gravitons?** [optional. I'm re-posting this problem in case you want to think about it more.] Photons are massless, and this means that the cross sections we measure actually include soft ones that we don't detect. If we don't include them we get IR-divergent nonsense.

   Gravitons are also massless. Why don't we need to worry about them in the same way? Here we'll sketch some hints for how to think about this question.

   (a) Consider the action
   $$S_0[h_{\mu\nu}] = \int d^4x \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu}.$$
   This is a kinetic term for (too many polarizations of a) two-index symmetric-tensor field $h_{\mu\nu} = h_{\nu\mu}$ (which we'll think of as a small fluctuation of the metric about flat space: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and this is where the coupling below comes from). Like with the photon, we'll rely on the couplings to matter to keep unphysical polarizations from being made. Write the propagator. We still raise and lower indices with $\eta_{\mu\nu}$. [1]

   (b) Couple the graviton to the electron field via
   $$S_G = \int d^4x \; G h^{\mu\nu} T_{\mu\nu}$$
   $$T_{\mu\nu} \equiv \bar{\psi} \left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu \right) \psi. \tag{1}$$
   What are the engineering dimensions of the coupling constant G? What is the new Feynman rule?

[1] A warning: I've done two misdeeds in the statement of this problem. First, the einstein-hilbert term is $\int d^4x \frac{1}{8\pi G_N} \sqrt{g} R = \int d^4x \frac{1}{8\pi G_N} (\partial h)^2 + \ldots$ has a factor of G_N in front of it. R has units of $\frac{1}{\text{length}^2}$, and g is dimensionless, so G_N has units of length^2 – it is $8\pi G_N = \frac{1}{M_{\text{Pl}}^2}$, where M_{Pl} is the Planck mass. I've absorbed a factor of $\sqrt{G_N}$ into h so that the coefficient of the kinetic term is unity. Second, the $(\partial h)^2$ here involves various index contractions, which I haven't written. Some gauge fixing (de Donder gauge) is required to arrive at the simple expression I wrote above, and one more thing – the $h_{\mu\nu}$ I've written is actually the 'trace-reversed' graviton field
$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$
where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$ is the trace. (I didn't write the bar.) For the details of this, see chapter 10 of my GR notes.

(c) Draw a (tree level) Feynman diagram which describes the creation of gravitational radiation from an electron as a result of its acceleration from the absorption of a photon ($e\gamma \to eh$). Evaluate it if you dare. Estimate or calculate the cross section (hint: use dimensional analysis).

(d) Now the main event: study the one-loop diagram by which the graviton corrects the QED vertex. Is it IR divergent? If not, why not?

(e) If you get stuck on the previous part, replace the graviton field by a massless scalar $\pi(x)$. Compare the case of derivative coupling $\lambda\partial_\mu\pi\bar{\psi}\gamma^\mu\psi$ with the more direct Yukawa coupling $y\pi\bar{\psi}\psi$.

(f) Quite a bit about the form of the coupling of gravity to matter is determined by the demand of coordinate invariance. This plays a role like gauge invariance in QED. Acting on the small fluctuation, the transformation is

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu\lambda_\nu(x) + \partial_\nu\lambda_\mu(x).$$

What condition does the invariance under this (infinitesimal) transformation impose on the object $T_{\mu\nu}$ appearing in (1).