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Physics 215B QFT Winter 2017
Assignment 8

Due 11am Thursday, March 10, 2017

In the next two problems, we will study free massless bosons in two dimensions. This system is solvable and has many physical applications – *e.g.* in string theory, and at the edge of quantum Hall systems. It is an example of a *conformal field theory.* It is a field theory where the excitations are *not particles.*

1. **There are no Goldstone bosons in two dimensions.**

(a) Consider a massless scalar X in 2d, with action

$$S[X] = -\frac{1}{4\pi}\int d^2\sigma \partial_a X \partial^a X. \tag{1}$$

Show that the euclidean Green function G_2 satisfies

$$\nabla^2 G_2(z, z') = -2\pi\delta^2(z - z') \tag{2}$$

($z = \sigma_1^E + \mathbf{i}\sigma_2^E$)[1] and is given by

$$G_2(z, z') = -\ln|z - z'|,$$

for example by Fourier transform.

(b) [Perhaps this part is more of a diatribe than a problem.] The long-distance behavior of G_2 has important implications for the physics of massless scalars in two dimensions. Thinking of G_2 as the two point function of the massless scalar

$$G_2(z, z') = \langle X(z)X(z')\rangle$$

let's ask the following question:

There is no potential energy for the field X in (1). Someone used to doing physics in 3+1 dimensions might think that this means that there is a vacuum for every value of X. Let's try to fix the expectation value of the scalar $\langle X\rangle = x$ and see what happens. Perturb the putative vacuum $|x\rangle$ a little bit at the position z by inserting the operator X there. To measure what happens, insert the operator X at z'. The correlator G_2 can thus be interpreted as a measurement of how

[1] Recall Schwinger-Dyson equations from problem set 2.

the effects of our perturbation fall off with distance. What happens? Contrast this with the behavior you would see for a scalar field with a flat potential in more than two dimensions. Note that the case of (0+1) dimensional QFT (*i.e.* quantum mechanics) is even more problematic in the infrared.

One way to arrive at an action like (1) is if the field X arises as a Goldstone boson associated with a symmetry $X \to X + a$, which would be broken by fixing the vacuum $|x\rangle$. Then it is guaranteed by Goldstone's theorem that the action can only depend on derivatives of X. We will say more about Goldstone's theorem later, but note that the Goldstone-ness of the massless bosons (*i.e.* whether they are massless because of a broken symmetry) is not crucial for this discussion. One might expect massless bosons whose masslessness is not protected by a symmetry to be lifted (to acquire a mass) quantumly, but there are special cases (such as in supersymmetric theories) where different points on the space of minima of the potential need not be related by a symmetry.

This result is called the Coleman-Mermin-Wagner (sometimes Hohenberg, too) Theorem. Coleman's paper on the subject is S. Coleman, "There are no Goldstone bosons in two-dimensions," *Commun. Math. Phys.* **31**:259-264 (1973).

2. **Correlators of composite operators made of free bosons in 1+1 dimensions.**

Consider a collection of n two-dimensional free bosons X^μ governed by the action

$$S = -\frac{1}{4\pi g} \int d^2\sigma \partial_a X_\mu \partial^a X^\mu.$$

[The coupling g can be absorbed into the definition of X if we prefer, but it is useful to leave this coupling constant arbitrary since different physicists use different conventions for the normalization and as you will see this affects the appearance of the final answer.]

Until further notice, we will assume that X takes values on the real line.

(a) Rotate $e^{\mathbf{i}S}$ to Euclidean space ($d^2\sigma = -\mathbf{i}(d^2\sigma)_E$) and compute the Euclidean generating functional

$$Z[J] = \left\langle e^{\int (d^2\sigma)_E J^\mu X_\mu} \right\rangle \equiv Z_0^{-1} \int [dX] e^{\mathbf{i}S} e^{\int (d^2\sigma)_E J^\mu X_\mu}$$

(where $Z_0^{-1} \equiv Z[J=0]$ but please don't worry too much about the normalization of the path integral).

[Hint: use the Green function from the previous problem, and Wick's theorem. Or use our general formula for Gaussian integrals with sources.]

[Warning: In the problem at hand, even the euclidean kinetic operator has a kernel, namely the zero-momentum mode. You will need to do this integral separately.]

[Cultural remark 1: this field theory describes the propagation of featureless strings in n-dimensional flat space $\mathbb{R}^n$ – think of $X^\mu(\sigma)$ as the parametrizing the position in $\mathbb{R}^n$ to which the point σ is mapped.

Cultural remark 2: this is an example of a *conformal field theory*. In particular recall that massless scalars in $D = 2$ have engineering dimension zero.]

(b) Show that

$$\left\langle \prod_{i=1}^{N} : e^{-i\sqrt{2\alpha'}k_i \cdot X(\sigma^{(i)})} : \right\rangle = \delta^n \left(\sum_i k_i^\mu \right) \prod_{i,j=1}^{N} |z_i - z_j|^{+\alpha' g k_i \cdot k_j} \tag{3}$$

where $\sigma^{(i)}$ label points in 2d Euclidean space, $z_i \equiv \sigma_1^{(i)} + i\sigma_2^{(i)}$, α' is a parameter with dimensions of $[X^2/g]$ (called the 'Regge slope'), and k_i^μ are a set of arbitrary n-vectors in the target space. The : ... : indicate the following prescription for *defining* composite operators. The prescription is simply to leave out Wick contractions of objects within a pair of : ... :. Give a symmetry explanation of the delta function in k.

[Cultural remark: this calculation is the central ingredient in the *Veneziano amplitude* for scattering of bosonic strings at tree level.]

(c) Conclude that the composite operator $\mathcal{O}_a \equiv : e^{\mathbf{i}aX} :$ has *scaling dimension* $\Delta_a = \frac{ga^2}{2}$, in the sense that

$$\left\langle \mathcal{O}_a(z) \mathcal{O}_b^\dagger(0) \right\rangle = \delta(a-b) \frac{1}{|z|^{2\Delta_a}}.$$

Notice that the correlation functions of these operators do not describe the propagation of *particles* in any sense. The operator $\mathcal{O}$ produces some power-law excitation of the CFT soup.

(d) Suppose we have one field ($n = 1$) X which takes values on the circle, that is, we identify

$$X \simeq X + 2\pi R \; .$$

What values of a label single-valued operators $: e^{\mathbf{i}aX} :$? How should we modify (3)?

3. **1, 2, many.**

   By integrating momentum shells, derive the RG equations for the quartic and quadratic operators of the relativistic O(N) vector model in D dimensions,

$$S_E[\phi^a] = \int d^D x \left(\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \frac{1}{2} r \phi^a \phi^a + u \left(\phi^a \phi^a \right)^2 \right) .$$

   Here $a = 1..N$. [If you've done this problem already, for example in the RG class, please don't duplicate effort.]

4. **Schwinger-Dyson equations.**

   Consider the path integral

$$\int [D\phi] e^{\mathbf{i} S[\phi]}.$$

   Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi] \frac{\delta}{\delta \phi(x)} \left(\text{anything} \right)$$

   (as long as 'anything' doesn't grow at large ϕ). So this equation says that we can integrate by parts in the functional integral.

   (Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int [D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial \phi_x} \left(\text{anything} \right)$$

   with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

   This trivial-seeming set of equations (we get to pick the 'anything') can be quite useful and are called Schwinger-Dyson equations. (Be warned that these equations are sometimes also called Ward identities.) Unlike many of the other things we've discussed, they are true non-perturbatively, *i.e.* are really true, even at finite coupling. They provide a quantum implementation of the equations of motion.

   (a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta \phi(x)} \left(\phi(y) e^{\mathbf{i} S[\phi]} \right)$$

to conclude that

$$\left\langle \mathcal{T} \frac{\delta S}{\delta \phi(x)} \phi(y) \right\rangle = +\mathbf{i}\delta(x-y). \tag{4}$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (4) for the case of a free massive scalar field to show that the (two-point) time-ordered correlation functions of ϕ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\left\langle \mathcal{T} \left(+\Box_x + m^2\right) \phi(x)\phi(y) \right\rangle = -\mathbf{i}\delta(x-y), \tag{5}$$

with $\Box_x \equiv \partial_{x^\mu}\partial^{x^\mu}$.

(c) Find the generalization of (5) satisfied by (time-ordered) three-point functions of the free field ϕ.

(d) Remind yourself that last quarter you derived the equation (4) (for a free theory) more arduously, from a more canonical (*i.e.* Hamiltonian) point of view, by considering what happens when you act with the wave operator $+\Box_x + m^2$ on the time-ordered two-point function.

[Hints: Use the canonical equal-time commutation relations:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\partial_{x^0}\phi(\vec{x}), \phi(\vec{y})] = -\mathbf{i}\delta^{D-1}(\vec{x}-\vec{y}).$$

Do not neglect the fact that $\partial_t \theta(t) = \delta(t)$: the time derivatives act on the time-ordering symbol!]