University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2017
Assignment 9

Due 11am Tuesday, March 21, 2017

This problem set may grow a little bit. It is the last problem set for the quarter.

1. **Abrikosov-Nielsen-Oleson vortex string.**

   Consider the Abelian Higgs model in $D = 3 + 1$:

$$\mathcal{L}_h \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D_\mu \phi|^2 - V(|\phi|)$$

   where ϕ is a scalar field of charge e whose covariant derivative is $D_\mu \phi = (\partial_\mu - \mathbf{i} q A_\mu)\phi$, and let's take

$$V(|\phi|) = \frac{\kappa}{2}(|\phi|^2 - v^2)^2$$

   for some couplings κ, v. Here we are going to do some interesting classical field theory. Set $q = 1$ for a bit.

   (a) Consider a configuration which is independent of x^3, one of the spatial coordinates, and static (independent of time). Show that its energy density (energy per unit length in x^3) is

$$U = \int d^2x \left(\frac{1}{2} F_{12}^2 + \frac{1}{2} |D_i \phi|^2 + V(|\phi|) \right).$$

   (b) Consider the special case where $\kappa = 1$. Assuming that the integrand falls off sufficiently quickly at large $x^{1,2}$, show that

$$U_{\kappa=1} = \int d^2x \left(\frac{1}{2} \left(F_{12} + |\phi|^2 - v^2 \right)^2 + \frac{1}{4} |D_i \phi + \mathbf{i} \epsilon_{ij} D_j \phi|^2 + v^2 F_{12} - \frac{1}{2} \mathbf{i} \epsilon_{k\ell} \partial_k \left(\phi^\star D_\ell \phi \right) \right).$$

   (c) The first two terms in the energy density of the previous part are squares and hence manifestly positive, so setting to zero the things being squared will minimize the energy density. Show that the resulting first-order equations (they are called BPS equations after people with those initials, Bogolmonyi, Prasad, Sommerfeld)

$$0 = (D_i + \mathbf{i} \epsilon_{ij} D_j)\phi, \quad F_{12} = -|\phi|^2 + v^2$$

are solved by ($x^1 + \mathbf{i}x^2 \equiv re^{\mathbf{i}\varphi}$)

$$\phi = e^{\mathbf{i}n\varphi} f(r), \quad A_1 + \mathbf{i}A_2 = -\mathbf{i}e^{\mathbf{i}\varphi}\frac{a(r) - n}{r}$$

if

$$f' = \frac{a}{r}f, \; a' = r^2(f^2 - v^2)$$

with boundary conditions

$$a \to 0, f \to v + \mathcal{O}\left(e^{-mr}\right), \quad \text{at } r \to \infty \tag{1}$$

$$a \to n + \mathcal{O}(r^2), f \to r^n(1 + \mathcal{O}(r^2)), \quad \text{at } r \to 0.$$

(For other values of κ, the story is not as simple, but there is a solution with the same qualitative properties. See for example §3.3 of E. Weinberg, *Classical solutions in Quantum Field Theory.*)

(d) The second BPS equation and (1) imply that all the action (in particular F_{12}) is localized near $r = 0$. Evaluate the magnetic flux through the $x^1 - x^2$ plane, $\Phi \equiv \int B \cdot da$ in the vortex configuration labelled by n. Show that the energy density is $U = \frac{v^2}{2} \cdot \Phi$.

(e) Show that the previous result for the flux follows from demanding that the two terms in $D_i\phi$ cancel at large r so that

$$D_i\phi \overset{r\to\infty}{\to} 0 \tag{2}$$

faster than $1/r$. Solve (2) for A_i in terms of ϕ and integrate $\int d^2x F_{12}$.

(f) Argue that a single vortex (string) in the ungauged theory (with global $\mathsf{U}(1)$ symmetry)

$$\mathcal{L} = |\partial\phi|^2 + V(|\phi|)$$

does not have finite energy (density). By a vortex, I mean a configuration where $\phi \overset{r\to\infty}{\to} ve^{\mathbf{i}\varphi}$, where $x^1 + \mathbf{i}x^2 = re^{\mathbf{i}\varphi}$.

(g) Consider now the case where the scalar field has charge q. Show that the magnetic flux in the core of the minimal ($n = 1$) vortex is now (restoring units) $\frac{hc}{qe}$.

2. **BPS conditions from supersymmetry.** [bonus!] What's special about $\kappa = 1$? For one thing, it is the boundary between type I and type II superconductors. More sharply, it means the mass of the scalar equals the mass of the vector, at least classically. Moreover, in the presence of some extra fermionic fields, the model with this coupling has an additional symmetry mixing bosons and fermions, namely supersymmetry. This symmetry underlies the special features we found above. Here is an outline (you can do some parts without doing others) of how this works. Please do not trust my numerical factors.

(a) Add to $\mathcal{L}_h$ a charged fermion Ψ (partner of ϕ) and a neutral Majorana fermion λ (partner of A_μ):

$$\mathcal{L}_f = \frac{1}{2}\mathbf{i}\bar{\Psi}\not{D}\Psi + \mathbf{i}\bar{\lambda}\not{D}\lambda + \bar{\lambda}\Psi\phi + h.c..$$

Consider the transformation rules

$$\delta_\epsilon A_\mu = \mathbf{i}\bar{\epsilon}\gamma_\mu\lambda, \delta_\epsilon\Psi = D_\mu\phi\gamma^\mu\epsilon, \quad \delta_\epsilon\phi = -\mathbf{i}\bar{\epsilon}\Psi, \delta_\epsilon\lambda = -\frac{1}{2}\mathbf{i}\sigma^{\mu\nu}F_{\mu\nu}\epsilon + \mathbf{i}(|\phi|^2 - v)\epsilon$$

where the transformation parameter ϵ is a Majorana spinor (and a grassmann variable). Show that (something like this) is a symmetry of $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_f$. This is $\mathcal{N} = 1$ supersymmetry in $D = 4$.

(b) Show that the conserved charges associated with these transformations Q_α (they are grassmann objects and spinors, since they generate the transformations, via $\delta_\epsilon \text{fields} = [\epsilon_\alpha Q_\alpha + h.c., \text{fields}]$), satisfy the algebra

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu + 2\gamma^\mu\Sigma_\mu \tag{3}$$

where P_μ is the usual generator of spacetime translations and Σ_μ is the *vortex string charge*, which is nonzero in a state with a vortex string stretching in the μ direction. $\bar{Q} \equiv Q^\dagger\gamma^0$ as usual.

(c) If we multiply (3) on the right by γ^0, we get the positive operator $\{Q_\alpha, Q_\beta^\dagger\}$. This operator annihilates states which satisfy $Q\,|BPS\rangle = 0$ for some components of Q. Such a state is therefore invariant under some subgroup of the superymmetry, and is called a BPS state. Now look at the right hand side of (3)$\times\gamma^0$ in a configuration where $\Sigma_3 = \pi n v^2$ and show that its energy density is $E \geq \pi|n|v^2$, with the inequality saturated only for BPS states.

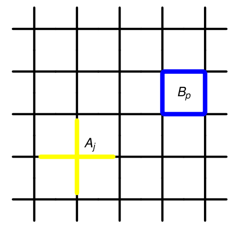
(d) To find BPS configurations, we can simply set to zero the relevant supersymmetry variations of the fields. Since we are going to get rid of the fermion fields anyway, we can set them to zero and consider just the (bosonic) variations of the fermionic fields. Show that this reproduces the BPS equations.

3. **Gauge theory can emerge from a local Hilbert space.**

The Hilbert space of a gauge theory is a funny thing: states related by a gauge transformation are physically equivalent. In particular, it is not a tensor product over independent local Hilbert spaces associated with regions of space. Because of this, there is much hand-wringing about defining entanglement in gauge theory. The following is helpful for thinking about this. It is a realization of $\mathbb{Z}_2$ lattice gauge theory, beginning from a model with no redundancy in its Hilbert space. In this avatar it is due to Kitaev and is called the *toric code*.

To define the Hilbert space, put a qbit on every link ℓ of a lattice, say the 2d square lattice, so that $\mathcal{H} = \otimes_\ell \mathcal{H}_\ell$. Let $\sigma^x_\ell, \sigma^z_\ell$ be the associated Pauli operators, and recall that $\{\sigma^x_\ell, \sigma^z_\ell\} = 0$. $\mathcal{H}_\ell = \text{span}\{|\sigma^z_\ell = 1\rangle, |\sigma^z_\ell = -1\rangle\}$ is a useful basis for the Hilbert space of a single link.

One term in the hamiltonian is associated with each site $j \to A_j \equiv \prod_{l \in j} \sigma^z_l$ and one with each plaquette $p \to B_p \equiv \prod_{l \in \partial p} \sigma^x_l$, as indicated in the figure at right.



$$\mathbf{H} = -\Gamma_e \sum_j A_j - \Gamma_m \sum_p B_p.$$

(a) Show that all these terms commute with each other.

(b) The previous result means we can diagonalize the Hamiltonian by minimizing one term at a time. Let's imagine that $\Gamma_e \gg \Gamma_m$ so we'll minimize the 'star' terms A_j first. Which states satisfy the 'star condition' $A_j = 1$? In the σ^x basis there is an extremely useful visualization: we say a link l of $\hat{\Gamma}$ is covered with a segment of string (an electric flux line) if $\sigma^x_l = -1$ (so the electric field on the link is $\mathbf{e}_l = 1$) and is not covered if $\sigma^x_l = +1$ (so the electric field on the link is $\mathbf{e}_l = 0$):  $\equiv (\sigma^z_\ell = -1)$. Draw all possible configurations incident on a single vertex j and characterize which ones satisfy $A_j = 1$.

(c) [bonus] What is the effect of adding a term $\Delta\mathbf{H} = \sum_\ell g\sigma^x$? Convince yourself that in the limit $\Gamma_e \gg \Gamma_m$, for energies $E \ll \Gamma_e$, this is identical to $\mathbb{Z}_2$ lattice gauge theory, where $A_j = 1$ is a discrete version of the Gauss law constraint. [This part is a bonus problem because I have not yet explained how to go from the euclidean lattice gauge theory to a Hamiltonian formulation. If you want to figure it out yourself, you can get help from section V.E of this Kogut review.]

(d) [bonus] Set $g = 0$ again. In the subspace of solutions of the star condition, find the groundstate(s) of the plaquette term. First consider a simply-connected region of lattice, then consider periodic boundary conditions.

4. **Brain-cooler.**

(a) Show that the *adjoint* representation matrices

$$\left(T^B\right)_{AC} \equiv -\mathbf{i} f_{ABC}$$

furnish a dim G-dimensional representation of the Lie algebra

$$[T^A, T^B] = \mathbf{i} f_{ABC} T^C \quad .$$

Hint: commutators satisfy the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

(b) From the transformation law for A, show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

Below here are some more optional problems.

5. **Wilson loops in abelian gauge theory at weak and strong coupling.**

(a) At weak coupling, the Wilson loop expectation value is a gaussian integral. In $D = 4$, study the continuum limit of a rectangular loop with time extent $T \gg R$, the spatial extent. Show that this reproduces the Coulomb force. For help, see VI.B of the Kogut paper linked above.

(b) Compute the combinatorial factors in the first few terms of the strong-coupling expansion of the same quantity.

(c) Consider the case of two spacetime dimensions. In this case, the plaquette variables are actually independent variables (since the relations between them arise from the boundaries of 3-volumes).

(d) Consider the weak coupling calculation again for a Wilson loop coupled to a massive vector field. Show that this reproduces an exponentially-decaying force between external static charges.