University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (130C) Winter 2014
Final exam

Please remember to put your name on your exam booklet. This is a closed-book exam. There are 6 problems, each with several parts, of varying levels of difficulty; make sure you try all of the parts. None of the problems require very extensive calculation; if you find yourself involved in a morass of calculation, step back and think. Good luck.

Possibly useful information:

$$\mathbf{U}(t) = e^{-\mathbf{i}\mathbf{H}t/\hbar} \quad \text{satisfies} \quad \mathbf{i}\hbar\partial_t \mathbf{U} = [\mathbf{H}, \mathbf{U}].$$

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\uparrow_{\hat{n}}\rangle = e^{-\mathbf{i}\varphi/2}\cos\frac{\theta}{2}|\uparrow_{\hat{z}}\rangle + e^{+\mathbf{i}\varphi/2}\sin\frac{\theta}{2}|\downarrow_{\hat{z}}\rangle \quad \text{satisfies} \quad \vec{\boldsymbol{\sigma}}\cdot\hat{n}|\uparrow_{\hat{n}}\rangle = |\uparrow_{\hat{n}}\rangle$$

$$e^{-i\alpha\hat{n}\cdot\vec{\boldsymbol{\sigma}}} = \mathbb{1}\cos\alpha - i\hat{n}\cdot\vec{\boldsymbol{\sigma}}\sin\alpha.$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{H}_{\text{SHO}} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right) = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2$$

$$\mathbf{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right), \quad \mathbf{p} = \frac{1}{\mathbf{i}}\sqrt{\frac{\hbar m\omega}{2}}\left(\mathbf{a} - \mathbf{a}^\dagger\right); \quad [\mathbf{q}, \mathbf{p}] = \mathbf{i}\hbar \implies [\mathbf{a}, \mathbf{a}^\dagger] = 1.$$

In Coulomb gauge, in vacuum ($\vec{\nabla}\cdot\vec{A} = 0, \Phi = 0$): $\quad \vec{E} = -\partial_t\vec{A}, \quad \vec{B} = \vec{\nabla}\times\vec{A}$.

$$\vec{\mathbf{A}}(\vec{r}) = \sum_{\vec{k}}\sqrt{\frac{\hbar}{2\epsilon_0\omega_k L^3}}\sum_{s=1,2}\left(\mathbf{a}_{\vec{k},s}\vec{e}_s(\hat{k})e^{\mathbf{i}\vec{k}\cdot\vec{r}} + \mathbf{a}^\dagger_{\vec{k},s}\vec{e}^{\,\star}_s(\hat{k})e^{-\mathbf{i}\vec{k}\cdot\vec{r}}\right),$$

$$\vec{\mathbf{E}}(\vec{r}) = \mathbf{i}\sum_{\vec{k}}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}}\sum_{s=1,2}\left(\mathbf{a}_{\vec{k},s}\vec{e}_s(\hat{k})e^{\mathbf{i}\vec{k}\cdot\vec{r}} - \mathbf{a}^\dagger_{\vec{k},s}\vec{e}^{\,\star}_s(\hat{k})e^{-\mathbf{i}\vec{k}\cdot\vec{r}}\right)$$

Chain of coupled springs:

$$\mathbf{q}(x) = \sqrt{\frac{\hbar}{2m}}\sum_k\frac{1}{\sqrt{\omega_k}}\left(e^{\mathbf{i}kx}\mathbf{a}_k + e^{-\mathbf{i}kx}\mathbf{a}^\dagger_k\right), \qquad \mathbf{p}(x) = \frac{1}{\mathbf{i}}\sqrt{\frac{\hbar m}{2}}\sum_k\sqrt{\omega_k}\left(e^{\mathbf{i}kx}\mathbf{a}_k - e^{-\mathbf{i}kx}\mathbf{a}^\dagger_k\right).$$

1. **Short answers and conceptual questions** [4 points each, except as noted]

For true or false questions: if the statement is false, you must explain what is wrong or correct it or give a counterexample; if the statement is true, you can simply say 'true'.

(a) [2 points] Recall that observables correspond to hermitian operators. in an interferometer (in terms of the upper-path and lower-path states $\{|\uparrow\rangle, |\downarrow\rangle\}$) what is the operator corresponding to the observable of looking for the photon in the upper path?

(b) A composite quantum system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is governed by the Hamiltonian

$$\mathbf{H} = \mathbf{H}_A \otimes 1\!\!1_B + 1\!\!1_A \otimes \mathbf{H}_B$$

where $1\!\!1_A, 1\!\!1_B$ are the identity operators on the respective parts of the system. Can this state

$$\frac{1}{\sqrt{2}} \left(|a_1\rangle \otimes |b_1\rangle + |a_2\rangle \otimes |b_2\rangle \right)$$

(with $|a_i\rangle$ orthonormal) evolve into this state under time evolution by $\mathbf{H}$:

$$|a_1\rangle \otimes |b_1\rangle \quad ?$$

Explain your reasoning.

(c) On a system of two qbits, the unitary operator

$$\mathcal{O} \equiv |00\rangle\langle 00| + |11\rangle\langle 10| + |10\rangle\langle 11| + |01\rangle\langle 01|$$

acts as

$$\mathcal{O}|00\rangle = |00\rangle, \quad \mathcal{O}|10\rangle = |11\rangle,$$

that is, it copies the state of the first qbit into the second qbit. Why doesn't this violate the no-cloning theorem?

For the next three parts, define a "pseudo-pure state" to be a density operator of the form

$$\boldsymbol{\rho} = \frac{1}{N}(1-\eta)1\!\!1 + \eta|\psi\rangle\langle\psi|$$

for a normalized vector $|\psi\rangle$. N is the dimension of the Hilbert space in question. I will refer to the parameter $\eta \in [0,1]$ as the "piety" of the state $\boldsymbol{\rho}$.

(d) Find a relation between the piety of $\boldsymbol{\rho}$ and the *purity* of $\boldsymbol{\rho}$, which is defined as $\mathrm{tr}\,(\boldsymbol{\rho}^2)$.

(e) [2 points] What are the piety and purity of a pure state?

(f) Suppose that our system is closed, in the sense that it does not interact with any larger system. Show that the piety of a pseudo-pure state is preserved by the time evolution of $\boldsymbol{\rho}$.

(g) [2 points] True or false: If $\mathbf{H}$ is a hamiltonian with discrete translation invariance, its distinct eigenvalues can be labelled by a momentum variable which is periodic.

(h) Suppose we have a qbit (whose Hilbert space we denote $\mathcal{H}_{\frac{1}{2}}$), which interacts with the electromagnetic field (whose Hilbert space we denote $\mathcal{H}_{EM}$). The qbit might be the spin of an electron in an atom (its position is fixed). Consider the following state in $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{EM}$:

$$|\psi\rangle = \cos\theta |\uparrow_z\rangle \otimes |0\rangle + \sin\theta |\downarrow_z\rangle \otimes \mathbf{a}_K^\dagger |0\rangle$$

where K labels some mode of the radiation field, and $|0\rangle$ is the ground state of the radiation field. Find the reduced density matrix that describes *measurements on the qbit.*

2. **Can you heat your coffee by looking at it?** [15 points]

[This problem is a response to a recent paper.]

Consider a spin-$\frac{1}{2}$ quantum system with Hamiltonian

$$\mathbf{H} = -\omega \boldsymbol{\sigma}^z \ .$$

At time $t = 0$ we prepare N (non-interacting) copies of the system in the state

$$|\psi(0)\rangle = |\uparrow_z\rangle$$

(which satisfies $\boldsymbol{\sigma}^z|\uparrow_z\rangle = +|\uparrow_z\rangle$.)

(a) What will we find if we measure the total energy of all the copies?

(b) How does this state evolve in time? That is: find $|\psi(t)\rangle$.

(c) At time t_1, we measure $\boldsymbol{\sigma}^x$ on every copy. On average, how many copies do we expect to give $+1$?

(d) If we obtain $+1$ when we measure $\boldsymbol{\sigma}^x$, what is the state $|\psi(t)\rangle$, at a time $t > t_1$?

(e) The final step of the protocol is to measure $\boldsymbol{\sigma}^z$, at time $t_2 > t_1$. What are the possible outcomes and their probabilities?

(f) Compute the expectation value of the total energy after the measurement, at time $t_3 > t_2$. Did it go up, compared to its value at $t = 0$?

(g) If the energy of the spin goes up, where does the additional energy come from?

(h) What happens if all the copies begin in the excited state $|\downarrow_z\rangle$ instead?

3. **Watching a quantum pot boil.** [15 points]

Consider a particle tunneling between a state $|L\rangle$ on the left-hand side of a symmetric barrier, and a state $|R\rangle$ on the right of the barrier. The states $|R\rangle, |L\rangle$ are orthonormal. The tunneling matrix element between the two states is Δ, that is, the Hamiltonian is

$$\mathbf{H} = -\Delta \left(|R\rangle\langle L| + |L\rangle\langle R| \right) \ .$$

(a) Find the groundstate and excited state energies and wavefunctions of the system.

(b) If the system is prepared at time $t = 0$ in $|L\rangle$, what is the probability of finding the particle on the right side of the barrier, when it is measured at some later time t? Sketch the resulting probability as a function of time.

Now prepare a *statistical ensemble* (not a superposition!) of such systems, with probability $p_R(t)$ and $p_L(t)$ of being on the left or the right, respectively, at time t. (Note that $p_R(t) + p_L(t) = 1$.)

Let the ensemble evolve according to the Schrödinger equation for a short time interval δt, and then 'measure' whether the particle is on the left or on the right, collapsing the wavefunction into $|L\rangle$ or $|R\rangle$ respectively.

This results in a *new* statistical ensemble of systems with probabilities $p_L(t + \delta t)$ and $p_R(t + \delta t)$.

(c) How are the probabilities at time $t + \delta t$ related to those at t? (You may assume that δt is small, $\delta t \ll \Delta^{-1}$.)

If you get stuck on the previous part, don't panic: instead, assume that there is a given probability per unit time q of transferring a particle from right to left, or vice versa.

(d) Find the differential equation relating $\frac{dp_L}{dt}$ to p_L. Solve it (use separation of variables). If a system is prepared at time $t = 0$ on the left (*i.e.* $p_L(t = 0) = 1, p_R(t = 0) = 0$), what is the probability that the particle is found on the right at time $t \gg \delta t$ if it is being 'watched' (*i.e.* regularly measured)? Sketch the result.

(e) How does the transition rate q depend on the length of the time interval δt between measurements, as this interval shrinks?

4. **Quantum interference versus measurement of which-way information**

[20 points]

Consider a double-slit interference experiment, described by a quantum system with two orthonormal states (call them $|\uparrow\rangle$ and $|\downarrow\rangle$), representing the possible paths taken by the particles. A particle emerging in state $|\uparrow\rangle$ produces a wavefunction at the screen of the form $\psi_\uparrow(x)$, (where x is a coordinate along the screen) while a particle emerging in state $|\downarrow\rangle$ produces the wavefunction $\psi_\downarrow(x)$. The evolution from the wall with the slits to the screen is linear in the input state.

As the source repeatedly spits out particles, the screen counts how many particles hit at each location x.

Suppose, for simplicity, that $\psi_\uparrow(x) = e^{\mathbf{i}k_\uparrow x}, \psi_\downarrow(x) = e^{\mathbf{i}k_\downarrow x}$, where $k_\uparrow, k_\downarrow$ are some constants.

(a) If the particles are all spat out in the state $|\uparrow\rangle$, what is the x-dependence of the resulting pattern $P_\uparrow(x)$?

(b) If the particles are all spat out in the (normalized) state

$$|\psi\rangle = \mu|\uparrow\rangle + \lambda|\downarrow\rangle \ ,$$

what is the x-dependence of the resulting pattern, $P_\psi(x)$? Assume μ, λ are real.

Now we wish to take into account interactions with the environment, which we will model by another two-state system, with Hilbert space $\mathcal{H}_E$. Suppose these interactions are described by the hamiltonian

$$\mathbf{H} = (\boldsymbol{\sigma}^z) \otimes \mathbf{M}$$

acting on $\mathcal{H}_2 \otimes \mathcal{H}_E$, where $\boldsymbol{\sigma}^z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ acting on the Hilbert space $\mathcal{H}_2$ of particle paths, and $\mathbf{M}$ is an operator acting on the Hilbert space of the environment.

Suppose the initial state of the whole system is

$$|\Psi_0\rangle \equiv (\mu|\uparrow\rangle + \lambda|\downarrow\rangle) \otimes |\uparrow\rangle_E \ ,$$

and that

$$\mathbf{M} = m\boldsymbol{\sigma}^x = m\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)_E .$$

(c) Find $|\Psi(t)\rangle$, the state of the whole system at time t.

(d) How does the interference pattern depend on x and t? For simplicity, consider the case where $\mu = \lambda = \frac{1}{\sqrt{2}}$.

(e) Interpret the previous result in terms of the time-dependence of the entanglement between the two qbits.

(f) What would happen if instead the initial state of the environment were an eigenvector of $\mathbf{M}$?

5. **Phase-flipping decoherence.** (from Schumacher) [12 points]

Consider the following model of decoherence on an N-state Hilbert space, with basis $\{|k\rangle, k = 1..N\}$.

Define the unitary operator

$$\mathbf{U}_\alpha \equiv \sum_k \alpha_k |k\rangle\langle k|$$

where α_k is an N-component vector of signs, ± 1 – it flips the signs of some of the basis states. There are 2^N distinct such operators.

Imagine that interactions with the environment act on any state of the system with the operator $\mathbf{U}_\alpha$, for some α, chosen randomly (with uniform probability from the 2^N choices).

[Hint: If you wish, set $N = 2$.]

(a) Warmup question: If the initial state is $|\psi\rangle$, what is the probability that the resulting output state is $\mathbf{U}_\alpha|\psi\rangle$?

(b) Write an expression for the resulting density matrix, $\mathcal{D}(\boldsymbol{\rho})$, in terms of $\boldsymbol{\rho}$.

(c) Think of $\mathcal{D}$ as a 'superoperator', an operator on density matrices. How does $\mathcal{D}$ act on a density matrix which is diagonal in the given basis,

$$\boldsymbol{\rho}_{\text{diagonal}} = \sum_k p_k |k\rangle\langle k| \quad ?$$

(d) The most general initial density matrix is not diagonal in the k-basis:

$$\boldsymbol{\rho}_{\text{general}} = \sum_{kl} \rho_{kl} |k\rangle\langle l| \quad .$$

what does $\mathcal{D}$ do to the off-diagonal elements of the density matrix?

6. **Questions about phonons.** [12 points]

Consider the model of a 1d crystalline solid that we discussed in class: It consists of N point masses, coupled to their neighbors:

$$\mathbf{H}_0 = \sum_n \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}\kappa \left(\mathbf{q}_n - \mathbf{q}_{n-1} \right)^2 \right) = \sum_{\{k\}} \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right) \quad . \tag{1}$$

The normal-mode frequencies are

$$\omega_k = 2\sqrt{\frac{\kappa}{m}} \sin \frac{|k|a}{2}.$$

Assume periodic boundary conditions $\mathbf{q}_n = \mathbf{q}_{n+N}$, so that the allowed wavenumbers are

$$\{k\} \equiv \{k_j = \frac{2\pi}{Na} j, \quad j = 1, 2...N\} \ .$$

(a) In terms of the ladder operators $\mathbf{a}_k$, $\mathbf{a}_k^\dagger$ and the vacuum $|0\rangle$ (which satisfies $\mathbf{a}_k|0\rangle = 0, \forall k$), write an expression for a (normalized) state $|x\rangle$ of one phonon at the position x.

(b) In the state $|k\rangle = \mathbf{a}_k^\dagger|0\rangle$ what is the probability of finding a phonon at a specific location x_1 ?

Consider the state

$$|k_1, k_2\rangle = \mathbf{a}_{k_1}^\dagger \mathbf{a}_{k_2}^\dagger |0\rangle \ .$$

(c) Is $|k_1, k_2\rangle$ an eigenstate of $\mathbf{H}_0$? If so, what is its energy?

(d) In the state $|k_1, k_2\rangle$, what is the probability of finding two phonons at the location x_1? Does the story change if $k_1 = k_2$?

(e) Based on what we've discussed in 130C, what are two disadvantages of keeping phonons as pets?

[Hints: 1) Here it may help to recall the anharmonic terms in the inter-atomic potentials that we've neglected.
2) Imagine what would happen if you let your pet phonons play with someone else's pet phonons.]