University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Winter 2016
Assignment 7

Due 2pm Tuesday, March 15, 2016

1. **Perturbative RG for worldsheet (Edwards-Flory) description of SAWs.**

[from Terry Hwa] Consider the Edwards hamiltonian

$$H[\vec{r}] = \frac{K}{2}\int_0^L ds \left(\frac{d\vec{r}}{ds}\right)^2 + \frac{u}{2}\int_{|s_1-s_2|>a} ds_1 ds_2 \; \delta^d[\vec{r}(s_1) - \vec{r}(s_2)]$$

with a self-avoidance coupling $u > 0$, a short-distance cutoff a, and an IR cutoff L. We would like to understand the large-L scaling of the polymer size, R,

$$R^2(L) \equiv \langle |\vec{r}(L) - \vec{r}(0)|^2 \rangle \sim L^{2\nu}.$$

(a) Consider the probability density for two points a distance $|s_1 - s_2|$ along the chain to be separated in space by a displacement $\vec{x}$,

$$P(\vec{x}; s_1 - s_2) = \langle \delta^d[\vec{r}(s_1) - \vec{r}(s_2) - \vec{x}] \rangle.$$

Show that the polymer size R can be obtained from its fourier transform $\tilde{P}(\vec{q}; s)$ by a

$$R^2(L) = -\nabla_q^2 \tilde{P}(\vec{q}; L)|_{q=0}.$$

(This does not involve a choice of hamiltonian.)

(b) For the free case $u = 0$, compute the polymer size $R_0(L)$ in terms of d, L, K. It may be helpful to derive a relation of the form

$$\langle e^{\mathbf{i}\int_0^L ds \vec{k}(s)\cdot\vec{r}(s)} \rangle_0 = e^{\frac{1}{2K}\int_0^L ds ds' \vec{k}(s)\cdot\vec{k}(s') G(s-s')}.$$

(c) Develop an expansion of $\tilde{P}(\vec{q}; L)$ to first order in u, using the cumulant expansion as in §6.6 of the lecture notes. You should find an expression of the form $R^2(L) = R_0^2(L)\left(1 + \delta R_1^2(L) + \mathcal{O}(u^2)\right)$ with

$$\delta R_1^2(L) = \frac{u}{L}\left(\frac{K}{2\pi}\right)^{d/2} \int_0^L ds_1 \int_{s_1+a}^L ds_2 \frac{A^2(s_1, s_2; L)}{|s_1 - s_2|^{\frac{d-2}{2}}}.$$

(d) Show that the integrals in the previous part diverge as $a/L \to 0$ below a certain dimension d_c. More precisely, by changing variables to $s = s_1 - s_2$ and $\bar{s} = (s_1 + s_2)/2$ (and ignoring stuff at the upper limit of integration, as appropriate for $L \gg a$) show that

$$\delta R_1^2(L) \simeq u \left(\frac{K}{2\pi}\right)^{d/2} \int_a^L ds s^{\frac{\epsilon}{2}-1}$$

with $\epsilon = d_c - d$.

(e) How does $\vec{r}$ scale with $s \mapsto bs$ if we demand that the free hamiltonian ($u = 0$) is a fixed point? What is ν at the free fixed point?

(f) Find d_c by power counting.

(g) We wish to integrate out the short distance fluctuations with wavelengths between a and ba, to find an effective Hamiltonian governing the remaining degrees of freedom:

$$\tilde{H}[\vec{r}] = \frac{\tilde{K}}{2} \int_0^L ds \left(\frac{d\vec{r}}{ds}\right)^2 + \frac{\tilde{u}}{2} \int_{|s_1 - s_2| > ba} ds_1 ds_2 \delta^d[\vec{r}(s_1) - \vec{r}(s_2)]$$

Using the first-order-in-u result for δR above, show that for small ϵ and small $\log b$, the coarse-grained 'stiffness' parameter is of the form

$$\tilde{K} = K(1 - \bar{v} \log b)$$

and find $\bar{v}$.

(h) A similar calculation yields $\tilde{v} = v(1 - 2\bar{v} \log b)$. Do rescaling step of the RG procedure, redefining s by a factor of $b = 1 + \ell + \mathcal{O}(\ell^2)$ and rescaling the $\vec{r} \to Z(b)\vec{r}$ to put the Hamiltonian back in the original form with the original cutoff and renormalized parameters K', v'.

(i) Find the beta functions for $K(\ell)$ and $v(\ell)$. Find ν to first order in ϵ.

2. **Self-avoiding membranes?**

[Slightly open-ended.] Consider redoing the Edwards-Flory analysis for a theory of membranes. The fields are now $\vec{r}(\sigma_1, \sigma_2, \sigma_D)$, vectors parametrizing the embedding of a D-dimensional object into $\mathbb{R}^d$. We might consider perturbing the Gaussian action

$$S_0[r] = \int d^D\sigma \sum_{\alpha=1}^{D} (\partial_{\sigma_\alpha} \vec{r})^2$$

by a self-avoidance term

$$S_u[r] = \int d^D\sigma \int d^D\sigma' \delta^d(\vec{r}(\sigma - \sigma')).$$

For various d and D, what does the Flory argument predict for the scaling exponent of the brane size with the linear size L of the base space? For which values is the excluded-volume term relevant?

Are there other terms we should consider in the action?

Try to resist googling before you think about this question.

3. **Random walk on a fractal.**

   Consider the adjacency matrix of the d-dimensional generalization of the Sierpinski triangle, with vertices at the corners of d-simplices. Find the transformation rule for the dimensionless hopping parameter $x \to x'(x)$ under the (generalization of the) decimation procedure defined in lecture.

   Use a computer to make a plots of the Nth iterate of x as a function of x, for some large enough N.

   What are the properties of the map which differ between translation-invariant spaces and fractals?