University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2017
Assignment 6

Due 11am Thursday, February 23, 2017

1. **Heavy leptons get real.**

   Consider the contribution of a single loop of a heavy lepton of mass M to the vacuum polarization. Find the imaginary part of $\text{Im}\,\Pi_L(q^2)$. Show that it is independent of the cutoff. Check that it agrees with the optical theorem result for the $e^+e^- \to L^+L^-$ cross section.

2. **Another consequence of unitarity of the S matrix.**

   (a) Show that unitarity of S, $S^\dagger S = \mathbb{1} = SS^\dagger$, implies that the transition matrix is *normal*:
   $$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{T} \; . \tag{1}$$

   (b) What does this mean for the amplitudes $\mathcal{M}_{\alpha\beta}$ (defined as usual by $\mathcal{T}_{\alpha\beta} = \not{\delta}(p_\alpha - p_\beta)\mathcal{M}_{\alpha\beta}$)?

   (c) The probability of a transition from α to β is
   $$P_{\alpha\to\beta} = |S_{\beta\alpha}|^2 = VT\not{\delta}(p_\alpha - p_\beta)|\mathcal{M}_{\alpha\beta}|^2$$
   which is IR divergent. More useful is the transition rate per unit time per unit volume:
   $$\Gamma_{\alpha\to\beta} \equiv \frac{P_{\alpha\to\beta}}{VT}.$$
   Show that the the total decay rate of the state α is
   $$\Gamma_\alpha \equiv \int d\beta \Gamma_{\alpha\to\beta} = 2\text{Im}\,\mathcal{M}_{\alpha\alpha}.$$

   (d) Consider an ensemble of states p_α evolving according to the evolution rule
   $$\partial_t p_\alpha = -p_\alpha \Gamma_\alpha + \int d\beta p_\beta \Gamma_{\beta\to\alpha}.$$
   $S[p] \equiv -\int d\alpha p_\alpha \ln p_\alpha$ is the Shannon entropy of the distribution. Show that
   $$\frac{dS}{dt} \geq 0$$
   as a consequence of (1). This is a version of the Boltzmann H-theorem.

(e) [Bonus] Notice that we are doing something weird in the previous part by using classical probabilities. This is a special case; more generally, we should describe such an ensemble by a density matrix $\rho_{\alpha\beta}$. Generalize the result of the previous part appropriately.

3. **Fundamental theorem of functional integrals.**

(a) Show that

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

[Hint: square the integral and go to polar coordinates.]

(b) Consider a collection of variables $x_i, i = 1..N$ and a hermitian matrix a_{ij}. Show that

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a_{ij}^{-1} J^j}.$$

(Summation convention in effect, as always.)

[Hint: change variables to diagonalize a. Recall that $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

(c) Consider a Gaussian field Q, governed by the (quadratic) euclidean action in one dimension:

$$S[x] = \int dt \frac{1}{2}\left(\dot{Q}^2 + \Omega^2 Q^2\right).$$

Show that

$$\left\langle e^{-\int ds J(s) Q(s)} \right\rangle_Q = \mathcal{N} e^{+\frac{1}{2}\int ds dt J(s) G(s,t) J(t)}$$

where G is the (Feynman) Green's function for Q, satisfying:

$$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t).$$

Here $\mathcal{N}$ is a normalization factor which is independent of J. Note the similarity with the previous problem, under the replacement

$$a = -\partial_s^2 + \Omega^2, \quad a^{-1} = G.$$

(d) Consider a Gaussian field ϕ, governed by the (quadratic) euclidean action in D dimensions

$$S[x] = \int dt \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\right).$$

Show that

$$\left\langle e^{-\int d^D x J(x) \phi(x)} \right\rangle_\phi = \mathcal{N} e^{+\frac{1}{2}\int d^D x d^D y J(x) G(x,y) J(y)}$$

where G is the (Feynman) Green's function for ϕ, satisfying:

$$\left(-\partial_\mu \partial^\mu + m^2\right) G(x,y) = \delta^D(x-y).$$