University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Winter 2016
Assignment 2 – Solutions

Due 2pm Thursday, January 28, 2016

This homework begins with two simple problems to warm up your brain.

1. **Reminder.** Check from the 'bridge' equation $F = -T \log Z$ that $F = E - TS$ where E is the average energy and $S = -\partial_T F$ is the entropy.

2. What is the relationship between the spin-spin correlation function $\langle s_i s_j \rangle$ and the probability that s_i and s_j are pointing in the same direction? (Hint: $(1 + s_i s_j)/2$ is a projector.)

3. **Real-space RG for the SAW.** Consider again the problem of a self-avoiding walk on the square lattice. Construct an RG scheme with zoom factor $\lambda = 3$ (so that nine sites of the fine lattice are represented by one site of the coarse lattice). Find the RG map $K'(K)$, find its fixed points and estimate the critical exponent at the nontrivial fixed point. Is it closer to the numerical result than the $\lambda = 2$ schemes discussed in lecture and by Creswick?

In the following problems we consider a closed (periodic) chain of N classical spins $s_i = \pm 1$ with Hamiltonian

$$H = -J \sum_i s_i s_{i+1} - h \sum_i s_i + \text{const}, \quad s_{N+1} = s_1$$

The partition function is $Z(\beta J, \beta h) = \sum_{\{s\}} e^{-\beta H}$; let's measure J, h in units of temperature, *i.e.* set $\beta = 1$.

4. **Ising model in 1d by transfer matrix.**

   (a) Show that the partition function can be written as

   $$Z = \text{tr}_2 T^N$$

   where T is the 2×2 matrix

   $$T = \begin{pmatrix} e^{J+h} & e^{-J} \\ e^{-J} & e^{J-h} \end{pmatrix}$$

(called the *transfer matrix*) and $\text{tr}_2 M = M_{11} + M_{22}$ denotes trace in this two-dimensional space. Express Z in terms of the eigenvalues of T and find the free energy density $f = -\frac{T}{N} \log Z$ in the thermodynamic ($N \to \infty$) limit. Plot the free energy for $h = 0$ as a function of $x = e^{-4J}$ for $0 \le x \le 1$.

(b) Find an expression for the correlation function

$$G(m) \equiv \langle s_i s_{m+i} \rangle - \langle s_i \rangle \langle s_{i+m} \rangle$$

using the transfer matrix. Show that as $N \to \infty$,

$$G(m) \sim e^{-m/\xi}$$

where $\xi = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)}$ where $\lambda_1 > \lambda_2$ are eigenvalues of T. Note that $\xi \to \infty$ when $\lambda_1 \to \lambda_2$. For what values of β, h, J does this happen?

5. **Decimation of 1d Ising model in a field.**

Now suppose that N is even.

(a) Decimate the even sites:

$$\sum_{s_{\text{even}}} e^{-H(s)} \equiv \Delta e^{H_{\text{eff}}(s_{\text{odd}})}.$$

More explicitly, identify the terms in $H(s)$ that depend on any one even site, $H_2(s)$ and define its contribution to H_{eff} by

$$\sum_{s_2} e^{-H_2(s)} \equiv \Delta e^{-\Delta H_{\text{eff}}(s_1, s_3)}$$

Rewriting $H_{\text{eff}}(s_{\text{odd}}) = -J' \sum ss - h' \sum s - \text{const}$ in the usual form, find J', h' and the constant in terms of the microscopic parameters J, h.

I find

$$J' = \frac{1}{4\beta} \log\left(\frac{\cosh\beta(2J+h)\cosh\beta(2J-h)}{\cosh^2\beta h}\right)$$
$$h' = h + \frac{1}{2\beta}\log\left(\frac{\cosh 2\beta(2J+h)}{\cosh 2\beta(2J-h)}\right)$$
$$\Delta = \cosh\beta h\sqrt{\cosh\beta(2J+h)\cosh\beta(2J-h)}.$$

(b) Let $w \equiv \tanh\beta J, v \equiv \tanh\beta h$. Plot some RG trajectories in the v, w plane.

(c) Find all the fixed points and compute the exponents near each of the fixed points.

The fixed points are $J^\star = \infty, h^\star = 0$: the ferromagnetic, strong coupling fixed point, and $J^\star = 0$, any h, the paramagnetic, high-temperature fixed point. Near the paramagnetic fixed point,

$$w' = w, v' = v^2(1 - w^2).$$

The two eigenvalues of the 2×2 matrix

$$R = \frac{\partial(v', w')}{\partial(v, w)}$$

are

$$\begin{aligned} y_v &= \frac{1}{\log 2} \log \partial_v v'|_{v_\star=0} = -\infty \qquad \textit{very} \text{ irrelevant.} \\ y_h &= \frac{1}{\log 2} \log \partial_w w'|_{v_\star=0} = 0 \qquad \text{marginal.} \end{aligned} \tag{1}$$

Near the ferromagnetic fixed point, $w' = 2w, t' = 2t$, where $t \equiv 1 - v$. Here we have $y_v = y_t = 1$, two relevant perturbations.