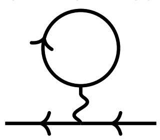
University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 8

Due 12:30pm Wednesday, March 6, 2019

1. **Brain-warmer.** Show that $(\Delta_T)^\mu_\rho \equiv \delta^\mu_\rho - \frac{q^\mu q_\rho}{q^2}$ is a projector onto momenta transverse to q^ρ. (This requires showing both that $\Delta q = 0$ and that $\Delta^2 = \Delta$.)

2. **Brain-warmer: tadpole diagram.** Why don't we worry about the following diagram  as a correction to the electron self-energy in QED?

3. **Bremsstrahlung.** Show that the number of photons per decade of wavenumber produced by the sudden acceleration of a charge is

$$f_{IR}(q^2) = \frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right),$$

where $q_\mu = p'_\mu - p_\mu$ is the change of momentum and m is the mass of the charge.

4. **Equivalent photon approximation.** Consider a process in which very high-energy electrons scatter off a target. At leading order in α, the electron line is connected to the rest of the diagram by a single photon propagator. If the initial and final energies of the electron are E and E', the photon will carry momentum q with $q^2 = -2EE'(1 - \cos\theta)$ (ignoring the electron mass $m \ll E$). In the limit of forward scattering ($\theta \to 0$), we have $q^2 \to 0$, so the photon approaches its mass shell. In this problem, we ask: To what extent can we treat it as a real photon?

   (a) The matrix element for the scattering process can be written as

   $$\mathcal{M} = -\mathbf{i} e \bar{u}(p')\gamma^\mu u(p) \frac{-\mathbf{i}\eta_{\mu\nu}}{q^2} \hat{\mathcal{M}}^\nu(q)$$

   where $\hat{\mathcal{M}}^\nu$ represents the coupling of the virtual photon to the target. Let $q = (q^0, \vec{q})$ and define $\tilde{q} = (q^0, -\vec{q})$. The electron line can be parametrized as

   $$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu$$

   where ϵ_α are unit vectors transverse to $\vec{q}$. Show that B is at most of order θ^2 (dot it with q), so we can ignore it at leading order in an expansion about forward scattering. Why do we not care about the coefficient A?

(b) Working in the frame with $p = (E, 0, 0, E)$, compute

$$\bar{u}(p')\gamma \cdot \epsilon_\alpha u(p)$$

explicitly using massless electrons, where $\bar{u}$ and u are spinors of definite helicity, and $\epsilon_{\alpha=\parallel,\perp}$ are unit vectors parallel and perpendicular to the plane of scattering. Keep only terms through order θ. Note that for $\epsilon_\parallel$, the (small) $\hat{3}$ component matters.

(c) Now write the expression for the electron scattering cross section, in terms of $|\hat{\mathcal{M}}^\mu|^2$ and the integral over phase space of the target. This expression must be integrated over the final electron momentum $\vec{p}'$. The integral over $p^{3'}$ is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

(d) The divergence as $\theta \to 0$ is regulated by the electron mass (which we've ignored above). Show that reintroducing the electron mass in the expression

$$q^2 = -2(EE' - pp'\cos\theta) + 2m^2$$

cuts off the divergence and gives a factor of $\log(s/m^2)$ in its place.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of a beam of real photons with energy distribution given by

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}(1 + (1-x)^2)\log\frac{s}{m^2}$$

where $x \equiv E_\gamma/E$. This is the Weiszäcker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD.

5. **Soft gravitons?** [optional] Photons are massless, and this means that the cross sections we measure actually include soft ones that we don't detect. If we don't include them we get IR-divergent nonsense.

Gravitons are also massless. Why don't we need to worry about them in the same way? Here we'll sketch some hints for how to think about this question.

(a) Consider the action

$$S_0[h_{\mu\nu}] = \int d^4x \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu}.$$

This is a kinetic term for (too many polarizations of a) two-index symmetric-tensor field $h_{\mu\nu} = h_{\nu\mu}$ (which we'll think of as a small fluctuation of the metric about flat space: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and this is where the coupling below comes from). Like with the photon, we'll rely on the couplings to matter to keep unphysical polarizations from being made. Write the propagator. We still raise and lower indices with $\eta_{\mu\nu}$. [1]

(b) Couple the graviton to the electron field via

$$S_G = \int d^4x \; G h^{\mu\nu} T_{\mu\nu}$$
$$T_{\mu\nu} \equiv \bar{\psi} \left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu \right) \psi. \tag{1}$$

What are the engineering dimensions of the coupling constant G? What is the new Feynman rule?

(c) Draw a (tree level) Feynman diagram which describes the creation of gravitational radiation from an electron as a result of its acceleration from the absorption of a photon ($e\gamma \to eh$). Evaluate it if you dare. Estimate or calculate the cross section (hint: use dimensional analysis).

(d) Now the main event: study the one-loop diagram by which the graviton corrects the QED vertex. Is it IR divergent? If not, why not?

(e) If you get stuck on the previous part, replace the graviton field by a massless scalar $\pi(x)$. Compare the case of derivative coupling $\lambda \partial_\mu \pi \bar{\psi} \gamma^\mu \psi$ with the more direct Yukawa coupling $y \pi \bar{\psi} \psi$.

(f) Quite a bit about the form of the coupling of gravity to matter is determined by the demand of coordinate invariance. This plays a role like gauge invariance in QED. Acting on the small fluctuation, the transformation is

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x).$$

What condition does the invariance under this (infinitesimal) transformation impose on the object $T_{\mu\nu}$ appearing in (1).

[1] A warning: I've done two misdeeds in the statement of this problem. First, the einstein-hilbert term is $\int d^4x \frac{1}{8\pi G_N} \sqrt{g} R = \int d^4x \frac{1}{8\pi G_N} (\partial h)^2 + ...$ has a factor of G_N in front of it. R has units of $\frac{1}{\text{length}^2}$, and g is dimensionless, so G_N has units of length^2 – it is $8\pi G_N = \frac{1}{M_{\text{Pl}}^2}$, where M_{Pl} is the Planck mass. I've absorbed a factor of $\sqrt{G_N}$ into h so that the coefficient of the kinetic term is unity. Second, the $(\partial h)^2$ here involves various index contractions, which I haven't written. Some gauge fixing (de Donder gauge) is required to arrive at the simple expression I wrote above, and one more thing – the $h_{\mu\nu}$ I've written is actually the 'trace-reversed' graviton field

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$ is the trace. (I didn't write the bar.) For the details of this, see chapter 10 of my GR notes.

6. ϕ^3 **theory.** Using dimensional regularization, renormalize the ϕ^3 theory

$$S[\phi] = \int d^D x \left(\frac{1}{2} Z (\partial\phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{g_0}{3!} \phi^3 \right)$$

at one loop, in arbitrary dimension D. As renormalization conditions, impose that the propagator has a pole at $p^2 = m_P^2$ with residue 1. Finally, expand $\epsilon \equiv 6 - D$ and subtract the $1/\epsilon$ pole in the three-point Green's function. I recommend redefining $g_0 \to g_0 \bar{\mu}^{\frac{D-6}{2}}$ to make it dimensionless.

7. **Yukawa couplings in QED.** Consider adding to QED an additional scalar field of (physical) mass m, coupled to the electron by

$$L_Y = \lambda \phi \bar{\psi} \psi.$$

Verify that the contribution to the electron wavefunction renormalization factor Z_2 from a virtual ϕ equals the contribution to the the QED vertex Z_1 from a vertex diagram with a virtual ϕ.