University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (Physics 130C) Winter 2015
Assignment 7

Posted March 4, 2015, revised March 10, 2015 **Due 11am Thursday, March 12, 2015**

Please note an important typo corrected in problem 4 (March 10), marked in red.

1. **Phase-flipping decoherence.** (from Schumacher)

   Consider the following model of decoherence on an N-state Hilbert space, with basis $\{|k\rangle, k = 1..N\}$.

   Define the unitary operator

   $$\mathbf{U}_\alpha \equiv \sum_k \alpha_k |k\rangle\langle k|$$

   where α_k is an N-component vector of signs, ± 1 – it flips the signs of some of the basis states. There are 2^N distinct such operators.

   Imagine that interactions with the environment act on any state of the system with the operator $\mathbf{U}_\alpha$, for some α, chosen randomly (with uniform probability from the 2^N choices).

   [Hint: If you wish, set $N = 2$.]

   (a) Warmup question: If the initial state is $|\psi\rangle$, what is the probability that the resulting output state is $\mathbf{U}_\alpha|\psi\rangle$?

   (b) Write an expression for the resulting density matrix, $\mathcal{D}(\boldsymbol{\rho})$, in terms of $\boldsymbol{\rho}$.

   (c) Think of $\mathcal{D}$ as a 'superoperator', an operator on density matrices. How does $\mathcal{D}$ act on a density matrix which is diagonal in the given basis,

   $$\boldsymbol{\rho}_{\text{diagonal}} = \sum_k p_k |k\rangle\langle k| \quad ?$$

   (d) The most general initial density matrix is not diagonal in the k-basis:

   $$\boldsymbol{\rho}_{\text{general}} = \sum_{kl} \rho_{kl} |k\rangle\langle l| \quad .$$

   what does $\mathcal{D}$ do to the off-diagonal elements of the density matrix?

2. **Decoherence by phase damping with non-orthogonal states** [from Preskill] [extra credit]

   Suppose that a heavy particle A begins its life in outer space in a superposition of two positions

   $$|\psi_0\rangle_A = a|x_0\rangle + b|x_1\rangle.$$

   These positions are not too far apart. The particle interacts with the electromagnetic field, and in time dt, the whole system evolves according to

   $$\mathbf{U}_{AE}|x_0\rangle_A \otimes |0\rangle_E = \sqrt{1-p}|x_0\rangle_A \otimes |0\rangle_E + \sqrt{p}|x_0\rangle_A \otimes |\gamma_0\rangle_E$$

   $$\mathbf{U}_{AE}|x_1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}|x_1\rangle_A \otimes |0\rangle_E + \sqrt{p}|x_1\rangle_A \otimes |\gamma_1\rangle_E$$

   But because x_0 and x_1 are close, the (normalized) photon states $|\gamma_0\rangle, |\gamma_1\rangle$ have a large overlap:

   $$\langle\gamma_0|\gamma_1\rangle_E = 1-\epsilon, \quad \text{with } 0 < \epsilon \ll 1.$$

   (a) Find the Kraus operators describing the time evolution of the reduced density matrix $\boldsymbol{\rho}_A$.

   (b) How long does it take the superposition to decohere? More precisely, at what time t is $(\rho_A)_{01}(t) = \frac{1}{e}(\rho_A)_{01}(t=0)$?

3. **Decoherence on the Bloch sphere** [from Preskill]

   Parametrize the density matrix of a single qubit as

   $$\boldsymbol{\rho}_A = \frac{1}{2}\left(\mathbb{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right).$$

   (a) **Polarization-damping channel.**

   Consider the (unitary) evolution of a qbit A coupled to a 4-state environment via

   $$\mathbf{U}_{AE}|\phi\rangle_A \otimes |0\rangle_E = \sqrt{1-p}|\phi\rangle_A \otimes |0\rangle_E + \sqrt{p/3}\sum_{i=1}^{3}\boldsymbol{\sigma}_A^i \otimes \mathbb{1}_E|\phi\rangle_A \otimes |i\rangle_E$$

   Show that this evolution can be accomplished with the Kraus operators

   $$\mathbf{M}_0 = \sqrt{1-p}\mathbb{1}, \qquad \mathbf{M}_i = \sqrt{p/3}\boldsymbol{\sigma}^i,$$

   and show that they obey the completeness relation requred by unitarity of $\mathbf{U}_{AE}$. Show that the polarization P_i of the qbit evolves according to

   $$\vec{P} \to \left(1 - \frac{4p}{3}\right)\vec{P}.$$

   [Hint: use the identity $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\boldsymbol{\sigma}_i = 2\boldsymbol{\sigma}_j\delta_{ij} - \boldsymbol{\sigma}_j$.]

   Describe this evolution in terms of what happens to the Bloch ball.

   What happens if $p > 3/4$?

(b) **Two-Pauli channel.** [extra credit]

Consider the (unitary) evolution of a qbit A coupled to a *three*-state environment via

$$\mathbf{U}_{AE}|\phi\rangle_A \otimes |0\rangle_E = \sqrt{1-p}|\phi\rangle_A \otimes |0\rangle_E + \sqrt{p/2}\sum_{i=1}^{2} \boldsymbol{\sigma}_A^i \otimes 1\!\!1_E |\phi\rangle_A \otimes |i\rangle_E$$

Show that this evolution can be accomplished with the Kraus operators

$$\mathbf{M}_0 = \sqrt{1-p}1\!\!1, \quad \mathbf{M}_i = \sqrt{p/2}\boldsymbol{\sigma}^i, i = 1, 2$$

and show that they obey the completeness relation requred by unitarity of $\mathbf{U}_{AE}$. Describe this evolution in terms of what happens to the Bloch ball.

(c) **Phase-damping channel.** [extra credit]

For the evolution of problem 2,

$$\mathbf{U}_{AE}|0\rangle_A \otimes |0\rangle_E = \sqrt{1-p}|0\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |\gamma_0\rangle_E$$

$$\mathbf{U}_{AE}|1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}|1\rangle_A \otimes |0\rangle_E + \sqrt{p}|1\rangle_A \otimes |\gamma_1\rangle_E$$

now thinking of A as a qbit, describe the evolution of its polarization vector on the Bloch ball.

4. **Near-derivation of Born rule**

This question is about a step in Hartle's near-derivation of the Born rule. We studied the Hilbert space of N copies of our system, $\mathcal{H} \otimes \mathcal{H} \cdots \otimes \mathcal{H} = \mathcal{H}^N$, and the state

$$|c\rangle \equiv \sum_{n_1..n_N} c_{n_1} \cdots c_{n_N} e^{\mathbf{i}\sum_i \varphi_{i,n_i}} |n_1 \cdots n_N\rangle \equiv \sum_{\nu} \left(\prod_n c_n^{N_{\nu n}} \right) e^{\mathbf{i}\varphi_\nu} |\nu\rangle$$

where we introduced the shorthand $\nu \equiv \{n_1 \cdots n_N\}$ and $N_{\nu n}$ is the number of elements of the set ν equal to n. Notice that for each ν $\sum_n N_{\nu n} = N$.

We defined the hermitian 'frequency' operators $\mathbf{P}_n$ by their eigenvalue equation:

$$\mathbf{P}_n|\nu\rangle = \frac{N_{\nu n}}{N}|\nu\rangle.$$

Show that

$$\| \left(\mathbf{P}_n - |c_n|^2\right) |c\rangle \|^2 = \frac{1}{N}|c_n|^2 \left(1 - |c_n|^2\right) \le \frac{1}{4N}.$$

Hints:

- For any fixed N_n, the number of νs with $N_{\nu n} = N_n$ is

$$\frac{N!}{N_1!N_2!\cdots}.$$

- $\langle \nu' | \nu \rangle = \delta_{\nu\nu'}$
- The multinomial theorem says

$$\left(\sum_m |c_m|^2 \right)^N = \sum_{N_1, N_2 \ldots | \sum_n N_n = N} \frac{N!}{N_1!N_2!\cdots} \prod_m |c_m|^{2N_m}$$

- Differentiating the BHS of the previous equation with respect to $|c_m|^2$ (or better, acting with $|c_m|^2 \frac{\partial}{\partial |c_m|^2}$) gives a formula for

$$\sum_{N_1, N_2 \ldots | \sum_n N_n = N} \frac{N!}{N_1!N_2!\cdots} \prod_m |c_m|^{2N_m} N_n$$

5. **Two coupled spins.** [based on Le Bellac problem 6.5.4]

Consider a four-state system consisting of two qbits,

$$\mathcal{H} = \text{span}\{|\epsilon_1\rangle \otimes |\epsilon_2\rangle \equiv |\epsilon_1 \epsilon_2\rangle, \epsilon = \uparrow_z, \downarrow_z\}.$$

(a) For each qbit, define $\boldsymbol{\sigma}^\pm \equiv \frac{1}{2}(\boldsymbol{\sigma}^x \pm \mathbf{i}\boldsymbol{\sigma}^y)$. (These are raising and lowering operators for $\boldsymbol{\sigma}^z$: $[\boldsymbol{\sigma}^z, \boldsymbol{\sigma}^\pm] = \pm 2\boldsymbol{\sigma}^\pm$. Show this.) Show that

$$\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2 = 2\left(\boldsymbol{\sigma}_1^+ \boldsymbol{\sigma}_2^- + \boldsymbol{\sigma}_1^- \boldsymbol{\sigma}_2^+\right) + \boldsymbol{\sigma}_1^z \boldsymbol{\sigma}_2^z.$$

(b) Determine the action of the operator $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ on the basis states

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle.$$

(c) Show that the four vectors

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right), \ |1,1\rangle \equiv |\uparrow\uparrow\rangle, \ |1,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \ |1,-1\rangle \equiv |\downarrow\downarrow\rangle$$

are orthonormal and are eigenvectors of $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ with eigenvalues 1 or -3.

(d) Show that they are also eigenvectors of $\mathbf{J}^2 \equiv (\vec{\boldsymbol{\sigma}}_1 + \vec{\boldsymbol{\sigma}}_2)^2$ and $\mathbf{J}^z \equiv \boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z$ and find their eigenvalues.

(e) Consider the operator

$$\mathcal{P}_{1,2} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2\right)$$

acting on the two spins. Show that $\mathcal{P}_{1,2}$ acts by exchanging the states of the two spins:

$$\mathcal{P}_{1,2}|\epsilon_1\epsilon_2\rangle = |\epsilon_2\epsilon_1\rangle \ .$$

6. **Coherent states.**

Consider a quantum harmonic oscillator. The creation and annihilation operators $\mathbf{a}^\dagger$ and $\mathbf{a}$ satisfy the algebra

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1$$

and the vacuum state $|0\rangle$ satisfies $\mathbf{a}|0\rangle = 0$.

Coherent states are eigenstates of the annihilation operator:

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

(a) Show that

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \mathbf{a}^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

is an eigenstate of $\mathbf{a}$ with eigenvalue α. ($\mathbf{a}$ is not hermitian, so its eigenvalues need not be real.)

(b) Coherent states with different α are not orthogonal. ($\mathbf{a}$ is not hermitian, so its eigenstates need not be orthogonal.) Show that $|\langle\alpha_1|\alpha_2\rangle|^2 = e^{-|\alpha_1-\alpha_2|^2}$.

(c) Compute the expectation value of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$ in the coherent state $|\alpha\rangle$.

(d) Time evolution acts nicely on coherent states. The hamiltonian is $\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$. Show that a coherent state evolves into a coherent state with an eigenvalue $\alpha(t)$:

$$e^{-\mathbf{i}\mathbf{H}t}|\alpha\rangle = e^{-\mathbf{i}\omega t/2}|\alpha(t)\rangle$$

where $\alpha(t) = e^{-\mathbf{i}\omega t}\alpha$.

7. **Spin chains and spin waves.** [Related to Le Bellac problem 6.5.5 on page 200]

A one-dimensional *ferromagnet* can be represented as a chain of N qbits (spin-1/2 particles) numbered $n = 0, ...N-1$, $N \gg 1$, fixed along a line with a spacing ℓ between each successive pair. It is convenient to use *periodic boundary conditions* (as in HW 2 problem 2), where the Nth spin is identified with the 0th spin: $n + N \equiv n$. Suppose that each spin interacts only with its two nearest neighbors, so the Hamiltonian can be written as

$$\mathbf{H} = \frac{1}{2}NJ\mathbb{1} - \frac{1}{2}J\sum_{n=0}^{N-1} \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1} \ .$$

where J is a *coupling constant* determining the strength of the interactions.

(a) Show that all eigenvalues E of $\mathbf{H}$ are non-negative, and that the minimum energy E_0 (the *ground state*) is obtained in the state where all the spins point in the same direction. A possible choice for the ground state $|\Phi_0\rangle$ is then

$$|\Phi_0\rangle = |\uparrow_z\rangle_{n=0} \otimes |\uparrow_z\rangle_{n=1} \otimes ... \otimes |\uparrow_z\rangle_{N-1} \equiv |\uparrow\uparrow ... \uparrow\rangle.$$

(b) Show that any state obtained from $|\Phi_0\rangle$ by rotating each of the spins by the same angle is also a possible ground state.

[Hint: the generator of spin rotations $\vec{\mathbf{J}} \equiv \sum_n \vec{\boldsymbol{\sigma}}_n$ commutes with the Hamiltonian.]

[Cultural remark: the phenomenon of a ground state which does not preserve a symmetry of the Hamiltonian is called *spontaneous symmetry breaking.*]

(c) Now we wish to find the low-energy excitations above the ground state $|\Phi_0\rangle$. Show that $\mathbf{H}$ can be written

$$\mathbf{H} = NJ\mathbb{1} - J\sum_{n=0}^{N-1} \mathcal{P}_{n,n+1} = J\sum_{n=0}^{N-1} \left(\mathbb{1} - \mathcal{P}_{n,n+1}\right).$$

where

$$\mathcal{P}_{n,n+1} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1}\right) \ .$$

Using the result of the previous problem, show that the eigenvectors of $\mathbf{H}$ are linear combinations of vectors in which the number of up spins minus the number of down spins is fixed. Let $|\Psi_n\rangle$ be the state in which the spin n is down with all the other spins up. What is the action of $\mathbf{H}$ on $|\Psi_n\rangle$?

(d) We are going to construct eigenvectors $|k_s\rangle$ of $\mathbf{H}$ out of linear combinations of the $|\Psi_n\rangle$. Let

$$|k_s\rangle = \sum_{n=0}^{N-1} e^{\mathbf{i}k_s n\ell}|\Psi_n\rangle$$

with

$$k_s = \frac{2\pi s}{N\ell}, \quad s = 0, 1, ...N-1 \ .$$

Show that $|k_s\rangle$ is an eigenvector of $\mathbf{H}$ and determine the energy eigenvalue E_k. Show that the energy is proportional to k_s^2 as $k_s \to 0$. This state describes an elementary excitation called a *spin wave* or *magnon* with wave-vector k_s.