University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (130C) Winter 2013
Final exam

Monday March 18, 2013, 3-6pm

Please remember to put your name on your exam booklet. This is a closed-book exam. There are 6 problems, each with several parts, of varying levels of difficulty; make sure you try all of the parts. None of the problems require very extensive calculation; if you find yourself involved in a morass of calculation, step back and think. Good luck.

Possibly useful information:

$$\mathbf{U}(t) = e^{-\mathbf{i}\mathbf{H}t/\hbar} \quad \text{satisfies} \quad \mathbf{i}\hbar\partial_t \mathbf{U} = [\mathbf{H}, \mathbf{U}].$$

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\uparrow_{\hat n}\rangle = e^{-\mathbf{i}\varphi/2}\cos\frac{\theta}{2}|\uparrow_{\hat z}\rangle + e^{+\mathbf{i}\varphi/2}\sin\frac{\theta}{2}|\downarrow_{\hat z}\rangle \quad \text{satisfies} \quad \vec{\boldsymbol{\sigma}}\cdot\hat n|\uparrow_{\hat n}\rangle = |\uparrow_{\hat n}\rangle$$

$$e^{-i\alpha\hat n\cdot\vec{\boldsymbol{\sigma}}} = \mathbb{1}\cos\alpha - i\hat n\cdot\vec{\boldsymbol{\sigma}}\sin\alpha.$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{H}_{\text{SHO}} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right) = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2$$

$$\mathbf{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right), \quad \mathbf{p} = \frac{1}{\mathbf{i}}\sqrt{\frac{\hbar m\omega}{2}}\left(\mathbf{a} - \mathbf{a}^\dagger\right); \quad [\mathbf{q}, \mathbf{p}] = \mathbf{i}\hbar \implies [\mathbf{a}, \mathbf{a}^\dagger] = 1.$$

In Coulomb gauge, in vacuum ($\vec\nabla\cdot\vec A = 0, \Phi = 0$): $\quad \vec E = -\partial_t\vec A, \quad \vec B = \vec\nabla\times\vec A.$

$$\vec{\mathbf{A}}(\vec r) = \sum_{\vec k}\sqrt{\frac{\hbar}{2\epsilon_0\omega_k L^3}}\sum_{s=1,2}\left(\mathbf{a}_{\vec k,s}\vec e_s(\hat k)e^{\mathbf{i}\vec k\cdot\vec r} + \mathbf{a}^\dagger_{\vec k,s}\vec e^{\,\star}_s(\hat k)e^{-\mathbf{i}\vec k\cdot\vec r}\right)$$

$$\vec{\mathbf{E}}(\vec r) = \mathbf{i}\sum_{\vec k}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 L^3}}\sum_{s=1,2}\left(\mathbf{a}_{\vec k,s}\vec e_s(\hat k)e^{\mathbf{i}\vec k\cdot\vec r} - \mathbf{a}^\dagger_{\vec k,s}\vec e^{\,\star}_s(\hat k)e^{-\mathbf{i}\vec k\cdot\vec r}\right)$$

1. **Short answers and conceptual questions** [5 points each, except as noted]

For true or false questions: if the statement is false, you must explain what is wrong or correct it or give a counterexample; if the statement is true, you can simply say 'true'.

(a) [2 points] True or false: Unperformed experiments have no results.

(b) [2 points] True or false: The statement that "a state is a superposition" is basis-dependent.

(c) [2 points] True or false: A statement of the form "subsystems A and B are entangled in the state $|\psi\rangle$" is basis-dependent.

(d) [2 points] True or false: Coupling a system to a large environment chooses a special basis of states which are the ones we tend to observe.

(e) [2 points] True or false: If $\mathbf{H}$ is a hamiltonian with discrete translation invariance, its distinct eigenvalues can be labelled by a discrete momentum variable which can take arbitrarily large, independent values.

(f) Consider two qbits, labelled A and B. Let

$$\boldsymbol{\rho}_2 \equiv \frac{1}{2}\left(\mathbb{1}_A \otimes \mathbb{1}_B + \boldsymbol{\sigma}^z_A \otimes \boldsymbol{\sigma}^z_B\right)$$

Is $\boldsymbol{\rho}_2$ the density matrix for a pure state? If not, why not?

For the next two problems, suppose we have a qbit (whose Hilbert space we denote $\mathcal{H}_{\frac{1}{2}}$), which interacts with the electromagnetic field (whose Hilbert space we denote $\mathcal{H}_{EM}$). The qbit might be the spin of an electron in an atom (its position is fixed).

(g) Consider the following state in $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{EM}$:

$$|\psi\rangle = \cos\theta |\uparrow_z\rangle \otimes |0\rangle + \sin\theta |\downarrow_z\rangle \otimes \mathbf{a}^\dagger_K |0\rangle$$

where K labels some mode of the radiation field, and $|0\rangle$ is the ground state of the radiation field. Find the reduced density matrix that describes measurements on the radiation field.

(h) The hamiltonian contains a term by which the qbit interacts with light:

$$\mathbf{H}_{\text{int}} = \mu \vec{\mathbf{B}}(x) \cdot \vec{\boldsymbol{\sigma}}$$

where μ is a constant, x is the location of the qbit. Suppose the system is initially in the state

$$|\downarrow_{\hat{z}}\rangle \otimes \mathbf{a}^\dagger_{\vec{k},\hat{x}}|0\rangle \ .$$

The photon described by the state $\mathbf{a}^\dagger_{\vec{k},\hat{x}}|0\rangle$ is *polarized* along $\hat{x}$ – this means that the associated electric field $\vec{\mathbf{E}}$ points along $\pm\hat{x}$. In which direction $\hat{k}$ must the photon propagate to be able to flip the spin of the electron to $|\uparrow_z\rangle$ if it is absorbed by the electron?

[Extra credit: estimate the dependence on $|\vec{k}|$ of the spin-flip probability.]

2. **Neutrino oscillations** [20 points]

Neutrinos are subatomic particles which come in several types; different types interact differently with matter. Consider a neutrino which can be an electron-neutrino ν_e or a muon-neutrino ν_μ. We can describe its wave function as $|\psi\rangle = a|\nu_e\rangle + b|\nu_\mu\rangle$, where $|\nu_e\rangle, |\nu_\mu\rangle$ are orthonormal. A neutrino in the state $|\nu_e\rangle$ or $|\nu_\mu\rangle$ is detected as an electron-neutrino or muon-neutrino, respectively, with certainty. The Hamiltonian is

$$\mathbf{H} = p + \frac{\mathbf{M}^2}{2p}$$

where p is the (magnitude of the) momentum, and $\mathbf{M}^2$ is a 2×2 matrix with eigenvectors

$$\begin{aligned} |\nu_1\rangle &= \cos\theta|\nu_e\rangle + \sin\theta|\nu_\mu\rangle \\ |\nu_2\rangle &= -\sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle \end{aligned}$$

with respective eigenvalues m_1^2 and m_2^2. (We work in units with $c = 1$ for this problem.)

(a) Evaluate

$$c_i^e \equiv \langle\nu_i|\nu_e\rangle \quad \text{and} \quad c_i^\mu \equiv \langle\nu_i|\nu_\mu\rangle, \qquad i = 1, 2$$

as functions of θ.

(b) At $t = 0$ a neutrino with momentum p is in the state $|\nu_e\rangle$. If we were to measure the observable $\mathbf{M}^2$, what are the possible outcomes and with what probabilities will they occur?

(c) At $t = 0$ a neutrino with momentum p is in the state $|\nu_e\rangle$. At time t it reacts with matter inside of a detector. What is the probability that it is detected as a muon-neutrino?
[Hint: I found it useful to introduce the notation $\alpha_i \equiv t\left(p + \frac{m_i^2}{2p}\right)$.]

(d) [1 point extra credit] Restore the factors of c.

3. **Poisson statistics of the radiation field** [10 points]

Consider a single mode $\mathbf{a}$ of the radiation field:

$$\vec{\mathbf{E}}(r) = \mathbf{i}\sqrt{\frac{\hbar\omega\epsilon_0}{2}}\vec{e}e^{\mathbf{i}kr}\mathbf{a} + \text{hermitian conjugate.}$$

(Note that there is no sum over k or polarization in this expression.)

Consider the following coherent state of this mode: $|\alpha\rangle \equiv e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}|0\rangle$.

(a) Find the expectation value of the number of photons in this state.

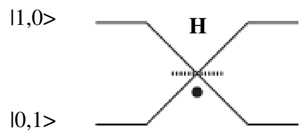
(b) Show that in this state the *variance* of the number of photons is equal to the average photon number.

The variance of any operator $\mathbf{X}$ is defined as $\langle\mathbf{X}^2\rangle - \langle\mathbf{X}\rangle^2$.

[This distribution is called *Poisson statistics*.]

4. **Multiple photons on paths of an interferometer** [10 points]

In lecture we made a qbit from the two states of a photon moving on the upper and lower paths of an interferometer, on which a half-silvered mirror $\mathbf{H}$ acts as a unitary gate:



But we also found out that photons are bosons. This means that if

$$\mathbf{a}^\dagger|0,0\rangle \equiv |1,0\rangle \text{ is a state with one photon on the upper path}$$

of the interferometer, then

$$\frac{(\mathbf{a}^\dagger)^n}{\sqrt{n!}}|0,0\rangle \equiv |n,0\rangle \text{ is a state with } n \text{ photons on the upper path.}$$

Similarly, define

$$\frac{(\mathbf{b}^\dagger)^n}{\sqrt{n!}}|0,0\rangle \equiv |0,n\rangle \text{ to be a state with } n \text{ photons on the lower path}$$

of the interferometer. (Note that $[\mathbf{a},\mathbf{b}] = 0 = [\mathbf{a},\mathbf{b}^\dagger]$.)

Now suppose we send these two paths through a half-silvered mirror, as in the figure. Recall that a half-silvered mirror acts as a Hadamard gate

$$\mathbf{H} \equiv \frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma}^x + \boldsymbol{\sigma}^z\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

on the qbit made from the one-photon states.

Some warm-up questions:

(a) How does $\mathbf{H}$ act on $|0,0\rangle$?

(b) How does $\mathbf{H}$ act on $|2,0\rangle$ and $|0,2\rangle$?

(c) How does $\mathbf{H}$ act on the operators $\mathbf{a}^\dagger$ and $\mathbf{b}^\dagger$?

Here's the interesting question:

(d) What is the state which results upon sending a coherent state of photons

$$|\alpha,\beta) \equiv \mathcal{N}_\alpha \mathcal{N}_\beta \; e^{\alpha\mathbf{a}^\dagger + \beta\mathbf{b}^\dagger}|0,0\rangle$$

through a half-silvered mirror? ($\mathcal{N}_\alpha \equiv e^{-|\alpha|^2/2}$ is a normalization constant.) [Hint: it may be useful to insert $\mathbb{1} = \mathbf{H}^2$ in between the $e^{\alpha\mathbf{a}^\dagger + \beta\mathbf{b}^\dagger}$ and the $|0,0\rangle$.]

5. **Harmonic oscillator coupled to qbit, with a hidden symmetry** [25 points]

Consider a quantum system whose hilbert space is $\mathcal{H} \equiv \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\text{SHO}}$ where

$$\mathcal{H}_{\frac{1}{2}} \equiv \text{span}\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$$

is a two-state hilbert space, and

$$\mathcal{H}_{\text{SHO}} = \text{span}\{|0\rangle, \mathbf{a}^\dagger|0\rangle \equiv |1\rangle, \ldots \frac{(\mathbf{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \equiv |n\rangle \ldots\}$$

is that of a harmonic oscillator. Consider the hamiltonian

$$\mathbf{H}_0 \equiv \mu \boldsymbol{\sigma}^z \otimes \mathbb{1} + \hbar\omega \mathbb{1} \otimes \left(\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2}\right),$$

where μ and ω are parameters.

(a) With this hamiltonian $\mathbf{H}_0$, do the qbit and the harmonic oscillator interact? That is, does time evolution with $\mathbf{H}_0$ create an entangled state starting from a product state such as $|\uparrow_z\rangle \otimes |0\rangle$?

(b) Describe the spectrum of $\mathbf{H}_0$, beginning with the state of lowest energy, the groundstate $|gs\rangle$. More precisely, find the five energy eigenstates with lowest energy and tell me their energies. Assume $0 < \mu \leq \hbar\omega/2$, and you may assume μ is close to $\hbar\omega/2$.

(c) What is the groundstate energy when $\mu = \hbar\omega/2$?

We would like to understand what is special about this value of parameters. To do so, we will consider the operator

$$\mathbf{Q} \equiv \boldsymbol{\sigma}^+ \otimes \mathbf{a}^\dagger$$

where $\boldsymbol{\sigma}^\pm \equiv \frac{1}{2}(\boldsymbol{\sigma}^x \mp \mathbf{i}\boldsymbol{\sigma}^y)$.

(d) Describe the action of $\boldsymbol{\sigma}^\pm$ on the basis states of the qbit.

(e) What is $[\boldsymbol{\sigma}^\pm, \boldsymbol{\sigma}^z]$?

(f) What is $[\mathbf{a}^\dagger\mathbf{a}, \mathbf{a}^\dagger]$?

(g) What is $\mathbf{Q}|gs\rangle$? What is $\mathbf{Q}^\dagger|gs\rangle$?

(h) Compute $[\mathbf{H}_0, \mathbf{Q}]$ and show that for the special parameter choice $\mu = \hbar\omega/2$

$$[\mathbf{H}_0, \mathbf{Q}] = 0 .$$

The previous result shows that when $\mu = \hbar\omega/2$, $\mathbf{Q}$ generates a symmetry of $\mathbf{H}_0$. In the next parts of this problem, we try to understand the nature of this symmetry.

(i) Show that $\mathbf{Q}^2 = 0$.

(j) Show that when $\mu = \hbar\omega/2$,

$$\left(\mathbf{Q} + \mathbf{Q}^\dagger\right)^2 = \mathbf{H}_0 \ .$$

[Cultural remarks: This relationship, which is called the *supersymmetry algebra*, explains why the ground state energy vanishes for $\mu = \hbar\omega/2$, given the result of part 5g. The generator $\mathbf{Q}$ is called the *supercharge*. The symmetry it generates is called supersymmetry and is weird because the generator squares to zero. It is therefore sometimes called a *fermionic* symmetry.]

6. **The black hole information problem** [10 points]

A black hole is a region of space from which nothing can escape, not even light. Suppose you have convenient access to both a black hole and an EPR pair – a pair of spin-$\frac{1}{2}$ particles in the state which is a singlet of the total spin. And suppose you throw one of the two particles into the black hole.

(a) Write an expression for the density matrix describing the spin state of your remaining particle.

(b) What is the probability that you will find the spin of the remaining particle to be up along the z-axis?

So far, there is no problem – the role played by the black hole so far could just as well be played by a well-secured safe. Here's the problem: black holes *evaporate*. Roughly, this means that they slowly disappear, and (in a certain limit) leave behind only a featureless gas of thermal radiation. The state of this radiation is labelled by one number, the temperature. On the other hand, black holes can be formed by gravitational collapse of anything: stars, trees, dictionaries, quantum computers.

(c) Here's the question: can this process, as just described, be consistent with the idea that the whole system (including the black hole) should be governed by the axioms of quantum mechanics of a closed system, as stated in Physics 130C? Explain your reasoning.