A gauge theory generalization of the fermion doubling theorem

John McGreevy, UCSD

work in progress with:

S. M. Kravec, UCSD

with help from:

T. Senthil and Brian Swingle

This talk is about (examples of) obstructions to symmetry-preserving regulators of QFT, in 3+1 dimensions.

Goal: understand such obstructions by thinking about certain states of matter in one higher dimension with an energy gap (*i.e.* $E_1 - E_{gs} > 0$ in thermodynamic limit).

More precisely: using their low-energy effective field theories (topological field theories (TFTs) in $D = 4 + 1$).

These will be difficult states to access in the lab!

---

[1] they live in $D = 4 + 1$

[2] they are 3+1 dimensional at least

[3] with some important disclaimers

This talk is about (examples of) obstructions to symmetry-preserving regulators of QFT, in 3+1 dimensions.

Goal: understand such obstructions by thinking about certain states of matter in one higher dimension with an energy gap
(*i.e.* $E_1 - E_{gs} > 0$ in thermodynamic limit).
More precisely: using their low-energy effective field theories
(topological field theories (TFTs) in $D = 4 + 1$).
These will be difficult states to access in the lab!

Strategy: use theories that obviously don't exist[1]
to prove that certain slightly more reasonable-looking theories[2]
don't exist even in principle[3].

One possible outcome: Constraints on SUSY regulators.

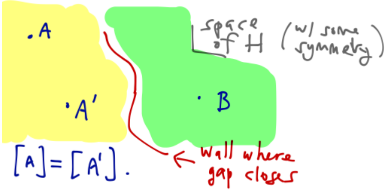
---

[1] they live in $D = 4 + 1$
[2] they are 3+1 dimensional at least
[3] with some important disclaimers

Realizations of symmetries in QFT and cond-mat

Basic Q: What are possible gapped phases of matter?

Def: Two gapped states are equivalent if they are adiabatically connected (varying the parameters in the **H** whose ground state they are to get from one to the other, without closing the energy gap).



One important distinguishing feature: how are the symmetries realized?

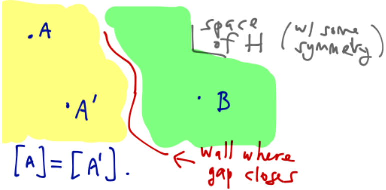
Landau distinction: characterize by *broken* symmetries

e.g. ferromagnet vs paramagnet, insulator vs SC.

Realizations of symmetries in QFT and cond-mat

Basic Q: What are possible gapped phases of matter?

Def: Two gapped states are equivalent if they are adiabatically connected (varying the parameters in the **H** whose ground state they are to get from one to the other, without closing the energy gap).



One important distinguishing feature: how are the symmetries realized?

Landau distinction: characterize by *broken* symmetries

e.g. ferromagnet vs paramagnet, insulator vs SC. ✓

Mod out by Landau: "What are possible (gapped) phases that don't break symmetries?" How do we distinguish them?

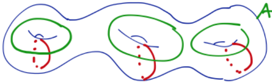
One (fancy) answer [Wen]: topological order.

Topological order

3 intimately-connected features:

(e.g. FQH)

1. *Fractionalization* of symmetries (i.e. emergent quasiparticle excitations carry quantum numbers which are fractions of those of the constituents)
   - *e.g.* quasiparticles are anyons of charge e/k
2. $\#$ of groundstates depends on the topology of space.
   connection to prev: pair-create qp-antiqp pair, move them around a spatial cycle and re-annihilate. This process maps one gs to another.
   - $\mathcal{F}_x \mathcal{F}_y = \mathcal{F}_y \mathcal{F}_x e^{2\pi i/k}$



$\longrightarrow k^g$ groundstates.

3. Requires *long-range entanglement*
   [Kitaev-Preskill, Levin-Wen]:
   $S(A) \equiv -\mathrm{tr}\, \boldsymbol{\rho}_A \log \boldsymbol{\rho}_A$, the EE of the subregion A in the state in question.
   $S(A) = \Lambda \ell(\partial A) - \gamma$ $\quad$ ($\Lambda =$UV cutoff)
   $\gamma \equiv$ "topological entanglement entropy"
   $\propto \log(\#\text{torus groundstates}) \geq 0$.
   (Deficit relative to area law.)

Topological order

3 intimately-connected features:

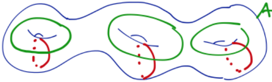
1. *Fractionalization* of symmetries (i.e. emergent quasiparticle excitations carry quantum numbers which are fractions of those of the constituents)
2. # of groundstates depends on the topology of space.

   connection to prev: pair-create qp-antiqp pair, move them around a spatial cycle and re-annihilate. This process maps one gs to another.
3. Requires *long-range entanglement*

   [Kitaev-Preskill, Levin-Wen]:
   $S(A) \equiv -\mathrm{tr}\, \boldsymbol{\rho}_A \log \boldsymbol{\rho}_A$, the EE of the subregion A in the state in question.
   $S(A) = \Lambda \ell(\partial A) - \gamma$ ($\Lambda =$UV cutoff)
   $\gamma \equiv$ "topological entanglement entropy"
   $\propto \log\left(\#\text{torus groundstates}\right) \geq 0$.
   (Deficit relative to area law.)

(e.g. FQH)

- *e.g.* quasiparticles are anyons of charge e/k
- $\mathcal{F}_x \mathcal{F}_y = \mathcal{F}_y \mathcal{F}_x e^{2\pi i/k}$



$\longrightarrow k^g$ groundstates.

- *c.f.*: For a state w/o LRE
  $S(A) = \oint_{\partial A} s d\ell$
  (local at bdy)
  $= \oint \left(\Lambda + bK + cK^2 + \ldots\right)$
  $= \Lambda \ell(\partial A) + \tilde{b} + \frac{\tilde{c}}{\ell(\partial A)}$
  Pure state:
  $S(A) = S(\bar{A}) \implies b = 0$.

  [Grover-Turner-Vishwanath]

Mod out by Wen, too

"What are possible (gapped) phases that don't break symmetries and don't have topological order?"

Mod out by Wen, too

"What are possible (gapped) phases that don't break symmetries and don't have topological order?"

[nice review: Turner-Vishwanath, 1301.0330]

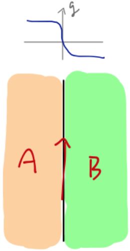
In the absence of topological order ('SRE', hence simpler), another answer: Put the model on the space with boundary.

A gapped state of matter in $d + 1$ dimensions
with short-range entanglement
can be (at least partially) characterized (within some symmetry class of hamiltonians) by (properties of) its edge states
(*i.e.* what happens at an interface with the vacuum,
or with another SRE state).

[Note: I am using the West-Coast definition of SRE
(vs deformable to product state by finite # of local unitaries)]

SRE states are characterized by their edge states

Idea: just like varying the Hamiltonian in time to another phase requires closing the gap $\mathbf{H} = \mathbf{H}_1 + g(t)\mathbf{H}_2$, so does varying the Hamiltonian in space $\mathbf{H} = \mathbf{H}_1 + g(x)\mathbf{H}_2$.



- Important role of SRE assumption: Here we are assuming that the bulk state has short-ranged correlations, so that changes we might make at the surface cannot have effects deep in the bulk.

SPT states

Def: An *SPT state* (symmetry-protected topological state), protected by a symmetry group G is:

a SRE state, which is not adiabatically connected to a product state by local hamiltonians preserving G.

e.g.: free fermion topological insulators in 3+1d, protected by $U(1)$ and $\mathcal{T}$, have an odd number of Dirac cones on the surface.

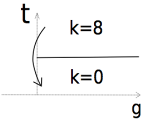
SPT states

Def: An *SPT state* (symmetry-protected topological state), protected by a symmetry group G is:
a SRE state, which is not adiabatically connected to a product state by local hamiltonians preserving G.

e.g.: free fermion topological insulators in 3+1d, protected by $U(1)$ and $\mathcal{T}$, have an odd number of Dirac cones on the surface.

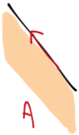
- Free fermion TIs classified [Kitaev: homotopy theory; Schneider et al: edge]

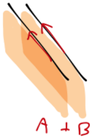
Interactions can affect the connectivity of the phase diagram:



- (e.g. states which are adiabatically connected only via interacting hamiltonians) [Fidkowski-Kitaev, 0904.2197]
- Bosonic SPT states require interactions, else superfluid.

Group structure of SPT states





Simplifying feature:

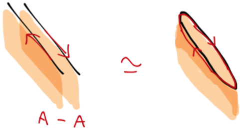
SPT states (for given G) form a group:

Group structure of SPT states

Simplifying feature:

SPT states (for given G) form a group:

-A : is the mirror image.

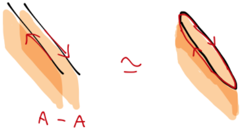


Group structure of SPT states

Simplifying feature:

SPT states (for given G) form a group:

-A : is the mirror image.



Note: with topological order, even if we can gap out the edge states, there is still stuff going on (e.g. fractional charges) in the bulk. Not a group.

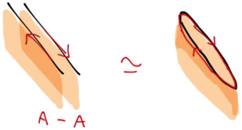
- [Chen-Gu-Wen, 1106.4772] conjecture: the group is $H^{D+1}(BG, U(1))$.
- $\exists$ 'beyond-cohomology' states in $D = 3 + 1$ [Senthil-Vishwanath]
- [Kitaev, unpublished] knows the correct construction of the group.

Group structure of SPT states

Simplifying feature:

SPT states (for given G) form a group:

-A : is the mirror image.



Note: with topological order, even if we can gap out the edge states, there is still stuff going on (e.g. fractional charges) in the bulk. Not a group.

- [Chen-Gu-Wen, 1106.4772] conjecture: the group is $H^{D+1}(BG, U(1))$.
- $\exists$ 'beyond-cohomology' states in $D = 3 + 1$ [Senthil-Vishwanath]
- [Kitaev, unpublished] knows the correct construction of the group.

This talk: an implication of this group structure

– which we can pursue by examples – is...

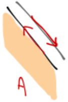
Surface-only models

Counterfactual:

Suppose the edge theory of an SPT state were realized *otherwise*

– intrinsically in D dimensions, with a local hamiltonian.

Then we could paint that the conjugate local theory on the surface without changing anything about the bulk state.



Surface-only models

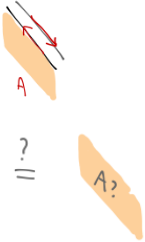
Counterfactual:

Suppose the edge theory of an SPT state were realized *otherwise*

– intrinsically in D dimensions, with a local hamiltonian.

Then we could paint that the conjugate local theory on the surface without changing anything about the bulk state.

And then small deformations of the surface Hamiltonian, localized on the surface, consistent with symmetries, can pair up the edge states.

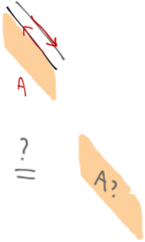


Surface-only models

Counterfactual:

Suppose the edge theory of an SPT state were realized *otherwise* – intrinsically in D dimensions, with a local hamiltonian.

Then we could paint that the conjugate local theory on the surface without changing anything about the bulk state. And then small deformations of the surface Hamiltonian, localized on the surface, consistent with symmetries, can pair up the edge states.



But this contradicts the claim that we could characterize the $D+1$-dimensional SPT state by its edge theory.

Conclusion: Edge theories of SPT_G states cannot be regularized intrinsically in D dims, preserving G – "surface-only models".

[Wang-Senthil, 1302.6234 – general idea, concrete surprising examples of 2+1 surface-only states
Wen, 1303.1803 – attempt to characterize the underlying mathematical structure, classify *all* such obstructions
Wen, 1305.1045 – use this perspective to regulate the Standard Model on a 5d slab

Metlitski-Kane-Fisher, 1302.6535; Burnell-Chen-Fidkowski-Vishwanath, 1302.7072]

Summary of Nielsen-Ninomiya result on fermion doubling

The most famous example of such an obstruction was articulated by Nielsen and Ninomiya:

It is not possible to regulate free fermions while preserving the chiral symmetry.

(In odd spacetime dimensions, 'chiral symmetry' means 'parity'.)

Summary of Nielsen-Ninomiya result on fermion doubling

The most famous example of such an obstruction was articulated by Nielsen and Ninomiya:

It is not possible to regulate free fermions while preserving the chiral symmetry.

(In odd spacetime dimensions, 'chiral symmetry' means 'parity'.)

More precise (lattice) statement: A fermion action

$$S = \int_{\mathrm{BZ}} \bar{d}^{2k} p \bar{\Psi}_p D(p) \Psi_p$$

cannot satisfy all three of these:

1. $D(p)$ is smooth and periodic in the BZ (*i.e.* the FT of a local kinetic term on the lattice)
2. A single Dirac cone, *i.e.* $D(p) \sim \gamma_\mu p^\mu$ for $|p_\mu| \ll 1$, and D invertible everywhere else.
3. $\{\Gamma, D(p)\} = 0$, where Γ is the chirality matrix (γ^5).

Illustration of fermion doubling

Simple illustration: attempt to regulate them on the lattice.

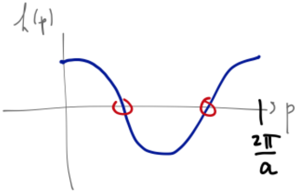
Then the momentum space is compact:

$$\text{for } n \in \mathbb{Z}, \quad e^{\mathbf{i}nap} = e^{\mathbf{i}na\left(p+\frac{2\pi}{a}\right)} \quad \Longrightarrow \{p\} \simeq T^d \quad \text{(the Brillouin zone)}.$$

The hamiltonian is of the form

$$\mathbf{H} = \int_{p \in \mathrm{BZ}} h_{ab}(p) c_a^\dagger(p) c_b(p)$$

where h is a *periodic* map.



e.g., in 1+1d: $\text{sign}\left(\frac{\partial h}{\partial p}\right) = \Gamma$

- Friedan refinement: in each irreducible representation of the internal symmetry group there are no chiral fermions.
- Consistent with ABJ anomaly, since an exact symmetry of the lattice model is a symmetry.

Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in

4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

$\pm m$ label distinct Lorentz-invariant

(P-broken) phases.

Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

$\pm m$ label distinct Lorentz-invariant (P-broken) phases.

One proof of this:

Couple to external gauge field $\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$

Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

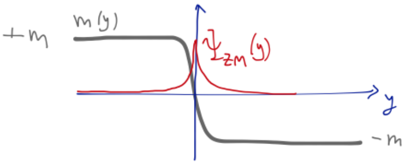
$\pm m$ label distinct Lorentz-invariant (P-broken) phases.

Domain wall between them hosts (exponentially-localized) 3+1 chiral fermions: [Jackiw-Rebbi, Callan-Harvey, Kaplan]

One proof of this:

Couple to external gauge field $\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$



Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

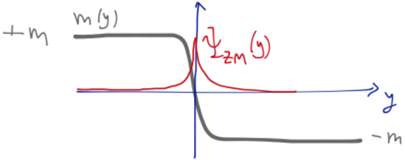
$\pm m$ label distinct Lorentz-invariant (P-broken) phases.

Domain wall between them hosts (exponentially-localized) 3+1 chiral fermions: [Jackiw-Rebbi, Callan-Harvey, Kaplan]

One proof of this:

Couple to external gauge field $\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$



More famous $D = 3 + 1$ analog:

$$S = \int d^{3+1}x \bar{\Psi} \left(\partial\!\!\!/ + m + \mathbf{i}\hat{m}\gamma^5 \right) \Psi$$

With $\mathcal{T}$-invariance, $\hat{m} = 0$.

Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

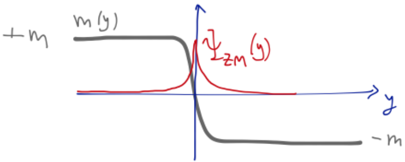
$\pm m$ label distinct Lorentz-invariant (P-broken) phases.

Domain wall between them hosts (exponentially-localized) 3+1 chiral fermions: [Jackiw-Rebbi, Callan-Harvey, Kaplan]

One proof of this:

Couple to external gauge field $\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$



More famous $D = 3+1$ analog:

$$S = \int d^{3+1}x \bar{\Psi} \left(\partial\!\!\!/ + m + \mathbf{i} \hat{m} \gamma^5 \right) \Psi$$

With $\mathcal{T}$-invariance, $\hat{m} = 0$.

$m = \pm m$ are distinct states ($\theta = 0, \pi$):

$$\log \int [D\Psi] e^{\mathbf{i} S_{3+1}[\Psi, A]} \propto \frac{m}{|m|} \int F \wedge F$$

Domain wall hosts a single Dirac cone in 2+1d.

Strategy

Study a simple (unitary) gapped or topological field theory in 4+1 dimensions without topological order, wth symmetry G.

Consider the model on the disk with some boundary conditions.

The resulting edge theory is
a "surface-only theory with respect to G"
– it cannot be regulated by a local $3+1$-dim'l model while preserving G.

What does it mean to be a surface-only state?

These theories are perfectly consistent and unitary – they *can* be realized as the edge theory of some gapped bulk. They just can't be regularized in a local way consistent with the symmetries without the bulk.

1. It (probably) means these QFTs will not be found as low-energy EFTs of solids or in cold atom lattice simulations.

What does it mean to be a surface-only state?

These theories are perfectly consistent and unitary – they *can* be realized as the edge theory of some gapped bulk. They just can't be regularized in a local way consistent with the symmetries without the bulk.

1. It (probably) means these QFTs will not be found as low-energy EFTs of solids or in cold atom lattice simulations.
2. Why 'probably'? This perspective does not rule out emergent ("accidental") symmetries, not explicitly preserved in the UV.

What does it mean to be a surface-only state?

These theories are perfectly consistent and unitary – they *can* be realized as the edge theory of some gapped bulk. They just can't be regularized in a local way consistent with the symmetries without the bulk.

1. It (probably) means these QFTs will not be found as low-energy EFTs of solids or in cold atom lattice simulations.
2. Why 'probably'? This perspective does not rule out emergent ("accidental") symmetries, not explicitly preserved in the UV.
3. It also does not rule out symmetric UV completions that include gravity, or decoupling limits of gravity/string theory.
   (UV completions of gravity have their own complications!)
   String theory strongly suggests the existence of Lorentz-invariant states of gravity with chiral fermions and lots of supersymmetry
   (the $E_8 \times E_8$ heterotic string, chiral matter on D-brane intersections, self-dual tensor fields...)
   some of which can be decoupled from gravity.

A simple example

The 4+1d analog of the K-matrix approach to 2+1d SPTs of [Lu-Vishwanath].

A simple topological field theory in 4+1 dimensions

Consider 2-forms B_{MN} in $4+1$ dimensions, with action

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma} B^I \wedge \mathrm{d}B^J$$

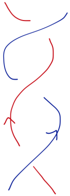
In $4\ell + 1$ dims, K is a *skew*-symmetric integer $2N_B \times 2N_B$ matrix.

Note: $B \wedge \mathrm{d}B = \frac{1}{2}\mathrm{d}(B \wedge B)$.

Independent of choice of metric on $\mathbb{R} \times \Sigma_{2p}$.

Related models studied in: [Horowitz 1989, Blau et al 1989, Witten 1998, Maldacena-Moore-Seiberg 2001, Belov-Moore 2005, Hartnoll 2006]

[Horowitz-Srednicki]: coupling to string sources $\Delta S = \int_{\Gamma_I} B^I$ computes linking $\#$ of conjugate species of worldsheets Γ^I.



A simple topological field theory in 4+1 dimensions

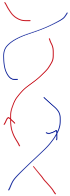
Consider 2-forms B_{MN} in $4+1$ dimensions, with action

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma} B^I \wedge \mathrm{d}B^J$$

In $4\ell + 1$ dims, K is a *skew*-symmetric integer $2N_B \times 2N_B$ matrix.

Note: $B \wedge \mathrm{d}B = \frac{1}{2}\mathrm{d}(B \wedge B)$.

Independent of choice of metric on $\mathbb{R} \times \Sigma_{2p}$.



Related models studied in: [Horowitz 1989, Blau et al 1989, Witten 1998, Maldacena-Moore-Seiberg 2001, Belov-Moore 2005, Hartnoll 2006]

[Horowitz-Srednicki]: coupling to string sources $\Delta S = \int_{\Gamma_I} B^I$ computes linking # of conjugate species of worldsheets Γ^I.

Simplest case is realized in IIB strings on $AdS_5 \times S^5$,

$B \equiv B_{NSNS}$, $C \equiv C_{RR}$:

$$S_{IIB} \ni \frac{1}{2\pi} \int_{AdS_5 \times S^5} F^{(5)}_{RR} \wedge B \wedge \mathrm{d}C = \frac{N}{2\pi} \int_{\mathbb{R}\times\Sigma} B \wedge \mathrm{d}C$$

(allows for baryon vertex of N F-strings [Gross-Ooguri, Witten 98])

‘Trivial but difficult’

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma} B^I \wedge \mathrm{d}B^J$$

gauge redundancies: $B^I \simeq B^I + \mathrm{d}\lambda^I$, λ^I are 1-forms

large gauge equivalences: $B^I \simeq B^I + n^\alpha \omega_\alpha$, $[\omega^\alpha] \in H^2(\Sigma, \mathbb{Z})$, $n^\alpha \in \mathbb{Z}^{b^2}$

‘Trivial but difficult’

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma} B^I \wedge \mathrm{d}B^J$$

gauge redundancies: $B^I \simeq B^I + \mathrm{d}\lambda^I$, λ^I are 1-forms

large gauge equivalences: $B^I \simeq B^I + n^\alpha \omega_\alpha$, $[\omega^\alpha] \in H^2(\Sigma, \mathbb{Z})$, $n^\alpha \in \mathbb{Z}^{b^2}$

- heavy machinery: [Freed-Moore-Segal 2006] I believe this machinery is not necessary if we consider only Σ_4 without torsion homology.

- This sort of model has been used

[Witten, 90s; Shatashvili, unpublished; Maldacena-Moore-Seiberg 01; Belov-Moore 03-06]

to ‘holographically’ *define* the partition function of the edge.

Mainly in $D = 4\ell + 3$: 1+1d chiral CFTs, conformal blocks of 5+1d (2,0) theory.

- The simplest model is equivalent to a $\mathbb{Z}_k$ *2-form* gauge theory.

(More below.)

Bulk physics

When does the 4+1d CS theory have topological order?

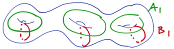
Consider p forms in $2p+1$ dimensions:
$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma_{2p}} B^I \wedge \mathrm{d}B^J$
For now, suppose that $\partial\Sigma = \emptyset$.
Gauge-inequivalent operators labelled by $[\omega_\alpha] \in H^p(\Sigma, \mathbb{Z})$:

$$\mathcal{F}_{\omega_\alpha}(m) \equiv e^{2\pi \mathbf{i} m_I^\alpha \int_{\omega_\alpha} B^I}$$

large gauge eq $\implies$ $m_I^\alpha \in \mathbb{Z}$. ETCRs $\implies$ Heisenberg algebra:

$$\mathcal{F}_{\omega_\alpha}(m)\mathcal{F}_{\omega_\beta}(m') = \mathcal{F}_{\omega_\beta}(m')\mathcal{F}_{\omega_\alpha}(m) e^{2\pi \mathbf{i} m_I^\alpha m_J'^\beta (K^{-1})^{IJ} \mathcal{I}_{\alpha\beta}}.$$

$\int_\Sigma \omega_\alpha \wedge \omega_\beta = \mathcal{I}_{\alpha\beta}$, intersection form (symmetric for Σ_4, AS for Σ_2).



In $2+1$: $\mathcal{I} \approx \mathbb{1}_{g\times g} \otimes \mathbf{i}\sigma^2$
the irrep of this algebra has dimension $|\det(K)|^g$.
In $4+1$: the irrep of this algebra has dimension $|\mathrm{Pfaff}(K \otimes \mathcal{I})|$.
Fact about 4-manifolds: $\mathcal{I}$ is unimodular $\implies$

$$\text{SRE} \quad \Leftrightarrow \quad |\mathrm{Pfaff}(K)| = 1 \quad .$$

Zeromode quantum mechanics

A more direct construction of the groundstates.

Expand in zeromodes $b^{I\alpha} \simeq b^{I\alpha} + 2\pi$:

$$B^I = \sum_{\alpha=1}^{b^2(\Sigma_4,\mathbb{Z})} \omega_\alpha b^{I\alpha}(t), \quad \text{span}\{[\omega_\alpha]\} = H^2(\Sigma_4, \mathbb{Z}),$$

$$S = \frac{K_{IJ}}{2\pi} \int dt \int_{\Sigma_4} \omega_\alpha \wedge \omega_\beta b^{I\alpha} \dot{b}^{J\beta} = \frac{K_{IJ}}{2\pi} \int dt \mathcal{I}_{\alpha\beta} b^{I\alpha} \dot{b}^{J\beta}$$

which describes a particle in $b^2(\Sigma)$ dimensions with a magnetic field in each pair of dimensions of strength k, in the LLL.

As in 2+1d, Maxwell-like terms

$$\Delta S = \int_{\Sigma\times\mathbb{R}} \frac{1}{m} \left(\mathrm{d}B \wedge \star\mathrm{d}B + \mathrm{d}C \wedge \star\mathrm{d}C\right) \propto \int dt \frac{1}{m} \dot{b}^2$$

$m < \infty$ brings down higher landau levels.

This is a model of bosons

Low-energy evidence: I did not have to choose a spin structure to put this on an arbitrary 4-manifold.

(unlike $U(1)_{k=1}$ CS theory in $d = 2 + 1$.)

Comment about spin structure:

On a manifold that admits spinors, the intersection form is even ($\mathcal{I}(v, v) \in 2\mathbb{Z}$)

$\implies$ to describe an EFT for a *fermionic* SPT state, we should consider $k \in \mathbb{Z}/2$.

[Belov-Moore] 'spin Chern-Simons theory'.

High-energy (i.e. cond-mat) evidence:

Conjecture for a lattice model of bosons which produces this EFT:

Which model of bosons?

[Thanks to Brian Swingle!]

- Put rotors $e^{\mathbf{i}b_p}$ on the *plaquettes* p of a 4d spatial lattice.

$e^{\mathbf{i}b_p}|n_p\rangle = |n_p + 1\rangle$.

$n_p \equiv \#$ of 'sheets' covering the plaquette.

- Put charge-k bosons $\Phi_\ell = \Phi^\dagger_{-\ell}$ on the *links* ℓ.

$[\Phi_\ell, \Phi^\dagger_\ell] = 1$

[Wegner, ..., Motrunich-Senthil, Levin-Wen, Walker-Wang, Burnell et al]

$\Phi^\dagger_\ell$ creates a string segment.

$\Phi^\dagger_\ell \Phi_\ell \equiv \#$ of strings covering the link.

$$\mathbf{H} = -\underbrace{\sum_{\text{links},\ell\in\Delta_1} \Big(\sum_{p\in s(\ell)} n_p - k\Phi^\dagger_\ell\Phi_\ell\Big)^2}_{\mathbf{H}_1,\ \text{gauss law. happy when sheets close, or end on strings}} - \underbrace{\sum_{\text{volumes},\ v\in\Delta_3} \prod_{p\in\partial v} e^{\mathbf{i}b_p} + h.c.}_{\mathbf{H}_3\sim B^2,\ \text{makes sheets hop.}}$$

$$- \underbrace{\Gamma \sum_{p\in\Delta_2} n_p^2}_{\mathbf{H}_2\sim E^2.\ \text{discourages sheets.}} \underbrace{- t \sum_{p\in\Delta_2} e^{\mathbf{i}kb_p} \prod_{\ell\in\partial p} \Phi^\dagger_\ell + h.c.}_{\mathbf{H}_{\text{strings}},\ \text{hopping term for matter strings}} + V\left(|\Phi|^2\right)$$

When $\Gamma = 0$, these terms all commute.

Soup of oriented closed 2d sheets, groups of k can end on strings.

Which model of bosons, cont'd

Condense $\Phi_\ell = ve^{\mathbf{i}\varphi_\ell}$: $\mathbf{H}_{\text{strings}} = -\sum_p tv^4 \cos\left(kb_p - \sum_{\ell\in\partial p}\varphi_\ell\right)$

$\implies \left(e^{\mathbf{i}b_p}\right)^k = \mathbb{1}$, $|n_p\rangle \simeq |n_p + k\rangle$.

Leaves behind k species of (unoriented) sheets.

Groundstates: equal-superposition sheet soup. k^{b_2} sectors for $\mathcal{I} = \mathbb{1}$.

Continuum limit.

$U(1) \overset{\text{Higgs}}{\rightarrow} \mathbb{Z}_k$ 2-form gauge theory:

$$L = \frac{tv^4}{2}\left(\mathrm{d}\varphi_1 + kB_2\right) \wedge \star\left(\mathrm{d}\varphi_1 + kB_2\right) + \frac{1}{g^2}\mathrm{d}B_2 \wedge \star \mathrm{d}B_2$$

$$\simeq \frac{k}{2\pi} B \wedge \mathrm{d}C + \frac{1}{8\pi tv^4} dC \wedge \star \mathrm{d}C + \frac{1}{g^2}\mathrm{d}B \wedge \star \mathrm{d}B$$

with $\mathrm{d}C \simeq 2\pi t \star (\mathrm{d}\varphi + kB)$.

[Maldacena-Moore-Seiberg hep-th/0108152, Hansson-Oganesyan-Sondhi cond-mat/0404327]

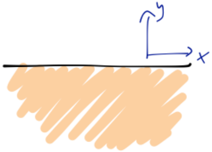
Edge states

Review of edge states of 2+1 CS theory.

Consider abelian CS theory on the LHP.

[Witten, Elitzur et al, Wen, ...

Belov-Moore]



$$S = \frac{k}{4\pi} \int_{\mathbb{R}\times \mathrm{LHP}} A \wedge \mathrm{d}A$$

EoM for A_0: $0 = F$

$\implies A = \mathbf{i} g^{-1} \mathrm{d}g = \mathrm{d}\phi \,,\, \phi \simeq \phi + 2\pi.$

Only gauge transfs which approach $\mathbb{1}$ at the bdy preserve S_{CS}

$\implies \phi$ is dynamical.

Boundary condition: $0 = A - v(\star_2 A)$ i.e.

$A_t = vA_x$. v is UV data.

$$S_{CS}[A = \mathrm{d}\phi] = \frac{k}{2\pi} \int dtdx \left(\partial_t \phi \partial_x \phi + v \left(\partial_x \phi\right)^2 \right).$$

Conclusion: ϕ is a chiral boson.

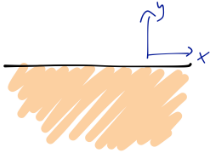
$kv > 0$ required for stability.

Review of edge states of 2+1 CS theory.

Consider abelian CS theory on the LHP.

[Witten, Elitzur et al, Wen, ... Belov-Moore]

$$S = \frac{k}{4\pi} \int_{\mathbb{R}\times \mathrm{LHP}} A \wedge \mathrm{d}A$$



EoM for A_0: $0 = F$

$\implies A = \mathbf{i}g^{-1}\mathrm{d}g = \mathrm{d}\phi$, $\phi \simeq \phi + 2\pi$.

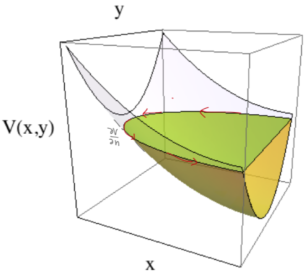
Only gauge transfs which approach $\mathbb{1}$ at the bdy preserve S_{CS}

$\implies \phi$ is dynamical.

Boundary condition: $0 = A - v(\star_2 A)$ i.e.

$A_t = vA_x$. v is UV data.

microscopic picture:



$$S_{CS}[A = \mathrm{d}\phi] = \frac{k}{2\pi} \int dtdx \left(\partial_t \phi \partial_x \phi + v \left(\partial_x \phi\right)^2 \right)$$

Conclusion: ϕ is a chiral boson.

$kv > 0$ required for stability.

Note: The Hamiltonian depends on the boundary conditions; the $\mathcal{H}$ does not.

Edge states of 4+1d CS theory.

Focus on the simplest case where $K = k\mathbf{i}\boldsymbol{\sigma}^2$, $S = \frac{k}{2\pi}\int B \wedge dC$.

$$S[B,C] = \frac{k}{2\pi}\int_{\mathbb{R}\times\Sigma_4} B \wedge \mathrm{d}C + \frac{1}{4g^2}\int_{\mathbb{R}\times\partial\Sigma_4} (C \wedge \star_4 C + B \wedge \star_4 C)$$

bc is: $\left(\frac{k}{2\pi}B - \frac{1}{2g^2}\star_4 C\right)|_{\partial\Sigma_4} = 0.$

$$\int DBe^{\mathbf{i}S} = \delta[\mathrm{d}C] \quad \Longrightarrow \quad C = \mathrm{d}a$$

Edge states of 4+1d CS theory.

Focus on the simplest case where $K = k\mathbf{i}\boldsymbol{\sigma}^2$, $S = \frac{k}{2\pi}\int B \wedge dC$.

$$S[B, C] = \frac{k}{2\pi}\int_{\mathbb{R}\times\Sigma_4} B \wedge \mathrm{d}C + \frac{1}{4g^2}\int_{\mathbb{R}\times\partial\Sigma_4} (C \wedge \star_4 C + B \wedge \star_4 C)$$

bc is: $\left(\frac{k}{2\pi}B - \frac{1}{2g^2} \star_4 C\right)|_{\partial\Sigma_4} = 0.$

$$\int DBe^{\mathbf{i}S} = \delta[\mathrm{d}C] \quad \Longrightarrow \quad C = \mathrm{d}a$$

$$S[C = \mathrm{d}a] = \boxed{\frac{1}{4g^2}\int_{\mathbb{R}\times\partial\Sigma_4} \mathrm{d}a \wedge \star_4 \mathrm{d}a}$$

Edge states of 4+1d CS theory.

Focus on the simplest case where $K = k\mathbf{i}\boldsymbol{\sigma}^2$, $S = \frac{k}{2\pi}\int B \wedge dC$.

$$S[B,C] = \frac{k}{2\pi}\int_{\mathbb{R}\times\Sigma_4} B \wedge \mathrm{d}C + \frac{1}{4g^2}\int_{\mathbb{R}\times\partial\Sigma_4} (C \wedge \star_4 C + B \wedge \star_4 C)$$

bc is: $\left(\frac{k}{2\pi}B - \frac{1}{2g^2}\star_4 C\right)|_{\partial\Sigma_4} = 0.$

$$\int DBe^{\mathbf{i}S} = \delta[\mathrm{d}C] \quad \Longrightarrow \quad C = \mathrm{d}a$$

$$S[C = \mathrm{d}a] = \boxed{\frac{1}{4g^2}\int_{\mathbb{R}\times\partial\Sigma_4} \mathrm{d}a \wedge \star_4 \mathrm{d}a}$$

This is ordinary Maxwell theory! We know how to regularize this! *e.g.*

$$H = -\sum_{\text{vertices},v\in\Delta_0}\left(\sum_{\ell\in s(v)} n_\ell - q_v\right)^2 - \sum_{p\in\Delta_2}\prod_{\ell\in\partial(p)} e^{\mathbf{i}b_\ell} + h.c. - \sum_{\ell\in\Delta_1}\Gamma n_\ell^2.$$

$\Delta_p \equiv \{p-\text{simplices}\}$. $s(v) \equiv \{\text{edges incident on } v \text{ (oriented ingoing)}\}$ and $[b_p, n_p] = \mathbf{i}$ is a number-phase representation. $b_p \equiv b_p + 2\pi, n_p \in \mathbb{Z}$.

Symmetries

Continuous symmetries

In 2+1d CS theory, as it arises from QH states, we have a conserved current (electron number):

$$0 = \partial^\mu J_\mu \quad \Longrightarrow \quad J_\mu \equiv \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu A_\rho$$

J is conserved iff A is single-valued.

In 4+1, the analog is pairs of string currents

$$J^I_{\mu\nu} \equiv \frac{1}{2\pi}\epsilon_{\mu\nu\rho\sigma\lambda}\partial_\rho B^I_{\sigma\lambda}$$

Demonstration that different K are different states

In $D = 2+1$: couple the particle currents $J^I = \star \mathrm{d}A^I$ to *external* 1-form potentials, $\mathcal{A}_I$:

$$\log \int [DA^I] e^{\mathbf{i} \int kA\mathrm{d}A + \mathbf{i} \int J_I \mathcal{A}^I} = \int_{2+1} \left(k^{-1}\right)_{IJ} \mathcal{A}^I \mathrm{d}\mathcal{A}^J$$

– quantized Hall response (integer if no topological order, $\det k = 1$).

Demonstration that different K are different states

In $D = 2+1$: couple the particle currents $J^I = \star \mathrm{d}A^I$ to *external* 1-form potentials, $\mathcal{A}_I$:

$$\log \int [DA^I] e^{\mathbf{i} \int kA\mathrm{d}A + \mathbf{i} \int J_I \mathcal{A}^I} = \int_{2+1} \left(k^{-1}\right)_{IJ} \mathcal{A}^I \mathrm{d}\mathcal{A}^J$$

– quantized Hall response (integer if no topological order, $\det k = 1$).

In $D = 4+1$: Couple the string currents $J^I = \star \mathrm{d}B^I$ to *external* 2-form potentials, $\mathcal{B}_I$:

$$\log \int [DB^I] e^{\mathbf{i} \int KB\mathrm{d}B + \mathbf{i} \int J_I \mathcal{B}^I} = \int \left(K^{-1}\right)_{IJ} \mathcal{B}^I \mathrm{d}\mathcal{B}^J \quad .$$

quantized 'string Hall' response (integer if no topological order, $\mathrm{Pfaff} K = 1$).

Symmetries

- Translation invariance is a red herring (I think!).

  The lattice model should have the same edge states.

- Stringy symmetries: $J^B_{\ell 0}|_{bdy} = E_\ell, J^C_{\ell 0}|_{bdy} = -B_\ell$.

  $E_\ell \equiv \partial_t a_\ell - \partial_\ell a_t \;\; B_\ell \equiv \epsilon_{\ell ij}(\partial_i a_j - \partial_j a_i)$ are ordinary E&M fields

$$J^C_{y0} = \epsilon_{ijk}\partial_i C_{jk} = \epsilon_{ijk}\partial_i\partial_j a_k = \vec{\nabla}\cdot\vec{B}$$

$$J^B_{y0} = \epsilon_{ijk}\partial_i B_{jk} = \epsilon_{ijk}\partial_i \epsilon_{jkl} E_\ell = \vec{\nabla}\cdot\vec{E}.$$

  This is ordinary charge, of course it has to be conserved.

- $\mathcal{C}$: $(B, C) \to -(B, C)$ is $(\vec{E}, \vec{B}) \to -(\vec{E}, \vec{B})$. Preserved in pure U(1) lattice gauge theory.

- $\mathcal{TP}$: $t \to -t, x^M \to -x^M, \mathbf{i} \to -\mathbf{i}$, $B \to -B, C \to C$ as two-forms. Acts in the usual way on the EM field as $(E, B) \to (E, -B)$.

Symmetries

- Translation invariance is a red herring (I think!).
  The lattice model should have the same edge states.
- Stringy symmetries: $J^B_{\ell 0}|_{bdy} = E_\ell, J^C_{\ell 0}|_{bdy} = -B_\ell$.
  $E_\ell \equiv \partial_t a_\ell - \partial_\ell a_t \;\; B_\ell \equiv \epsilon_{\ell ij}(\partial_i a_j - \partial_j a_i)$ are ordinary E&M fields

$$J^C_{y0} = \epsilon_{ijk}\partial_i C_{jk} = \epsilon_{ijk}\partial_i\partial_j a_k = \vec{\nabla} \cdot \vec{B}$$

$$J^B_{y0} = \epsilon_{ijk}\partial_i B_{jk} = \epsilon_{ijk}\partial_i \epsilon_{jkl} E_\ell = \vec{\nabla} \cdot \vec{E}.$$

  This is ordinary charge, of course it has to be conserved.
- $\mathcal{C}$: $(B, C) \to -(B, C)$ is $(\vec{E}, \vec{B}) \to -(\vec{E}, \vec{B})$. Preserved in pure U(1) lattice gauge theory.
- $\mathcal{TP}$: $t \to -t, x^M \to -x^M, \mathbf{i} \to -\mathbf{i}$, $B \to -B, C \to C$ as two-forms. Acts in the usual way on the EM field as $(E, B) \to (E, -B)$.
-

$$\boxed{\textbf{EM DUALITY!:}\ (B, C) \to (C, B)}$$

  is a manifest symmetry of the bulk theory.
  Unbreakable in the IR.

7d CS theory and the (2,0) superconformal theory

While we're at it, consider the following 6+1d TFT:

$$S_7[C^{(3)}] = \frac{k}{4\pi}\int_{\mathbb{R}\times\Sigma_6} C^{(3)} \wedge \mathrm{d}C^{(3)}$$

For $k = 1$, no topological order.
gauss law: $C^{(3)} = dc^{(2)}$

$$S_7[C^{(3)} = dc^{(2)}] = \frac{k}{4\pi}\int_{\mathbb{R}\times\partial\Sigma_6} \epsilon\partial c^{(2)} \cdot \left(\partial_t c^{(2)} + v\epsilon\partial c^{(2)}\right).$$

(boundary condition: $C_{0ij} = v(\star_6 C)_{ij}$.)

Conclusion: c is a self-dual 2-form potential in 5+1d.
'Topological sector' of the A_0 *(2,0) superconformal theory in 6d*
– the worldvolume theory of M5-branes.

The conjecture that it can be consistently decoupled from gravity underlies much recent progress in the field formerly known as strings [Witten, Gaiotto...].
e.g. it makes various deep 4d QFT dualities manifest.

Concluding remarks

Are all obstructions would-be-gauge-anomalies?

Many surface-only obstructions are *anomalies*: gauge anomalies; gravitational anomalies; discrete gauge anomalies (e.g. Witten SU(2) anomaly)

They would be gauge anomalies if we tried to gauge the protecting symmetry.

Are all obstructions would-be-gauge-anomalies?

Many surface-only obstructions are *anomalies*: gauge anomalies; gravitational anomalies; discrete gauge anomalies (e.g. Witten SU(2) anomaly)

They would be gauge anomalies if we tried to gauge the protecting symmetry.

Obstructions more general than obstructions to gauging:

1. [Senthil, Swingle]: SPT states protected by time-reversal $\mathcal{T}$.

   What would it mean to gauge $\mathbf{i} \to -\mathbf{i}$??

Are all obstructions would-be-gauge-anomalies?

Many surface-only obstructions are *anomalies*: gauge anomalies; gravitational anomalies; discrete gauge anomalies (e.g. Witten SU(2) anomaly)

They would be gauge anomalies if we tried to gauge the protecting symmetry.

Obstructions more general than obstructions to gauging:

1. [Senthil, Swingle]: SPT states protected by time-reversal $\mathcal{T}$.

   What would it mean to gauge $\mathbf{i} \to -\mathbf{i}$??

2. We found an obstruction to regularizing Maxwell theory preserving EM duality. [Previous literature suggesting it's impossible: Deser 1012.5109, Bunster 1101.3927, Saa 1101.6064] [in other cases, it is possible to gauge EM duality: Barkeshli-Wen]

Are all obstructions would-be-gauge-anomalies?

Many surface-only obstructions are *anomalies*: gauge anomalies; gravitational anomalies; discrete gauge anomalies (e.g. Witten SU(2) anomaly)

They would be gauge anomalies if we tried to gauge the protecting symmetry.

Obstructions more general than obstructions to gauging:

1. [Senthil, Swingle]: SPT states protected by time-reversal $\mathcal{T}$.

   What would it mean to gauge $\mathbf{i} \to -\mathbf{i}$??

2. We found an obstruction to regularizing Maxwell theory preserving EM duality. [Previous literature suggesting it's impossible: Deser 1012.5109, Bunster 1101.3927, Saa 1101.6064] [in other cases, it is possible to gauge EM duality: Barkeshli-Wen]

3. We found an obstruction to regularizing a self-dual 2-form theory in $D = 5 + 1$. One might have thought by analogy with chiral CFTs (chiral boson: $\mathrm{d}\phi = \star\mathrm{d}\phi$) that a *gravitational anomaly* was relevant here. In 1+1 dimensions, $c_L - c_R$ measures an anomaly that arises upon coupling the theory to gravity.

   $D = 5 + 1 \neq 2\text{mod}8$: no gravitational anomalies [AlvarezGaume-Witten, 1983].

Are all obstructions would-be-gauge-anomalies?

Many surface-only obstructions are *anomalies*: gauge anomalies; gravitational anomalies; discrete gauge anomalies (e.g. Witten SU(2) anomaly)

They would be gauge anomalies if we tried to gauge the protecting symmetry.

Obstructions more general than obstructions to gauging:

1. [Senthil, Swingle]: SPT states protected by time-reversal $\mathcal{T}$.

   What would it mean to gauge $\mathbf{i} \to -\mathbf{i}$??
2. We found an obstruction to regularizing Maxwell theory preserving EM duality. [Previous literature suggesting it's impossible: Deser 1012.5109, Bunster 1101.3927, Saa 1101.6064] [in other cases, it is possible to gauge EM duality: Barkeshli-Wen]
3. We found an obstruction to regularizing a self-dual 2-form theory in $D = 5 + 1$. One might have thought by analogy with chiral CFTs (chiral boson: $\mathrm{d}\phi = \star \mathrm{d}\phi$) that a *gravitational anomaly* was relevant here. In 1+1 dimensions, $c_L - c_R$ measures an anomaly that arises upon coupling the theory to gravity.

   $D = 5 + 1 \neq 2\mathrm{mod}8$: no gravitational anomalies [AlvarezGaume-Witten, 1983].
4. And what about supersymmetry?

   Gauging this leads to supergravity!

The end

Thanks for listening.