New dimensions for wound strings

based on:

E. Silverstein, hep-th/0510044

JM, E. Silverstein, D. Starr, hep-th/0612121

D. Green, A. Lawrence, JM, D. Morrison, E. Silverstein, in progress.

March 28, 2007

Some motivation

- we know a lot about Ricci-flat compactifications.

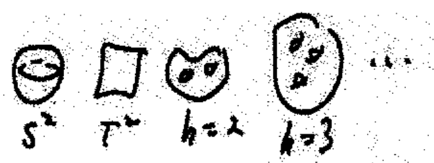
we know to parametrize light modes of the metric,
we know how to count them,
we know their holomorphic couplings,
(not the kahler potential...)
we know about brane probes.

low-energy supersymmetry is appealing and helpful.
it's not a prediction of string theory.

- is this lamp-post phenomenon?

or are they really preferred somehow?
to answer this we have to show that others are not preferred.

most spaces are not flat.



in 2d, only the torus is flat ......and only $\mathbf{P}^1$ has net positive curvature.
All the others have net negative curvature.

And notice that they also have many one-cycles.
More important and impressively large is their π_1.
This generalizes to $d > 2$.

Perturbative string theory propagating on a space with π_1 has stable winding modes.
These are the protagonists of this talk.

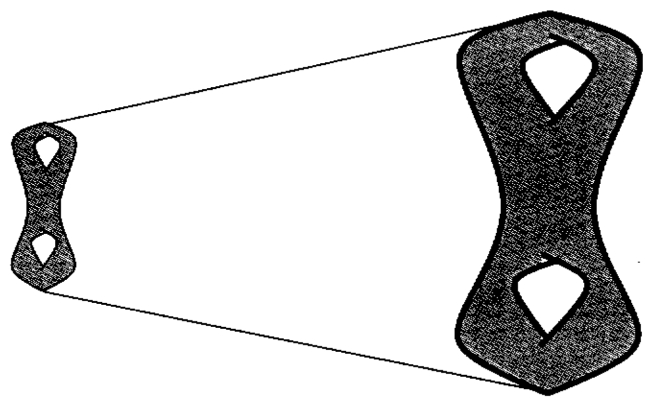
How to make ~~on-shell~~ solutions which contain such spaces?

Option 1: Add fluxes etc to stabilize compactification data.

[Saltman-Silverstein, much more to do]

Option 2: We'll make cosmologies out of them – treat them as initial data and evolve.

(not all choices are possible, these are).



At large radius, they expand slowly in a way controlled by GR: *e.g.*

$$ds^2 = -dt^2 + t^2 ds^2_{H^n/\Gamma}$$

For simplicity: at large dimension, time evolution approximates RG flow [Polchinski 89, ...].

There is a solution of string theory (exact in α') with linear dilaton. weakly coupled on a semi-infinite interval.

(relevant both for understanding space of compactifications and space of possible cosmological histories.)

why should we do this? huge uncharted territory of landscape. perhaps qualitatively different.

Observation ES, hep-th/0510...

moduli potential for generic string compactification:

(in string units, in string frame)

$$U_{4\text{d einstein}} = g_s^2 (D - D_c) V^{-1} + \frac{g_s^2}{V} \int_X \sqrt{G_X} \frac{(-R)}{V}.$$

+fluxes+orientifolds+branes +loops+nonperturbative effects....

On the worldsheet:

$$\beta_\Phi \sim D - D_c + \frac{\int (-R)}{V}.$$

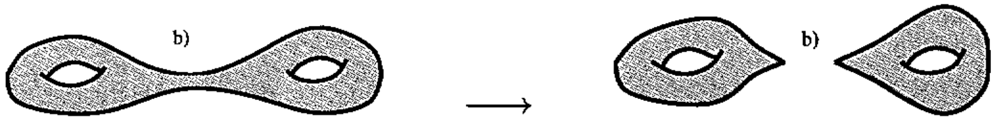
$D - D_c$ and $g - 1$ appear in the same way.

Are they dual??

Previous work:

["Things Fall Apart," Allan Adams, Xiao Liu, JM, Saltman, ES hep-th/0501...

"The tachyon at the end of the universe" JM, ES hep-th/0506...]



for this, the spin structure matters:
it's important to have APBCs on the spacetime fermions.
$\Longrightarrow$ tachyons.

Today: no pinching.

Plan

I. a check on the consistency of string propagation in such spaces

1. a theorem of Milnor relating negative curvature to large π_1
2. implications of modular invariance
3. check in examples

II. what is the supercritical space?

4. what do D-branes think it is?
5. Buscher argument
6. input from AdS/CFT

a result of Milnor relating geometry and topology.

Milnor (1968): For compact spaces $\mathcal{M}$ with negative sectional curvature, $\pi_1(\mathcal{M})$ has exponential growth, in the word metric.

$$g_1 g_2 g_1^{-1} g_2 g_3^{-1} \cdots g_1$$

$\Longrightarrow$ the number of closed geodesics (up to homotopy equivalence) grows exponentially with length:

$$\rho(L) \sim e^{L/L_0}.$$

Why:
the volume of a ball in the covering space $\tilde{\mathcal{M}}$

$$V(r) \geq e^{(n-1)r/r_0}$$

but this ball must be covered by many translates of smaller balls by the fundamental group of $\mathcal{M}$.

Put weakly-coupled strings on such a space:

each element of π_1 guarantees a string state with $m \propto L$

which is stable when $g_s = 0$

$$\rho(L) \sim e^{L/L_0}$$

this leads to an exponential growth in the number of winding strings

$$\Longrightarrow$$

extra contribution to hagedorn behavior of the density of states.

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review of modular invariance [Kutasov-Seiberg]

$$\int \frac{d^2\tau}{\tau_2^2} Z_1(\tau) = \mathrm{tr} \int \frac{d^2\tau}{4\tau_2} q^{L_0} \bar{q}^{\tilde{L}_0}.$$

$$Z_1(\tau) = Z_1(\tau+1) = Z_1\left(\frac{-1}{\tau}\right).$$

def: $\rho(m) \sim e^{\sqrt{\frac{2}{3} c_{eff}} \pi m \sqrt{\alpha'}}$.

$$Z_1(\tau_2 \to 0) \to Z_{UV} \sim e^{\pi c_{eff}/6\tau_2}.$$

c_{eff} counts effective dimensions (b-f)

e.g. type 0 $c = 12$, bosonic $c = 24$, with spacetime susy $c = 0$.

Since $Z_1(\tau_2 \to \infty) \to Z_{IR} \sim e^{-\pi\alpha'\tau_2 m^2_{\min}}$:

$$\underset{\text{(UV)}}{c_{eff} > 0} \quad \Leftrightarrow \quad \underset{\text{(IR)}}{\text{negative modes.}}$$

pseudotachyons [Silverstein-Aharony, Hellerman-Swanson]

in time-dependent backgrounds, there can be unstable modes which don't destroy the solution.

(which don't grow because of Hubble friction,
whose back-reaction is small)

the basic example: tachyons in supercritical strings can slow-roll.

Example

A. flat space with hyperbolic slices:

$$ds^2 = -dt^2 + t^2 ds^2_{H^n}$$

$$ds^2_{H_n} = dy^2 + \sinh^2 y d\Omega^2_{n-1}$$

B. we can make the spatial slices compact by quotienting by a discrete isometry group:

$$ds^2 = -dt^2 + t^2 ds^2_{H^n/\Gamma}$$

Consider a superstring in this background.

Assume initial data that leads to large smooth slowly growing solution at large t.

Preview:

	UV	IR
A.	$c_{eff} = 0$ (by susy)	$m^2_{min} = 0$
B.	$c_{eff} > 0$ (winding modes)	$m^2_{min} < 0$

IR behavior.

As a proxy for all the light string modes, consider a massless scalar in this background. $ds^2 = -dt^2 + t^2 ds^2_{H^n}$

Covering space first. EOM:

$$\nabla^2 \eta = 0$$

$$\nabla^2 = \frac{1}{t^2}\left(-(t^2\partial_t^2 + nt\partial_t) + \nabla^2_{H_n}\right)$$

Separate vars: $\eta = u(t)Y(\Omega)f(y)$.

Solve

$$\nabla^2 f = -k^2 f \quad \longrightarrow \quad f_k \sim e^{\lambda_\pm y},$$

$$\lambda_\pm = \frac{1}{2}(-(n-1) \pm \sqrt{(n-1)^2 - 4k^2})$$

Which are normalizable?

$$\int^\infty dy \sqrt{-G} f^*(y) f(y) \sim \int^\infty dy \; e^{y\sqrt{(n-1)^2-4k^2}}.$$

- $k^2 < (n-1)^2/4 \implies$ continuum normalizable.

That is, there is a gap in the spectrum:

$$k^2 > \frac{(n-1)^2}{4}$$

But this is just flat space in funny coordinates.

This must be compensated by the time dependence.

The massless on-shell mode equation boils down to

$$-\frac{1}{t^n}\partial_t(t^n \partial_t u_k) = \frac{k^2}{t^2} u_k(t)$$

This has solution

$$u \sim t^\alpha, \qquad \alpha = \frac{1}{2}\left[-(n-1) \pm \sqrt{(n-1)^2 - 4k^2}\right].$$

$$\eta = t^{1-n/2} e^{i\omega \ln t} f_{k,L}(y) Y_L(\Omega).$$

IR limit of partition function:

A useful basis of modes has

$$t^2\nabla^2\eta = (\omega^2 - k^2 + (n-1)^2/4)\eta$$

$$\int d^dx\sqrt{-G}\Lambda(t) = \int d^dx\sqrt{-G}\mathrm{tr}\ \ln H$$

$$= \int dt dy d\Omega t^n \sinh^{n-1} y V_{S^{n-1}} \int d\tilde{\omega}\frac{1}{2\pi}\sum_L \int_{k^2>(n-1)^2/4} dk|f_k|^2|Y_L|^2$$

$$\times \int^{\infty} \frac{d\tau_2}{\tau_2} \exp\left(-\pi\alpha'\tau_2 t^{-2}(\omega^2 - k^2 + (n-1)^2/4)\right).$$

so far: expected behavior for flat space.

check: no exponential divergence in $\tau_2 \to \infty$ IR limit.

Compactify

In any orbifold of H_n which makes it compact, the 'discrete states' below the gap (such as $k = 0$) become normalizible.

Final result:

$$Z_1(\tau_2 \to \infty) \sim \int dt t^n e^{-\alpha' m^2_{\text{min}}\tau_2\pi}.$$

$$m^2_{\text{min}} = -\frac{(n-1)^2}{4t^2}.$$

Does this mean death?

Pseudotachyon: $S_0 = \int dt t^n (\dot{\eta})^2$

$$\eta = \eta_0 + \eta_1/t^{n-1}.$$

energy density $\rho_\eta \sim \frac{1}{t^{2n}}$, curvature $R \sim \frac{1}{t^2}$.

UV behavior

we want to check

$$c_{eff} = \frac{3}{2}(n-1)^2 \frac{\alpha'}{t^2}.$$

winding modes dominate.

$$m^2_{\text{winding}} = \frac{R^2}{\alpha'^2}.$$

Unlike torus case, there is an isolated minimal geodesic: nearby ones are heavier.

The massive fluctuations about the minimal curve contribute to Z!

Tracking the τ, l-dependence

dual channel: space coord $\sigma_1 \in [0, 2\pi\tilde{\tau}_2]$, time coord $\sigma_2 \in [0, 2\pi]$.

(set $\tau_1 = 0$ for laziness.) $\tilde{\tau}_2 = \frac{1}{\tau_2}$.

$$X(\sigma + 2\pi\tilde{\tau}_2) = X(\sigma) + l, \qquad l \equiv 2\pi R.$$

$$H = \frac{1}{2}\left(p_\perp^2 \tilde{\tau}_2 + \frac{l[\delta X_\perp]^2}{\tilde{\tau}_2 (2\pi\alpha')^2}\right) + \cdots$$

From the geometry:

$$l[X_\perp]^2 \sim l^2 \left(1 + \frac{\delta X_\perp^2}{t^2}\right)$$

t is the radius of curvature of the hyperbolic space.

Small fluctuation sum

Each transverse dimension is an SHO with

$$\omega = \frac{l}{2\pi t}.$$

$$\mathrm{tr}_l\ e^{-(2\pi)H\sqrt{\alpha'}} = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})l/t} = \frac{1}{2\sinh(l/2t)}.$$

2π is the length of the worldsheet time direction σ_2.

How many winding modes?

for any Γ which makes H/Γ compact: (growth of Γ in word metric)

$$\rho(l) \sim e^{(n-1)l/t}$$

The extreme UV limit of the partition function is then:

$$e^{-\pi m_{min}^2 \tilde{\tau}_2} \sim Z_{UV} \sim \int dl \rho(l) \left(\frac{1}{\sinh(l/2t)} \right)^{n-1} e^{-l^2/(4\pi\alpha'\tilde{\tau}_2)}.$$

this is the Selberg/Gutzwiller trace formula.

Do the l integral by saddle point:

$$Z_{UV} \sim \int dl e^{(n-1)l/t} e^{-(n-1)l/2t} e^{-l^2/\zeta} \sim e^{-2\zeta\left(\frac{n-1}{2t}\right)^2}.$$

$$\zeta \equiv 4\pi\alpha'\tilde{\tau}_2.$$

Modular invariance works for each t! (sufficient, not necessary)

Summary

$$\boxed{\text{IR:}} \quad Z_1(\tau_2 \to \infty) \sim \int dt t^n e^{-\alpha' m^2_{min} \tau_2 \pi}.$$

$m^2_{min} = -\frac{(n-1)^2}{4t^2}$, a pseudotachyon.

$$\boxed{\text{UV:}} \quad Z_1(\tau_2 \to 0) \sim \int dt t^n e^{-\alpha' m^2_{min} \pi / \tau_2}.$$

hagedorn density of winding modes, exponential growth.

$$\begin{array}{ccc} \text{geometry} & \overset{\text{Milnor}}{\longleftrightarrow} & \text{topology} \\ \downarrow & & \downarrow \\ \text{pseudotachyon} & \overset{\tau \to -1/\tau}{\longleftrightarrow} & c_{\text{eff}} > 0 \end{array}$$

some other examples.

nil

$$ds^2_{nil} = (dx + \frac{1}{2}ydz - \frac{1}{2}zdy)^2 + (dy^2 + dz^2)$$

has constant curvature, $R < 0$, but indefinite sectional curvatures.

π_1 has power law growth, but bigger power than any non-negatively curved space.

late time ricci flow (= time evolution for $D \gg D_c$)

the RG scale is μe^{Qt}, the extra dimensions are flat with coordinates $x_\perp$.

$g_s \sim g_0 e^{-Qt}$.

$$ds^2_S = -dt^2 + \frac{1}{3t^{1/3}}(dx + \frac{1}{2}ydz - \frac{1}{2}zdy)^2 + t^{1/3}(dy^2 + dz^2) + dx^2_\perp$$

gives no enhanced IR divergence of 1-loop amplitude.

sol

$$ds^2_{sol} = dz^2 + \frac{1}{2}e^{-2z}dx^2 + \frac{1}{2}e^{2z}dy^2$$

π_1 has exponential growth,

but no gap in the spectrum of the laplacian (no discrete states).
still: we find an enhanced IR divergence.

late-time Ricci-flow, gives a supercritical solution in string frame:

$$ds^2_S = -dt^2 + 4tdz^2 + \frac{1}{2}e^{-2z}dx^2 + \frac{1}{2}e^{2z}dy^2 + dx^2_\perp$$

Summary of Part I

string backgrounds with negative sectional curvature are effectively supercritical.

positive potential energy in target space $\leftrightarrow$ more worldsheet degrees of freedom.

Part II: What's the supercritical space?

small spaces.

strings on small spaces do interesting things.

examples:

- small CY = LG
- topology can change (flops, conifold transitions, winding tachyon condensation)
- small circle is T-dual to big circle:

winding modes build up continuum of momentum modes on dual
D-branes on small tori have big spaces of wilson lines.

Hint: A Dp-brane on our $\mathcal{M}_p$ with $b_1(\mathcal{M}_p) = h$ has a moduli space of flat connections which is a T^{2h}.

Focus on Riemann surface (p=2) case.

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review of abel-jacobi

On the RS Σ, genus h:

holomorphic one forms ω_i, $i = 1...h$. Pick a basepoint $z_0 \in \Sigma$

$$X : \Sigma \to \text{Jac}(\Sigma)$$

$$X : z \mapsto (X_1(z), ..., X_h(z)) = \left(\int_{z_0}^{z} \omega_1, ..., \int_{z_0}^{z} \omega_h \right) \in \mathbb{C}^h.$$

$$X \equiv X + \left(\oint_\gamma \omega_1, ..., \oint_\gamma \omega_h \right), \ \forall \gamma \in H_1(\Sigma).$$

$$\text{Jac}(\Sigma) \equiv \mathbb{C}^h / \langle \text{periods} \rangle$$

Period matrix: $\Omega_{ab} \equiv \oint_{b^a} \omega^b$ in the basis where $\oint_{a^a} \omega^b = \delta_a^b$.

$2h$ winding currents:

$$J^a_\alpha = \epsilon^\beta_\alpha \omega^a_i \frac{\partial z^i}{\partial \sigma^\beta}, \quad a = 1..h$$

Conserved $\partial^\alpha J^a_\alpha = 0$ since $d\omega = 0$.

A metric on the RS which makes them explicit is the pullback of the flat metric on the Jacobian by the Abel-Jacobi map:

$$ds^2 = \omega^a dz \gamma_{ab} \bar{\omega}^b d\bar{z}, \quad \gamma_{ab} \equiv \mathrm{Im}\Omega^{-1}_{ab}$$

Note: there are many possible UV completions of the RS NLSM.

Claim: The embedding in the Jacobian is the minimal description which preserves the winding symmetry.

At a putative UV fixed point, the winding currents themselves contribute $c = 2h$.

Recall: Buscher derivation of T-duality

$$S_{S^1}[\theta, A, X] = \int d^2\sigma \Big(R^2 (\partial\theta - A)^2 + XF \Big)$$

Integrate out X first $\Longrightarrow$ A is pure gauge.

$$\text{original theory}: \quad S[\theta] = \int d^2\sigma R^2 (\partial\theta)^2.$$

Sum over winding sectors of X constrain the gauge fields' Wilson lines to vanish on worldsheets of arbitrary genus. [Rocek-Verlinde].

Integrate out A first, gauge $\theta = 0$

$$\mathrm{T-dual\ theory}: \quad S[X] = \int d^2\sigma \frac{1}{R^2} (\partial X)^2$$

A dilaton shift $\Phi \to \Phi - \frac{1}{2} \log R^2$ arises from the path integration over A.

Under T-duality, momentum and winding modes are exchanged.

Buscher Variations

What if we gauge the axial symmetry generated by the winding current $*\partial\theta$?

$$S_{S^1}^{\text{axial}} = \int d^2\sigma \Big(R^2(\partial\theta - *A)^2 + R^2 X F \Big)$$

$X \equiv X + 2\pi$

Integrate out X first $\Longrightarrow$ A is pure gauge.

$$\text{original theory :} \qquad S[\theta] = \int d^2\sigma R^2 (\partial\theta)^2.$$

We have chosen the radius of the $U(1)$ gauge group to be such that the quantization condition for flux on the worldsheet is $\int F = n/R^2$ for integer n.

Integrate out A first $\longrightarrow$ the original description, too!

(Momenta and windings are not interchanged.)

Buscher Variations, cont'd

More explicitly, consider

$$\langle e^{ipX(\sigma')} \rangle = Z^{-1} \int [DX][Dz] \frac{[DA]}{\text{Vol}(\mathcal{G})} e^{-S} e^{ipX(\sigma')}$$

$$\int [DX] e^{-\int (XF - ipX(\sigma')} = \delta[F - ip\delta^{(2)}(\sigma)].$$

$$\Longrightarrow \quad \langle e^{ipX(\sigma')} \rangle = \langle e^{ip\theta(\sigma')} \rangle.$$

For S^1 target space this path integral transform is trivial.
Its generalization to negatively curved target spaces will generate a dual description.

Buscherizing the Riemann Surface

Like NS5-ALE duality, the RS only has conserved winding currents.

Bosonically for now, and ignoring dressing by time-dependence:

$$S_\Sigma = \int d^2\sigma \Big(\left(\epsilon_\mu^\nu \omega^a(z)\partial_\nu z - A_\mu^a\right) \gamma_{ab} \left(\epsilon^{\mu\delta}\bar\omega^b(\bar z)\partial_\delta \bar z - \bar A^{b\mu}\right) + i[\bar X^a F^a + X^a \bar F^a] \Big) \qquad (1)$$

convenient gauge choice: $A_1^a = \omega^a \partial_0 z + \partial_0 X^a$. Integrate A:

$$S_\Sigma[X,z] = \int d^2\sigma \Big(\partial_\mu X^a \gamma_{ab} \partial^\mu X^{b\prime} + [\bar X^a \partial_\mu J_V^{b\mu} + X^a \partial_\mu \bar J_V^{b\mu}] \Big)$$

momentum modes of $X \leftrightarrow$ momentum modes of z

Momentum violation

$$S_\Sigma[X,z] = \int d^2\sigma \Big(\partial_\mu X^a \gamma_{ab} \partial^\mu X^{b\prime} + [\bar X^a \partial_\mu J_V^{b\mu} + X^a \partial_\mu \bar J_V^{b\mu}] \Big)$$

Unlike circle case: $\partial_\mu J_V^\mu \neq 0$.

$\Longrightarrow$ potential for X [like Gregory-Harvey-Moore; Hori-Vafa; Tong]

This potential is a rolling tachyon which lets the system reproduce its late-time central charge.

What can we say about this potential?

We can argue that it has critical points on the locus

$$X^a = \int^z \omega^a.$$

$\Longrightarrow$ Time evolution cuts RS out of its Jacobian.

Superstring

(1,1) version of Buscherization exists, works.

GSO is confusing [Hellerman 04, Adams et al "Things Fall Apart" 0501...].

AdS/CFT embedding

Poincare patch of AdS_5, choose coords to slice by hyperboloids:

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + t^2 ds^2_{H_3}) + \frac{L^2}{r^2}dr^2 + d\Omega_5^2$$

$$L^4 = 4\pi g_s N l_s^4.$$

Now orbifold by $\Gamma \subset SL(3, \mathbb{R})$:

$\mathcal{M}_3 = H_3/\Gamma$ compact.

Curvature radius of $\mathcal{M}_3$ is $l(t) \sim t$.

Proper size is $l_{proper} \sim rl/L$.

But this has singularities, and further, a mass gap in the IR (the classical moduli, which in fact parametrize $(T^{b_1})^N/S_N$ are lifted quantumly). something we don't understand.

Also: RR flux! (scary)

Solve these problems by higgsing at a higher scale [Horowitz-Silverstein]

$$ds^2 = h^{-1}(r)(-dt^2 + t^2 ds^2_{H_3}) + h(r)dr^2 + d\Omega_5^2$$

$$h(r) = \frac{L^2}{r^2}, r > R$$

$$h(r) = \frac{L^2}{R^2}, r \leq R$$

i.e. consider the $\mathcal{N} = 4$ theory (on the space in blue) in an $SO(6)$-invariant point on its Coulomb branch.

There is a relatively flat potential for R; the shell collapses slowly.

Return of the torus

$U(1)^N$ gauge theory on a space with one-cycles reduces to QM on the space of Wilson lines:

$$\text{Sym}^N T^{b_1(\mathcal{M})} = \left(T^{b_1(\mathcal{M})}\right)^N / S_N$$

This moduli space is lifted by a potential generated by modes with masses determined by the scale R of the higgsing shell.

Here the torus of Wilson lines arises in the IR of the gauge theory.
(No funny business with choice of UV completion like in worldsheet sigma model.)
(The funny business is mapped to the choice of state of field theory on Milne space.)

Conclusions

A novel mechanism for growing dimensions.

Positive potential energy in target space ↔ more worldsheet degrees of freedom.

Tension is Dimension, Harvey-Kachru-Moore-Silverstein

String theory probes geometry, topology, and even dimensionality differently from point particle theories.

Applications to geometric group theory?

The End

Comments on the potential

Expand the system about a point in the image of Abel, $X^a = \int^z \omega^a$

$$\int DX'Dz \prod_n \mathcal{O}^{(n)} \quad Exp\Big\{ i \int d^2\sigma \Big(\partial_\mu [\int^z \omega^a + X^{a\prime}] \gamma_{ab} \partial^\mu [\int^{\bar{z}} \bar{\omega}^b + \bar{X}^{b\prime}]$$
$$+ [(\bar{X}'^a + \int^{\bar{z}} \bar{\omega}^a) \partial_\mu J_V^{a\mu} + (X'^a + \int^z \omega^a) \partial_\mu \bar{J}_V^{a\mu}] \Big) \Big\} \quad (2)$$

where we have shifted $X'^a \equiv X^a - \int^z \omega^a$.

Contributions to the potential for X (equivalently for X') arise via momentum-violating configurations for which

$\int d^2\sigma \partial_\mu J_V^{a\mu} \equiv \Delta Q^a \neq 0.$

expand the system about a given point z_{0a} lying on the ath handle in the target space RS.

We can estimate the level of momentum violation ΔQ^b from the one-point functions $\langle e^{i\Delta Q^c \int^z \omega^c} \rangle$.

Since we are on the ath handle, the one-forms satisfy

$$\omega^{c\neq a}(z_{0a}) \ll \omega^a(z_{0a}).$$

Thus the one-point functions $\langle e^{i\Delta Q^c \int^z \omega^c} \rangle$ are larger for ΔQ^c in the transverse directions $c \neq a$ than in the ath direction:

$$\Delta Q^{c\neq a} \gg \Delta Q^a$$

In the path integral over z, this indicates that the potential for X' will be much stronger in the $2h-2$ directions transverse to the Riemann surface than in the 2 directions along it.

scale of potential

RSNLSM is not conformal, and is strongly coupled in the UV.

regulate the path integration over z w/ cutoff at the scale of the corresponding Landau pole

$$\Lambda \sim e^{+R^2/\alpha'}$$

where R is the smallest curvature radius in the Riemann surface.

addenda

$$\partial_\mu j^\mu = \omega^a \partial_\mu j_a^\mu$$

?

Setup

Consider a target space containing a Riemann Surface Σ.

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$$\nabla^2 = \frac{1}{t^2}(-(t^2\partial_t^2 + nt\partial_t) + \nabla^2_{\mathbb{H}_n}),$$

with $\nabla^2_{\mathbb{H}_n}$ the spatial Laplacian:

$$\nabla^2_{\mathbb{H}_n} = \frac{1}{\sinh^2 y}\left[\sinh^2 y\,\partial_y^2 + \left(\frac{n-1}{2}\right)\sinh(2y)\partial_y + \nabla^2_{\Omega_{n-1}}\right].$$

$$e^{-(n-1)y}\partial_y(e^{(n-1)y}\partial_y f) = -k^2 f$$

a

[a]COMPLETE SET OF MODES:

Let $Y_L(\Omega)$ be a spherical harmonic satisfying $\nabla^2_{\Omega_{n-1}} Y_L = -L^2 Y_L$ and with f as above. It proves convenient to consider a complete set of functions

$$\psi(t, y, \Omega) = u_\omega(t) f_{k,L}(y) Y_L(\Omega),$$

with $u_\omega(t)$ given by

$$u_\omega(t) = t^{\frac{1-n}{2}} e^{i\omega \ln t}$$

which satisfy

$$-(t^2\partial_t^2 + nt\partial_t)u_\omega = \left[\omega^2 + \left(\frac{n-1}{2}\right)^2\right] u_\omega.$$

$$\psi(t, y, \Omega) \sim t^{\frac{1-n}{2}} e^{i\omega \ln t} e^{\frac{1-n}{2} y} e^{ipy} Y_L(\Omega).$$