

## Quantum Mechanics C (Physics 130C) Winter 2015 Assignment 6

Posted February 24, 2015

Due 11am Thursday, March 5, 2015

Problem 6 of the earlier version of this problem set has been postponed until next week.

### 1. Copying classical information is OK.

Construct a *linear* operator  $\mathcal{O}$  on a system of two qbits which acts as follows on the computational basis states of the first qbit:

$$\mathcal{O}|0\rangle \otimes |a\rangle = |0\rangle \otimes |0\rangle, \quad \mathcal{O}|1\rangle \otimes |a\rangle = |1\rangle \otimes |1\rangle,$$

for any computational-basis state  $|a\rangle$  of the second qbit.

What is  $\mathcal{O}(\cos\theta|0\rangle + \sin\theta|1\rangle) \otimes |a\rangle$  ?

[We can think of this operator as copying *classical* information: if we are forced (*e.g.* by decoherence) to remain in the computational basis, the information in the first qbit is just a classical bit; the operator  $\mathcal{O}$  copies this classical bit into the second register. This shows that the quantum no-cloning theorem does not forbid the cloning of classical information.]

Extra credit: show that it is not possible to construct such a linear operator which acts as above for *arbitrary* states  $|a\rangle$  of the second qbit.

### 2. Deutsch's Algorithm

This is an example of a (contrived) problem which can be solved by a quantum computer in half the number of operations you would need classically. The problem is to distinguish between functions acting on a single bit, which return a single bit:

$$f : \{0, 1\} \rightarrow \{0, 1\} .$$

For purposes of this problem, when we add bits, we do so mod 2:

$$0 + 1 = 1, 1 + 1 = 0.$$

There are  $2^2$  = four such functions altogether, and they fall into two classes: *constant* ( $f(0) = f(1)$ ) and *balanced* ( $f(0) = f(1) + 1$ ).

The problem is: given a black box which eats a bit  $x$  and spits out a bit  $f(x)$ , decide whether  $f$  is constant or balanced as quickly as possible. Imagine that each operation of the box takes a long time, so the thing we want to minimize is the number of operations of the box.

- (a) How can you possibly answer this question without finding out both  $f(0)$  and  $f(1)$ ? Convince yourself that there is no way to do this classically in fewer than two operations of the box.

Now suppose we have a system of two qbits:  $\mathcal{H} = \text{span}\{|x\rangle \otimes |y\rangle, x, y \in \{0, 1\}\}$ . We are given a quantum black box operation ('oracle')  $\mathbf{U}_f$  which acts like this:

$$\mathbf{U}_f|x\rangle|y\rangle = |x\rangle|y + f(x)\rangle$$

on the given basis states.

To build the quantum computer we will use as the basic building block the Hadamard gate,  $\mathbf{H} \equiv \frac{1}{\sqrt{2}} (|0\rangle\langle 0| - |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|)$ . Its role in life is to create and destroy (uniform) superpositions of the given basis states.

- (b) Show that  $\mathbf{H}|x\rangle = \frac{1}{\sqrt{2}} \sum_{y=0,1} (-1)^{xy} |y\rangle$ .

Here's the design of the quantum computer which solves the problem:

step i. Feed in the state  $|0\rangle \otimes |1\rangle$ .

Now act successively with Hadamard gates on each of the two qbits:

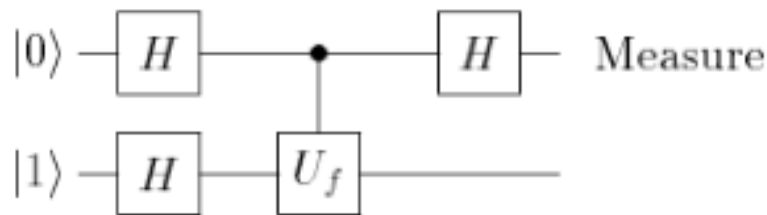
step ii.  $\mathbb{1} \otimes \mathbf{H}$

step iii.  $\mathbf{H} \otimes \mathbb{1}$

step iv. Next 'query the oracle', *i.e.* act with  $\mathbf{U}_f$ .

step v. Now act with the Hadamard on the first qbit:  $(\mathbf{H} \otimes \mathbb{1})$

The following figure summarizes the steps (time runs from left to right):



[From Preskill]

- (c) Find the output state of the quantum circuit:

$$|out\rangle \equiv (\mathbf{H} \otimes \mathbb{1}) \mathbf{U}_f (\mathbf{H} \otimes \mathbb{1}) (\mathbb{1} \otimes \mathbf{H}) |0\rangle \otimes |1\rangle$$

- (d) Why can we draw steps ii and iii on top of each other in the circuit diagram?

- (e) We are going to measure the value of the first qbit in the output state in the preferred basis. (That is, we measure  $\frac{1}{2}(\mathbb{1} - \sigma^z) = |1\rangle\langle 1|$ .) What are the probabilities for the outcomes 0 and 1 of this measurement in the output state  $|out\rangle$ ?
- (f) Show that from the outcome of this measurement you can learn whether  $f$  is constant or balanced. Notice that we only need to run the quantum black box once!
- (g) [Extra credit] Consider the case of a map from  $n$  bits

$$f : \{0, 1\}^n \rightarrow \{0, 1\} .$$

Such a map is constant if it is independent of its input, and is balanced if the average over all the inputs is  $\frac{1}{2}$  (for  $n > 1$ , these are no longer the only two possibilities). Suppose we are guaranteed that  $f$  is either constant or balanced. Describe a quantum circuit acting on a collection of (probably  $n + 1$ ) qbits which allows you to distinguish between the two options in a single shot.

### 3. An adaptation of Grover's algorithm [extra credit]

Suppose we have a table of  $N$  items and  $r$  of them  $a_i$  are marked. You are given a quantum computer and an oracle which acts by a reflection in the hyperplane whose normal vector is:  $|\tilde{a}\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^r |a_i\rangle$ .

[These words mean that the operator is  $\mathcal{O} = \mathbb{1} - 2|\tilde{a}\rangle\langle\tilde{a}| = \mathbb{1} - \frac{2}{r} \sum_{i=1}^r |a_i\rangle\langle a_i|$ .]

Construct a quantum algorithm to efficiently identify the marked items, and estimate its average time of operation as a function of  $r$  and  $N$ .

### 4. Stern-Gerlach for spin 1 particles

Consider a particle with spin one, with spin states which can be labelled by eigenvalues of  $\mathbf{J}^z$ :

$$\mathbf{J}^z|\mathbf{m}\rangle = \mathbf{m}|\mathbf{m}\rangle, \quad \mathbf{m} = -1, 0, 1.$$

A Stern-Gerlach device to measure its spin couples its position and its spin via a non-uniform magnetic field in the  $\hat{z}$  direction,

$$\Delta\mathbf{H} = \lambda\mathbf{z} \otimes \mathbf{J}^z ,$$

where  $\mathbf{z}$  is the position operator in the  $z$  direction.

If the particle spends a time  $t$  in the field, and begins in the state  $|\psi_z\rangle \otimes (a|-1\rangle + b|0\rangle + c|1\rangle)$  (where  $|\psi_z\rangle$  is a wave packet centered about position  $z$ ), describe the locations and relative intensities of the spots on the detector (as functions of  $a, b, c, \lambda, t, z$ ). The intensity of a spot on the detector is proportional to the number particles that have hit it.

## 5. Quantum interference versus measurement of which-way information

Consider a double-slit interference experiment, described by a quantum system with two orthonormal states (call them  $|\uparrow\rangle$  and  $|\downarrow\rangle$ ), representing the possible paths taken by the particles. A particle emerging in state  $|\uparrow\rangle$  produces a wavefunction at the screen of the form  $\psi_\uparrow(x)$ , (where  $x$  is a coordinate along the screen) while a particle emerging in state  $|\downarrow\rangle$  produces the wavefunction  $\psi_\downarrow(x)$ . The evolution from the wall with the slits to the screen is linear in the input state.

As the source repeatedly spits out particles, the screen counts how many particles hit at each location  $x$ .

Suppose, for simplicity, that  $\psi_\uparrow(x) = e^{ik_\uparrow x}$ ,  $\psi_\downarrow(x) = e^{ik_\downarrow x}$ , where  $k_\uparrow, k_\downarrow$  are some constants.

- If the particles are all spat out in the state  $|\uparrow\rangle$ , what is the  $x$ -dependence of the resulting pattern  $P_\uparrow(x)$ ?
- If the particles are all spat out in the (normalized) state

$$|\psi\rangle = \mu|\uparrow\rangle + \lambda|\downarrow\rangle,$$

what is the  $x$ -dependence of the resulting pattern,  $P_\psi(x)$ ? Assume  $\mu, \lambda$  are real.

Now we wish to take into account interactions with the environment, which we will model by another two-state system, with Hilbert space  $\mathcal{H}_E$ . Suppose these interactions are described by the hamiltonian

$$\mathbf{H} = (\boldsymbol{\sigma}^z) \otimes \mathbf{M}$$

acting on  $\mathcal{H}_2 \otimes \mathcal{H}_E$ , where  $\boldsymbol{\sigma}^z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$  acting on the Hilbert space  $\mathcal{H}_2$  of particle paths, and  $\mathbf{M}$  is an operator acting on the Hilbert space of the environment.

Suppose the initial state of the whole system is

$$|\Psi_0\rangle \equiv (\mu|\uparrow\rangle + \lambda|\downarrow\rangle) \otimes |\uparrow\rangle_E,$$

and that

$$\mathbf{M} = m\boldsymbol{\sigma}^x = m(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)_E.$$

- Find  $|\Psi(t)\rangle$ , the state of the whole system at time  $t$ .
- How does the interference pattern depend on  $x$  and  $t$ ? For simplicity, consider the case where  $\mu = \lambda = \frac{1}{\sqrt{2}}$ .
- Interpret the previous result in terms of the time-dependence of the entanglement between the two qbits.
- What would happen if instead the initial state of the environment were an eigenvector of  $\mathbf{M}$ ?