# Holographic duality of topological strings

in progress with:

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## Introduction

Consider the emergence from the DV matrix model of the geometry describing the low energy gauge theory dynamics.
(special geometry, deformations of singularities, period integrals)

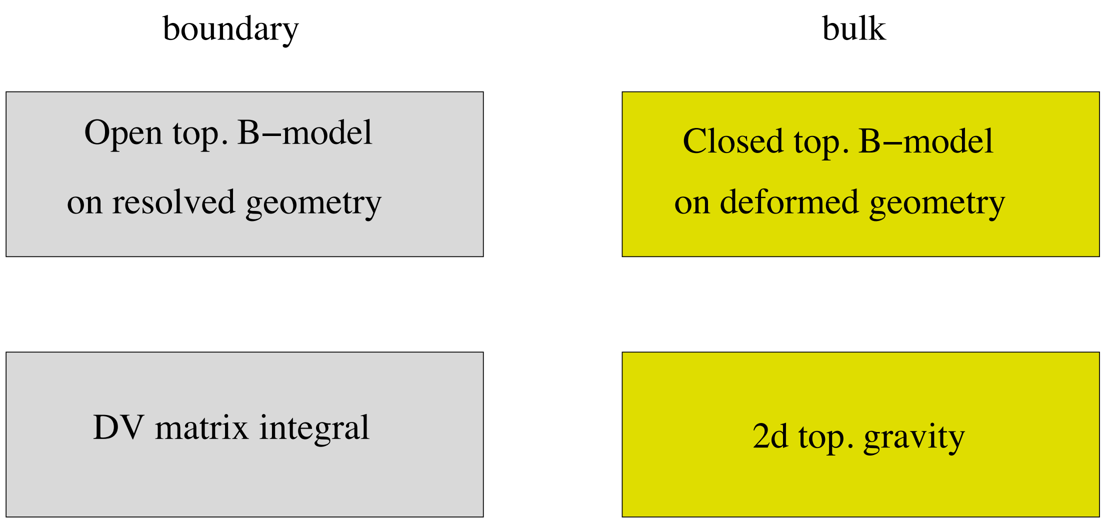
In the 't Hooft limit one takes, a description in terms of some classical macroscopic variables emerges
(in particular a chiral boson $\varphi$ on the SW curve)

I will try to argue that this is actually a gravitational description.
either 2d topological gravity on the SW curve, or the 6d KS theory of CS deformations of a CY $X_g$.

This is an example of a large-N theory where we can really identify the master-field with the string field of the 't Hooft dual string.

(Note: this is a string theory with no $\alpha'$ effects, so the 'supergravity limit' no-oscillators description is exact.)

$$u^2 + v^2 + y^2 = W'(x)^2$$




There exist many relationships between 2d gravity and matrix models (both 0d and $0 + 1$d).

Our 2d gravity will be in the *target space*.

## Review of the relevant geometry

Consider the singular, noncompact CY threefold $X_g$

$$u^2 + v^2 + y^2 = W'(x)^2.$$

$$W(x) = \sum_{r=1}^{g+2} t_r x^r$$

($g$ is the genus of tthe hyperelliptic curve $\Sigma_g$ that forgets $u, v$.)

this geometry can be made smooth by resolving (changing kahler structure) or deforming (changing complex structure).

its nowhere-vanishing holomorphic threeform looks like

$$\Omega = \frac{dx \wedge du \wedge dv}{y(x,u,v)}$$

and this encodes the complex structure.

the relationship between CS in the sense of Beltrami differentials

$$\bar{\partial}_{\bar{i}} \mapsto \bar{\partial}_{\bar{i}} + A_{\bar{i}}^{j} \partial_j$$

and in the sense of the holomorphic threeform (which always looks holomorphic) is as follows:

take a particular beltrami differential, $A_{\bar{i}}^{(I)\ j}$, contract with $\Omega \longrightarrow$ $(2,1)$ form $A^{(I)\prime} = A \cdot \Omega$.
There is a basis of coordinates $S_I$ on the CS moduli space where this is $\partial_{S_I}\Omega + k_I\Omega$.

So the complex structure is encoded in $\Omega$, in particular in its periods around compact three-cycles:

$$S_I = \oint_{A_I} \Omega$$

the norm of a CS deformation is

$$||m||^2 = \int_{X_g} \partial_m \Omega \wedge \bar{\partial}_{\bar{m}} \bar{\Omega}.$$

infiniteness of this can be diagnosed by calculating

$$\int_{\sum_i A_i} \partial_{t_r} \Omega = \int_\infty dx \partial_{t_r} y dx = \int_\infty dx \frac{r x^{g+1+r}}{y} = \infty.$$

This geometry can be *deformed* to

$$u^2 + v^2 + y^2 = W'(x)^2 + f(x)$$

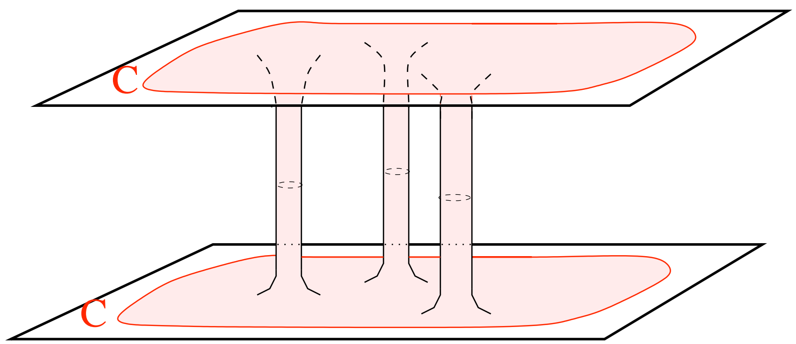
where $f(x) = \sum_{r=0}^{g} u_r x^r$ is of degree $g$. the $u_r$ are linearly related to the $S_I$ by the period matrix.

The $u$s are normalizible:

$$\int_{\sum A_i} \partial_{u_r} \Omega \propto \int_\infty dx \frac{x^r}{y} < \infty.$$

The $t$s are not.

**the point:** this silly CY only has $h^{2,1} = g + 1$ normalizible CS moduli, and the information about them is encoded in cycles which are visible on $\Sigma_g$, which also has this many CS moduli (actually it has $3g - 3$, these preserve hyperellipticity).




Alternatively, all of the CY moduli are even visible as moduli of the plane with $2g + 2$ punctures. (which are paired at weak coupling.)

## resolution and the open string side

if $W \equiv 0$ (this is of course not related through allowed fluctuations), this is an $A_1$ surface singularity times the complex $x$-plane. In this case, we know how to resolve the $A_1$ singularity, and discover a $\mathbb{P}^1$ where the singularity was

$$s^2(u^2 + v^2) + y^2 = 0$$

with an extra $C^*$ action where

$$(u, v, s) \mapsto (\lambda u, \lambda v, \lambda^{-2} s)$$

if we were to consider type IIB string theory on this space, we could wrap spacefilling D5 branes on the $\mathbb{P}^1$ at any value of $x$, and there would be a moduli space of the D5-brane theory parametrized by the vev of an adjoint scalar $\Phi$.

turning on $W(x)$ is a (non-normalizible) change in the complex structure of $X$ which leads to a superpotential $\mathrm{tr}W(\Phi)$ for this scalar. in terms of the geometry, it obstructs the deformations of the curve, leaving only $g+1$ isolated holomorphic $\mathbb{P}^1$s at the critical points of $W$.

I will be interested in the related open topological B-model, related by twist on the worldsheet.

$$\mathrm{tr}\lambda\lambda \leftrightarrow \mathrm{tr}1 = M$$

this is encoded in the holomorphic chern-simons action:

$$S_{\mathcal{C}} = \frac{1}{g_s}\int_{\mathcal{C}} \mathrm{tr}\Phi_1 \bar{\partial}_{\mathcal{A}} \Phi_0$$

The resolved $A_1$ times the $x$-plane is the total space of the normal bundle $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ of the curve $\mathcal{C} = \mathbb{P}^1$ at a reference value of $x$. Here we've picked coords $z, z_0, z_1$ for the directions

$\mathbb{P}^1 \leftarrow \mathcal{O}(0) \oplus \mathcal{O}(-2)$,

$\bar{\partial}_{\mathcal{A}} = \bar{\partial} + [\mathcal{A}, \cdot]$ where $\mathcal{A} = d\bar{z}^{\bar{j}} A^{j}_{\bar{i}} \frac{\partial}{\partial z_j} + A$ where $A^{j}_{\bar{i}}$ is the Beltrami differential encoding the change in complex structure, and $A$ is a $U(M)$ connection on $\mathcal{C}$.

Important fact: this action can be derived as (Witten's cubic) open SFT of the topological B-model with worldsheet boundaries on the $\mathbb{P}^1$'s.

but it also gives the right eom.
eom of cs = susy condition of physical theory.

picking a gauge where $A^{1}_{\bar{z}} = W'(z_0)$ this gives

$$S_{\mathcal{C}} = \frac{1}{g_s} \int_{\mathcal{C}} \left( \Phi_1 \bar{\partial}_A \Phi_0 + W(\Phi_0) dz \wedge d\bar{z} \right)$$

$\Phi_1$ appears linearly. integrating it out yields

$$\bar{\partial} \Phi_0 = 0$$

which says that $\Phi_0$ is a holomorphic function on $\mathbb{P}^1$
i.e. a constant matrix

$$\Phi_0(z) = \Phi$$

and so

$$\int D\Phi_0 D\Phi_1 DA e^{S_{\mathcal{C}}} = \int d\Phi e^{\frac{1}{g_s}\mathrm{tr}W(\Phi)}$$

- this is a holomorphic matrix integral

- 't Hooft couplings $S_i$ are fixed. (by chemical potentials if you like.)

**matrix integrals**

$\Phi$ is $M \times M$, complex.

$$Z = \int d\Phi e^{-\frac{1}{g_s}\mathrm{tr}W(\Phi)}$$

is a shorthand for

$$Z(S,t) = \int_{\Gamma_S} \prod_a d\lambda_a \Delta(\lambda)^2 e^{-\frac{1}{g_s}\sum_a W(\lambda_a)}$$

$$W(x) = \sum_{r=1}^{g+2} t_r x^r.$$

the integrals over the $\lambda_a$ are contour integrals over a complex plane with features a priori defined only by the potential $W$,
and to define this integral i need to tell you what the contours are.

the features specified by $W$ are saddle points NEAR critical points of $W$:

$$W'(x) \propto \prod_{i=1}^{g+1} (x - \alpha_i).$$

the geometry mentioned above emerges in an 't Hooft limit of this model, $g_s \to 0, M \to \infty, S = g_s M$ fixed. before explaining how to specify these contours, let me review the solution of the model in the 't Hooft limit.

(which, being classical, doesn't care about the definition of the measure and such.)

the most illustrative way to do this is to demand that the integral is invariant under reasonable changes of variables, e.g. under the change

$$\delta\Phi = \frac{\epsilon}{x - \Phi}.$$

where $x$ is bigger than any of the eigenvalues of $\Phi$.

note that this is all field redefs that don't require inverting $\Phi$.

$$\hat{\omega}(x) \equiv \mathrm{tr}\frac{1}{x-\Phi}, \quad \omega(x) \equiv \langle \mathrm{tr}\frac{1}{x-\Phi}\rangle_{mm}$$

the importance of the resolvent is that its discontinuities as a function of $x$ give the density of eigenvalues:

$$\rho(x) \equiv \frac{1}{N}\langle \sum_a \delta(x-\lambda_a)\rangle_{mm} \propto \omega(x+i\epsilon) - \omega(x-i\epsilon)$$

$$\omega(x) = N\int \frac{\rho(z)dz}{x-z}$$

the condition that this implies is called the loop equation

$$0 = \langle \hat{\omega}(x)^2 - \frac{1}{g_s}\mathrm{tr}\left(\frac{W'(\Phi)}{x-\Phi}\right)\rangle_{mm}$$

introducing

$$W(x) = -\sum_{n=1}^{\infty} t_n x^n$$

(to be set to $(t_n) = (t_1 ... t_{g+2}, 0..)$ later) this is equivalent to the Virasoro constraint:

$$0 = \oint_\infty \frac{dz}{x-z} \langle T(z) \rangle_{mm} \qquad (*)$$

with

$$T(x) = \frac{1}{2} (\partial \varphi(x))^2$$

$$\varphi(x) = W(x) + 2 g_s \mathrm{tr} \log (x - \Phi)$$

and $x$ is outside the contour, $|x| > |\lambda_a|, \quad \forall a$ .

$(*)$ can be rewritten as

$$\mathcal{L}_n Z = 0, \qquad n \geq -1$$

$$\mathcal{L}_n = \sum_{k=0}^{n} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k}} + \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}}$$

(note: $\frac{\partial}{\partial t_0} Z \equiv NZ$)

These $\mathcal{L}_n$'s satisfy a Virasoro algebra

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}.$$

The partition function furnishes a representation of the 2d conformal group.

Which CFT?

in terms of a 2d NS fermion living on the eigenvalue plane:

$$\psi(\lambda) = \sum_{r \in Z + \frac{1}{2}} \psi_r \lambda^r$$

$$\Delta(\lambda) = \prod_{a<b} (\lambda_a - \lambda_b) = \langle 0 | \psi_{\frac{1}{2}} \psi_{\frac{3}{2}} \ldots \psi_{N-\frac{1}{2}} \psi(\lambda_1) \cdots \psi(\lambda_N) | 0 \rangle \frac{1}{N!}$$

Using this fact, and with two fermions, (ITEP group, Kostov 9907060)

$$Z = \langle N, -N | e^{H[W]} e^{Q_+} | 0 \rangle$$

The two fermions generate a $u(2)$ current algebra.
The COM $U(1)$ decouples.

$$H(\lambda) \equiv : \psi^{(1)\star}(\lambda)\psi^{(1)}(\lambda) - \psi^{(2)\star}(\lambda)\psi^{(2)}(\lambda) :$$

$$J_+(\lambda) \equiv \psi^{(1)\star}(\lambda)\psi^{(2)}(\lambda) \quad J_-(\lambda) \equiv \psi^{(2)\star}(\lambda)\psi^{(1)}(\lambda)$$

$$H[W] \equiv \oint_\infty d\lambda W(\lambda) H(\lambda)$$

$$Q_+ \equiv \int_{-\infty}^{\infty} J_+(\lambda).$$

and

$$\langle N, -N| = \langle 0| \prod_{r=1}^{N} \psi^{(1)}_{r-\frac{1}{2}} \psi^{(2)\star}_{r-\frac{1}{2}}$$

is the lowest-energy state of fermion number $\int dz H(z) = 2N$.

Note: only the $Q_+^N/N!$ term in the expansion of the screening term contributes.

Any matrix model expectation value is

$$\langle \mathcal{O} \rangle_{mm} = \frac{\langle N, -N| e^{H[W]} \mathcal{O} e^{Q_+} |0\rangle}{Z}.$$

In particular the Virasoro constraints are

$$\langle N, -N| e^{H[W]} T(z) e^{Q_+} |0\rangle = (\text{reg. function at } z = 0)$$

where $T$ is the Sugawara stress tensor for the $u(2)$ current algebra (only the $su(2)$ piece matters).

## The return of the boson

The $u(2)$ current algebra also has a bosonized description.

$$\psi^{(\alpha)}(z) =: e^{\varphi^{(\alpha)}(z)} : \quad \psi^{(\alpha)\star}(z) =: e^{-\varphi^{(\alpha)}(z)} :$$

In terms of $\varphi = \frac{1}{\sqrt{2}}(\varphi^{(1)} - \varphi^{(2)})$

$$H(z) = \frac{1}{\sqrt{2}}\partial\varphi(z) \quad J_{\pm}(z) =: e^{\pm\sqrt{2}\varphi(z)} :$$

Note that $\varphi$ is periodic.
Restoring $g_s$ will tell us that $\varphi \simeq \varphi + g_s$.
The resulting quantization of momenta is quantization of eigenvalue number.

In the semiclassical limit, the matrix model partition function is

$$Z = e^{\int d^2x \partial\varphi\bar{\partial}\varphi + \oint dx W(x)\partial\varphi(x)}$$

## back to defining the contours

pick the $\alpha_i$ to be well-separated, and $g_s$ to be small compared to this separation. let $a_i$ be a contour in the $x$-plane surrounding only the $i$th critical point.

then specifying how many of the eigenvalues $M_i$ go over the $i$th contour is the same as specifying

$$M_i = \oint_{a_i} \omega(x)$$

show that this is infinitesimally true: making an infinitesimal change of the contour at infinity for a probe eigenvalue $x$ going over the $i$th pass is

$$\frac{\delta Z}{\delta b_i} = \int_{\delta b_i} dx e^{S_{eff}(x)} = \int_{b_i} dx \; \frac{\partial}{\partial x} \exp\left( W(x) + \int d\lambda \rho(\lambda) \ln(x - \lambda) \right)$$

$$= \int_{b_i} y dx \; e^{S_{eff}(x)} = \partial_{S_i} \mathcal{F} \; e^{\mathcal{F}} = \partial_{S_i} e^{\mathcal{F}} = \partial_{S_i} Z$$

this can be accomplished by a lagrange multiplier

$$\delta \left( g_s (M_i - \oint_{a_i} \mathrm{tr} \frac{1}{x - \Phi}) \right) = \int d\pi_i e^{i\pi_i \left( S_i - g_s \oint_{a_i} \omega(x) \right)}$$

Including also the chemical potentials in the CFT description, we have

$$Z(t, S) = \int d\pi_i e^{\mu_i S_i} \langle N, -N | \exp \left( H[W + \pi_i \mathbf{g}_{a_i}] \right) e^{Q_+} |0\rangle$$

where $\mathbf{g}_{a_i}$ is a function of x which satisfies

$$\int_{\infty} \mathbf{g}_{a_i}(x) f(x) = \int_{a_i} f(x).$$

## the bulk theory

One of the Gopakumar-Vafa dualities (recently given a worldsheet derivation by Ooguri and Vafa) says that

the open topological B-model on the resolved $X_{g=0}$ with $M$ branes

is

the closed topological B-model on $X_{g=0}$ deformed by $f(x) = g_s M$.

More generally
the open B-model on the resolved $X_g$ with $M_i$ branes on the $\mathbb{P}^1$ at the $i$th critical point of $W'$

is

the closed B-model on $X_g$ deformed by $f(x)$, such that
$g_s M_i = \oint_{a_i} dx\ \sqrt{W'(x)^2 + f(x)}$ .

## The KS theory

The string field theory of the closed B-model is the Kodaira-Spencer theory of gravity (BCOV 9309140)
which is a theory of deformations of complex structure.

The string field is a Beltrami differential on $X_g$

$$A = A_{\bar{i}}^{j} d\bar{z}^{\bar{i}} \frac{\partial}{\partial z^{j}} \in \Gamma(TX_g \otimes \Omega^{0,1})$$

The change in the metric associated with this change in the complex structure is

$$\delta g_{\bar{i}\bar{j}} = A_{\bar{i}}^{j} g_{j\bar{j}}.$$

A CY comes with a nowhere-vanishing holomorphic threeform $\Omega_0$, so we can use instead

$$(A')_{\bar{i}jk} \equiv (A \cdot \Omega_0)_{\bar{i}jk} \equiv A_{\bar{i}}^{i} (\Omega_0)_{ijk}.$$

The KS theory is a machine which takes some data $S, t, \bar{t}$ and produces a function $\mathcal{F}(S, t, \bar{t})$

which is also a function of $g_S$.

The input data is a base point in the CS moduli space, specified by the 3-form $\Omega_0$

In terms of $t$, it is the point where $\bar{t} = t$.

and a tangent vector to the moduli space

$$\mathbf{x} \in H^{(0,1)}(TX).$$

The KS equation

$$0 = \bar{\partial} A + \frac{1}{2}[A, A]$$

expresses the condition that the deformation away from $\Omega_0$ specified by $\mathbf{x}$ is integrable, i.e. that the deformed dolbeault operator $\bar{\partial} + A \cdot \partial$ still squares to zero.

## interjection about the point of this

Which modes of the KS field are physical in this case?

$$h^{(2,1)}(X_g) = g + 1$$

is the number of independent (normalizable) complex structure moduli of this CY.

Good coordinates on the CS moduli space are

$$S_i = \oint_{A_i} \Omega = \oint_{a_i} y dx.$$

The Riemann Surface $\Sigma_g$ encodes all of the data about the complex structure deformation of the CY.

Deformations of the complex structure of a CY threefold correspond to $(2,1)$ forms according to

$$\frac{\partial\Omega}{\partial S_I} = k_I \Omega + \chi_I$$

$$\oint_{A_J} \chi_I = k_I S_J + \delta_{IJ}.$$

The A-cycles of our specific class of CY manifolds can be described as fibrations of the 2-sphere the the $A_1$ fiber (in $u, v$) over *lines* in the $x$-plane.

A basis can be found where each generator is special Lagrangian,

(though they will not in general be mutually supersymmetric).

Their volume form is therefore of the form

$$(dx + e^{i\theta_x} d\bar{x}) \wedge (du + e^{i\theta_u} d\bar{u}) \wedge (du + e^{i\theta_v} d\bar{v})$$

$$\int_{A_J} \left( \Omega_{ijk} A^k_{\bar{k}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{k}} \right)$$

where $A = \sum_{I=1}^{g+1} A^{(I)} \delta S_I$; only components of the KS field which contribute to this integral affect the complex structure of the CY.

let $\alpha, \beta = u, v$ On the CY $X_g$ $A$ can be decomposed as

$$A^j_{\bar{i}} = \begin{pmatrix} A^\alpha_{\bar{x}} & A^\beta_{\bar{\alpha}} \\ A^x_{\bar{x}} & A^x_{\bar{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} C^\alpha_{\bar{x}} & A^\beta_{\bar{\alpha}} \\ \mu^x_{\bar{x}} & B^x_{\bar{\alpha}} \end{pmatrix}$$

The integral of merit is

$$\int_{A_J} \left( \Omega_{0\alpha\beta x} \mu^x_{\bar{x}} dz^\alpha \wedge dz^\beta \wedge d\bar{x} + \Omega_{0\alpha x \beta} A^\beta_{\bar{\alpha}} dz^\alpha \wedge dx \wedge d\bar{z}^{\bar{\alpha}} \right).$$

Therefore, *$B$ and $C$ do not change the complex structure of the Calabi-Yau manifold $X_g$.* This makes us feel much better about the fact that they do not appear in the effective theory we are about to derive.

## back to regularly-scheduled programming

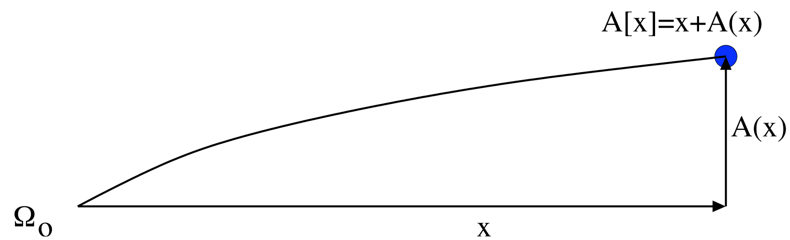
The input data, through the KS equation, determines a unique new holomorphic threeform

$$\Omega[x] = \Omega_0 + A' + (A \wedge A)' + (A \wedge A \wedge A)'$$

where $A$ here is

$$A[x] \equiv \mathbf{x} + A(x)$$

($A[x]$ satisfies the KS equation).




The fluctuating modes of $A$ are the "massive" modes, namely those in the complement of the kernel of $\bar{\partial}$.

This condition can be expressed as

$$0 = \int_{X_g} A' \wedge \bar{z}'$$

for all $\bar{z} \in H_{\partial}^{(1,0)}(T^{\star}X)$.

This system has a big gauge invariance, from reparametrizations of the CY:

$$A \mapsto A + \bar{\partial}\epsilon - [\epsilon, (\mathbf{x} + A)]$$

where $\epsilon$ is a holomorphic vector field.

This gauge symmetry can be partly fixed by imposing the Tian condition

$$0 = \partial A'.$$

This says that the deformed three-form remains $\partial$-closed.

With this condition, one can write an action for the massive modes whose eom is the ($\partial$ of) KS equation

$$\mathcal{S}_{KS} = \frac{1}{2g_s^2}\int_{X_g}\left(A'\frac{\bar\partial}{\partial}A' + \frac{1}{3}(\mathbf{x}+A)'\wedge((\mathbf{x}+A)\wedge(\mathbf{x}+A))'\right)$$

The nonlocality can locally be removed by solving the Tian condition

$$A' = \partial\Phi$$

for a $(1,1)$ form $\Phi$.

This introduces extra gauge symmetry which can be fixed by demanding that in terms of $\Phi$, the KS equation is

$$0 = \bar\partial\Phi + \frac{1}{2}((\mathbf{x}+W'\epsilon\cdot\partial\Phi)\wedge(\mathbf{x}+W'\epsilon\cdot\partial\Phi))'$$

$$(\mathbf{x}+W'\epsilon\cdot\partial\Phi)^{j}_{\bar i} \equiv \mathbf{x}^{j}_{\bar i} + W'\epsilon_{jmn}\partial_m\Phi_{\bar i n}$$

A convenient base point to choose for relating to the matrix model is the singular CY, where $S_i = 0$.

Use unfixed gauge symmetry to eliminate $B, C$.

Imposing the components of the KS equation which arise by varying $B, C$, we learn that we can write

$$\int_{\mathcal{C}_x} \Phi = \varphi(x)$$

where $\mathcal{C}_x$ denotes the $\mathbb{P}^1$ in the fiber over $x$.

Using this, a piece of the kinetic term is

$$\int_{X_g} \partial\Phi \wedge \bar{\partial}\Phi = \int d^2x \partial\varphi\bar{\partial}\varphi$$

There is one other piece of $A$ that survives, which is $A^x_{\bar{x}} = \mu$.

This contributes through the cubic term as

$$\int_{X_g} (\partial_x \Phi_{\bar{\alpha}\gamma} \partial_x \Phi_{\bar{\beta}\delta} (\Omega_0)_{\gamma x \delta}\; dx \wedge d\bar{x} \wedge dz^\alpha \wedge d\bar{z}^{\bar{\alpha}} \wedge dz^\beta \wedge d\bar{z}^{\bar{\beta}} A^x_{\bar{x}} (\Omega_0)_{x\alpha\beta}$$

$$= \int d^2 x \mu (\partial \varphi)^2$$

Another term which persists is even a boundary term in the $x$-plane

$$\oint dx W(x) \partial \varphi(x)$$

Knowing that the deformations of the CY $X_g$ can be encoded in a degree $g$ polynomial $f(x)$ as

$$0 = u^2 + v^2 + y^2 = W'(x)^2 + f(x)$$

so that the deformed 3-form will be

$$\Omega = \frac{dx \wedge du \wedge dv}{y(x, u, v)}$$

and identifying $y = \partial\varphi$ ne can get

$$\mathcal{S} = \int \partial\varphi\bar{\partial}\varphi + \mu(\partial\varphi)^2$$

by using the integration formula

$$\int_{u,v} \Omega = y dx$$

with $y = \sqrt{W'(x)^2 + f(x)}$.

## GKPW

I hope to have motivated the statement that the KS theory reduces to

$$\mathcal{S}(\varphi, \mu) = \int d^2x \left( \partial\varphi\bar{\partial}\varphi + \frac{1}{2}\mu(\partial\varphi)^2 \right) + \oint_\infty dx W(x)\partial\varphi(x)$$

and that $\varphi$ is the remnant of the second-quantized string field.

$\varphi$ is a chiral boson because the KS theory is a theory of deformations of complex structure, but not anti-complex structure.

Because of the massiveness condition, the fluctuating modes of $\mu$ are those with non-negative powers of $x$. these generate the transformations

$$x \mapsto x + \epsilon_0 + \epsilon_1 x + \ldots$$

so the equation of motion of $\mu$ implies that

$$0 = (\partial\varphi(x)^2)_<$$

the solution for $\mu$ is the 'beltrami equation'

$$\mu = -\frac{\bar{\partial}\varphi}{\partial\varphi}$$

the boundary, where the matrix model lives, is at infinity.

$\varphi$ determines the geometry of the space it lives on. a theorem about the existence of holomorphic 1-forms and the locations of their zeros implies that the solution for the curve is the one determined by $t, S$.

at leading order in $1/M$, the partition function of the matrix integral

$$e^{\mathcal{F}(t,S)} = \exp \mathcal{S}[\varphi_{cl}(x|t,S)]$$

where $\mathcal{S}$ is the action evaluated on the classical solution for $\varphi$ satisfying the Virasoro constraints, and the boundary conditions

$$S_i = \oint_\infty dx\ \mathbf{g}_{a_i}(x)\partial\varphi(x) \qquad \lim_{x\to\infty}\varphi(x) = W(x)$$

recall:

$$\oint_{\infty} \mathbf{g}_{a_i}(x) f(x) = \oint_{a_i} f(x).$$

the fact that the 'source' $W$ is at $x \to \infty$ is the sense in which the matrix theory lives there.

normalizible and non-normalizible.

the virasoro conditions can be rewritten using the operator description of the boson

$$e^{\mathcal{F}(t,S)} = \langle t_1, \ldots, t_{g+2}, 0 | \Sigma_g, t, S \rangle$$

($\langle t_n | \alpha_n = \langle t_n | t_n, n > 0$ is a coherent state of the creation modes of $\varphi$) as

$$L_n | \Sigma_{\tilde{t}} \rangle = \sum_m m \tilde{t}_m \alpha_{n+m} | \Sigma_{\tilde{t}} \rangle, n \geq -1$$

which can be read as the statement that $W'(x)$ spontanteously breaks reparametrization invariance.

## Quiver theories

The $A_{p-1}$ case arises from the CY

$$u^2 + v^2 + \prod_{r=1}^{p} (y - W_r'(x)) = 0$$

Considering $x$ as a base parameter, the fiber over each point is a (resolved/deformed) $Z_p$ orbifold of $C^2$.

A constraint on the geometry is

$$\sum_{r=1}^{p} W_r(x) = 0$$

A simple example ($\zeta_p^p = 1$) is $W_r(x) = \zeta_p^r W(x)$ in which case we get

$$u^2 + v^2 + y^p + W'(x)^p = 0.$$

The resulting matrix integral is

$$\int d\Phi_r dQ_{r,r+1} d\tilde{Q}_{r+1,r} \exp\left(\sum_r \tilde{Q}_{r+1,r}\Phi Q_{r,r+1} + \sum_r W_i(\Phi_i)\right)$$

The $u(2)$ current algebra is replaced by a $u(p)$ current algebra.

The constraint algebra $Vir$ is extended to $\mathcal{W}_p$.

We need $p-1$ bosons to determine the complex structure.

These descend from the reduction of $A'$ on the $(1,1)$ cohomology of the fiber.

The $\mathcal{W}_p$ constraints should again arise from the KS equations.

For only slightly more general CYs, the cohomology cannot be summarized by that of a curve

(e.g. Laufer geometry of $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ curves).

## Conclusions

The reason special geometry appears in the matrix integral is because it is a microscopic description of the KS theory.

The strings whose worldsheets are 't Hooft dual to the matrix model double-line diagrams are closed topological B-model strings on this CY.

Counting of degrees of freedom

The sizes of features of the geometry is determined by the number of bits of which they are made.

At finite $M$ the values which these sizes take are quantized in units of $g_s$, and they have a maximum value.

$\hbar \to 0$ says that the cycles all shrink and $\Sigma_g \to$ disc.

in general, in the matrix model, it is clear that the degrees of freedom are localized to the cuts.

the space on which the gravity lives, $\Sigma_g$, is (a cover of) the field space of the matrix model (eigenvalue plane).
the virasoro constraints expressing general covariance of the gravity theory emerge in the boundary theory simply because the integral doesn't depend on its integration variable.

## Non-classical effects

$1/N$ corrections to the matrix model free energy generate the loop expansion of the KS theory. e.g. Dijkgraaf, Sinkovic, Temurhan

Periodicity of the $su(2)$ boson $\varphi$ arises from the KS theory as a large gauge transformation of the KS field

$$\Phi \simeq \Phi + \alpha$$

where $\alpha$ is a generator of the integer cohomology of the $A_1$ fiber.

$\longrightarrow$ eigenvalue quantization.

Single-eigenvalue tunneling effects in the matrix integral go like $e^{-1/g_s}$ are nonperturbative effects in the topological B-model.