University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 5

Due 12:30pm Wednesday, February 13, 2019

1. **Electron-photon scattering at low energy.**

   Consider the process $e\gamma \to e\gamma$ in QED at leading order.

   (a) Draw and evaluate the two diagrams.

   (b) Find $\frac{1}{4}\sum_{\text{spins/polarizations}} |\mathcal{M}|^2$.

   (c) Construct the two-body final-state phase space measure in the limit where the photon frequency is $\omega \ll m$ (the electron mass), in the rest frame of the electron. I suggest the following kinematical variables:

   $$p_1 = (\omega, 0, 0, \omega), p_2 = (m, 0, 0, 0), p_4 = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta), p_3 = p_1 + p_2 - p_4 = (E', p')$$

   for the incoming photon, incoming electron, outgoing photon and outgoing electron respectively.

   (d) Find the differential cross section $\frac{d\sigma}{d\cos\theta}$ as a function of ω, θ, m. (The expression can be prettified by using the on-shell condition $p_3^2 = m^2$ to relate ω' to ω, θ. It is named after Klein and Nishina.)

   (e) Show that the limit $E \ll m$ gives the (Thomson) scattering cross section for classical electromagnetic radiation from a free electron.

2. **Non-relativistic limits.**

   (a) **Coulomb potential.**

   Derive from QED that the force between non-relativistic electrons is a repulsive $1/r^2$ force law!

   (b) **Pseudoscalar Yukawa theory.**

   Consider the theory of a massive Dirac fermion Ψ and a massive pseudoscalar φ interacting via the term

   $$V_5 \equiv g_5 \bar{\Psi} \gamma^5 \Psi \varphi.$$

   Convince yourself that this theory is parity invariant.

   List the Feynman rules.

Draw and evaluate the diagrams contributing to $\Psi\Psi \to \Psi\Psi$ scattering at leading order in g_5.

Consider the non-relativistic limit, $m \gg |\vec{p}|$ and find the effective interaction hamiltonian. If you happen to find zero for the leading term, then it's not the leading term.

3. **Scale invariant quantum mechanics.**

Consider the action for one quantum variable r with $r > 0$ and

$$S[r] = \int dt \left(\frac{1}{2} m \dot{r}^2 - V(r) \right), \qquad V(r) = \frac{\lambda}{r^2} .$$

(a) Show that the (non-relativistic) mass parameter m can be eliminated by a multiplicative redefinition of the field r or of the time t. As a result, convince yourself that the physics of interest here should only depend on the combination $m\lambda$. Show that the coupling $m\lambda$ is dimensionless: $[m\lambda] = 0$.

(b) Show that this action is *scale invariant*, *i.e.* show that the transformation

$$r(t) \to s^{\alpha} \cdot r(st) \tag{1}$$

(for some α which you must determine), (with $s \in \mathbb{R}^+$) is a symmetry. Find the associated Noether charge $\mathcal{D}$. For this last step, it will be useful to note that the infinitesimal version of (1) is ($s = e^a, a \ll 1$)

$$\delta r(t) = a \left(\alpha + t \frac{d}{dt} \right) r(t).$$

(c) Find the position-space Hamiltonian $\mathbf{H}$ governing the dynamics of r. Show that the Schrödinger equation is Bessel's equation

$$\left(-\frac{\partial_r^2}{2m} + \frac{\lambda}{r^2} \right) \psi_E(r) = E \psi_E(r).$$

Show that the Noether charge associated $\mathcal{D}$ with scale transformations ($\equiv$ dilatations) satisfies: $[\mathcal{D}, \mathbf{H}] = \mathbf{i}\mathbf{H}$. This equation says that the Hamiltonian has a definite scaling dimension, *i.e.* that its scale tranformation is $\delta \mathbf{H} = \mathbf{i} a [\mathcal{D}, \mathbf{H}] = -a\mathbf{H}$. Note that you should not need to use arcane facts about Bessel functions, only the asymptotic analysis of the equation, in subsequent parts of the problem.

(d) Describe the behavior of the solutions to this equation as $r \to 0$. [Hint: in this limit you can ignore the RHS. Make a power-law ansatz: $\psi(r) \sim r^{\Delta}$ and find Δ.]

(e) What happens if $2m\lambda < -\frac{1}{4}$? It looks like there is a continuum of negative-energy solutions (boundstates). This is another example of a *too-attractive* potential.

(f) A hermitian operator has orthogonal eigenvectors. We will show next that to make $\mathbf{H}$ hermitian when $2m\lambda < -\frac{1}{4}$, we must impose a constraint on the wavefunctions:

$$\left(\psi_E^\star \partial_r \psi_E - \psi_E \partial_r \psi_E^\star\right)|_{r=0} = 0 \tag{2}$$

There are two useful perspectives on this condition: one is that the LHS is the probability current passing through the point $r = 0$.

The other perspective is the following. Consider two eigenfunctions:

$$\mathbf{H}\psi_E = E\psi_E, \quad \mathbf{H}\psi_{E'} = E'\psi_{E'}.$$

Multiply the first equation by $\psi_{E'}^\star$ and integrate; multiply the second by $\psi_E^\star$ and integrate; take the difference. Show that the result is a boundary term which must vanish when $E = E'$.

(g) Show that the condition (2) is empty for $2m\lambda > -\frac{1}{4}$. Impose the condition (2) on the eigenfunctions for $2m\lambda < -\frac{1}{4}$. Show that the resulting spectrum of boundstates has a *discrete* scale invariance.

[Cultural remark: For some reason I don't know, restricting the Hilbert space in this way is called a *self-adjoint extension*.] [1]

(h) [Extra credit] Consider instead a particle moving in $\mathbb{R}^d$ with a central $1/r^2$ potential, $r^2 \equiv \vec{x} \cdot \vec{x}$,

$$S[\vec{x}] = \int dt \left(\frac{1}{2} m \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{\lambda}{r^2} \right).$$

Show that the same analysis applies (*e.g.* to the s-wave states) with minor modifications.

[A useful intermediate result is the following representation of (minus) the laplacian in $\mathbb{R}^d$:

$$\vec{p}^2 = -\frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r \right) + \frac{\hat{L}^2}{r^2}, \quad \hat{L}^2 \equiv \frac{1}{2} \hat{L}_{ij} \hat{L}_{ij}, \quad L_{ij} = -\mathbf{i} \left(x_i \partial_j - x_j \partial_i \right),$$

where $r^2 \equiv x^i x^i$. By 's-wave states' I mean those annihilated by $\hat{L}^2$.]

[1] This model has been studied extensively, beginning, I think, with K.M. Case, *Phys Rev* **80** (1950) 797. More recent literature includes Hammer and Swingle, arXiv:quant-ph/0503074, Annals Phys. 321 (2006) 306-317. The associated Schrödinger equation also arises as the scalar wave equation for a field in anti de Sitter space. A recent paper which discusses connections with the renormalization group in more detail is this one, by S. Paik.