

Physics 215B QFT Winter 2017 Assignment 1

Due 11am Tuesday, January 17, 2017

1. Scale invariant quantum mechanics.

Consider the action for one quantum variable r with $r > 0$ and

$$S[r] = \int dt \left(\frac{1}{2} m \dot{r}^2 - V(r) \right), \quad V(r) = \frac{\lambda}{r^2}.$$

- (a) Show that the (non-relativistic) mass parameter m can be eliminated by a multiplicative redefinition of the field r or of the time t . As a result, convince yourself that the physics of interest here should only depend on the combination $m\lambda$. Show that the coupling $m\lambda$ is dimensionless: $[m\lambda] = 0$.
- (b) Show that this action is *scale invariant*, *i.e.* show that the transformation

$$r(t) \rightarrow s^\alpha \cdot r(st) \tag{1}$$

(for some α which you must determine), (with $s \in \mathbb{R}^+$) is a symmetry. Find the associated Noether charge \mathcal{D} . For this last step, it will be useful to note that the infinitesimal version of (1) is ($s = e^a$, $a \ll 1$)

$$\delta r(t) = a \left(\alpha + t \frac{d}{dt} \right) r(t).$$

- (c) Find the position-space Hamiltonian \mathbf{H} governing the dynamics of r . Show that the Schrödinger equation is Bessel's equation

$$\left(-\frac{\partial_r^2}{2m} + \frac{\lambda}{r^2} \right) \psi_E(r) = E \psi_E(r).$$

Show that the Noether charge associated \mathcal{D} with scale transformations (\equiv dilatations) satisfies: $[\mathcal{D}, \mathbf{H}] = \mathbf{iH}$. This equation says that the Hamiltonian has a definite scaling dimension, *i.e.* that its scale transformation is $\delta \mathbf{H} = \mathbf{ia}[\mathcal{D}, \mathbf{H}] = -\mathbf{aH}$. Note that you should not need to use arcane facts about Bessel functions, only the asymptotic analysis of the equation, in subsequent parts of the problem.

- (d) Describe the behavior of the solutions to this equation as $r \rightarrow 0$. [Hint: in this limit you can ignore the RHS. Make a power-law ansatz: $\psi(r) \sim r^\Delta$ and find Δ .]

- (e) What happens if $2m\lambda < -\frac{1}{4}$? It looks like there is a continuum of negative-energy solutions (boundstates). This is another example of a *too-attractive* potential.
- (f) A hermitian operator has orthogonal eigenvectors. We will show next that to make \mathbf{H} hermitian when $2m\lambda < -\frac{1}{4}$, we must impose a constraint on the wavefunctions:

$$(\psi_E^* \partial_r \psi_E - \psi_E \partial_r \psi_E^*)|_{r=0} = 0 \quad (2)$$

There are two useful perspectives on this condition: one is that the LHS is the probability current passing through the point $r = 0$.

The other perspective is the following. Consider two eigenfunctions:

$$\mathbf{H}\psi_E = E\psi_E, \quad \mathbf{H}\psi_{E'} = E'\psi_{E'}.$$

Multiply the first equation by $\psi_{E'}^*$ and integrate; multiply the second by ψ_E^* and integrate; take the difference. Show that the result is a boundary term which must vanish when $E = E'$.

- (g) Show that the condition (2) is empty for $2m\lambda > -\frac{1}{4}$. Impose the condition (2) on the eigenfunctions for $2m\lambda < -\frac{1}{4}$. Show that the resulting spectrum of boundstates has a *discrete* scale invariance.

[Cultural remark: For some reason I don't know, restricting the Hilbert space in this way is called a *self-adjoint extension*.]¹

- (h) [Extra credit] Consider instead a particle moving in \mathbb{R}^d with a central $1/r^2$ potential, $r^2 \equiv \vec{x} \cdot \vec{x}$,

$$S[\vec{x}] = \int dt \left(\frac{1}{2} m \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{\lambda}{r^2} \right).$$

Show that the same analysis applies (*e.g.* to the s-wave states) with minor modifications.

[A useful intermediate result is the following representation of (minus) the laplacian in \mathbb{R}^d :

$$\vec{p}^2 = -\frac{1}{r^{d-1}} \partial_r (r^{d-1} \partial_r) + \frac{\hat{L}^2}{r^2}, \quad \hat{L}^2 \equiv \frac{1}{2} \hat{L}_{ij} \hat{L}_{ij}, \quad L_{ij} = -\mathbf{i} (x_i \partial_j - x_j \partial_i),$$

where $r^2 \equiv x^i x^i$. By 's-wave states' I mean those annihilated by \hat{L}^2 .]

¹This model has been studied extensively, beginning, I think, with K.M. Case, *Phys Rev* **80** (1950) 797. More recent literature includes Hammer and Swingle, [arXiv:quant-ph/0503074](https://arxiv.org/abs/quant-ph/0503074), *Annals Phys.* 321 (2006) 306-317. The associated Schrödinger equation also arises as the scalar wave equation for a field in anti de Sitter space.