University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (Physics 130C) Winter 2015
Assignment 1

Posted January 5, 2015 **Due 11am Thursday, January 15, 2015**

Please remember to put your name at the top of your homework.

Announcements

- The 130C web site is:

  http://physics.ucsd.edu/∼mcgreevy/w15/ .

  Please check it regularly for announcements and updates to the lecture notes and other fun things.

- Recitation sections begin Wednesday January 7.

Readings

- Read the lecture notes! I will mark our progress in the notes as we go. All other reading assignments can be regarded as supplementary.
- Chapter 2.1 of Preskill's Quantum Information Notes, Chapter 2. Chapter 1 of 130C will begin by following the logic this discussion, with occasional simplifications.
- You should read Chapter 1 of Le Bellac quickly – it provides some useful cultural background. Read chapter 2 of Le Bellac.

Problem Set 1

1. **I am curious about this.** What is currently your favorite field of physics? Possibilites might include: experimental or theoretical particle physics, string theory, condensed-matter physics (soft or hard, experiment or theory), astrophysics, atomic physics...

2. **Complex numbers reminders.**

   An essential feature of quantum mechanics is the fact that states are vectors over the complex numbers. Since some of our complex numbers may be rusty, let's do these exercises:

   (a) What real number is this:
   $$\left|\sqrt{5}+\mathbf{i}\right|$$
   (recall that the norm of a complex number z is $|z| \equiv \sqrt{zz^\star}$).

   (b) What is the real part of :
   $$e^{\mathbf{i}\frac{2\pi}{3}}$$

   (c) What is the imaginary part of :
   $$\frac{2+3\mathbf{i}}{1-4\mathbf{i}}$$

3. **Quick, I hope, exercises in linear algebra definitions.**

   (a) Show that the determinant of a unitary matrix is a phase (that is, has absolute value 1).
   [Hint: use the fact that $\det(AB) = \det A \det B$.]

   (b) Consider some operators acting on a Hilbert space with a resolution of the identity of the form
   $$\mathbb{1} = \sum_n |n\rangle\langle n| \ .$$
   Recall that the matrix representation of an operator in this basis is $A_{nm} = \langle n|\hat{A}|m\rangle$. Using Dirac notation, show that the matrix representation of a product of operators $\left(\hat{A}\hat{B}\right)_{nr}$ is given by the matrix product of the associated matrices $\sum_m A_{nm}B_{mr}$.

For the purposes of the following problems, recall that the *spectral representation* of an operator $\hat{H}$ on an N-dimensional vector space is an expression of the form

$$\hat{H} = \sum_{i=1}^{N} \nu_i \hat{P}_i$$

where $\hat{P}_i\hat{P}_j = \delta_{ij}\hat{P}_i$ is a set of N orthogonal projection operators.

4. **Do it by hand if you have to.** Consider an operator $\hat{M}$ with the following matrix representation :
   $$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} ,$$

(a) Find the eigenvalues and eigenvectors of $\hat{M}$.

(b) Find the spectral representation for $\hat{M}$. Relate $\hat{P}_i$ to the ith eigenvector of $\hat{M}$ and use Dirac notation.

5. **Normal matrices.**

An operator (or matrix) $\hat{A}$ is *normal* if it satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

(a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal.

(b) Show that any operator can be written as $\hat{A} = \hat{H} + \mathbf{i}\hat{G}$ where $\hat{H}, \hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger, \hat{A} - \hat{A}^\dagger$.] Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

(c) Show that a normal operator $\hat{A}$ admits a spectral representation

$$\hat{A} = \sum_{i=1}^{N} \lambda_i \hat{P}_i$$

for a set of projectors $\hat{P}_i$, and complex numbers λ_i.

6. **Clock and shift operators.**

Consider an N-dimensional Hilbert space, with orthonormal basis $\{|n\rangle, n = 0, \ldots, N-1\}$. Consider operators $\mathbf{T}$ and $\mathbf{U}$ which act on this N-state system by

$$\mathbf{T}|n\rangle = |n+1\rangle, \qquad \mathbf{U}|n\rangle = e^{\frac{2\pi \mathbf{i} n}{N}}|n\rangle \ .$$

In the definition of $\mathbf{T}$, the label on the ket should be understood as its value modulo N, so $N + n \equiv n$ (like a clock).

(a) Find the matrix representations of $\mathbf{T}$ and $\mathbf{U}$ in the basis $\{|n\rangle\}$.

(b) What are the eigenvalues of $\mathbf{U}$? What are the eigenvalues of its adjoint, $\mathbf{U}^\dagger$?

(c) Show that

$$\mathbf{UT} = e^{\frac{2\pi \mathbf{i}}{N}} \mathbf{TU}.$$

(d) From the definition of adjoint, how does $\mathbf{T}^\dagger$ act?

$$\mathbf{T}^\dagger|n\rangle = ?$$

(e) Show that the 'clock operator' $\mathbf{T}$ is normal – that is, commutes with its adjoint – and therefore can be diagonalized by a unitary basis rotation.

(f) Find the eigenvalues and eigenvectors of $\mathbf{T}$.
[Hint: consider states of the form $|\theta\rangle \equiv \sum_n e^{\mathbf{i} n\theta}|n\rangle$.]