TASI Lectures on Symmetries in Quantum Matter

John McGreevy
Department of Physics
University of California San Diego
mcgreevy@physics.ucsd.edu

Abstract

These notes are about aspects of symmetry in condensed matter, generalized and emergent. First I review some apparent exceptions to the Landau Paradigm for understanding phases of matter, namely topological phases. Then I describe a generalized symmetry perspective on phases of matter, generalizing the Landau Paradigm to incorporate these exceptions. The key ingredients are generalized symmetries and anomalies. I then discuss a more austere perspective on states of matter, called Entanglement Bootstrap, that begins with a single wavefunction. I use this perspective to understand generalized symmetries of the associated state of matter. Then I discuss extensions of this perspective to conformal field theory groundstates, from which we can understand the emergence of conformal invariance from a single quantum state.

June 2023. last update: 2024/01/15

1 Introduction

Here is my plan: My first goal is to talk about quantum phases of matter, and how symmetries (particularly newfangled ones) are helpful for thinking about them. An important characteristic of generalized symmetries is that they are not often microscopic symmetries of condensed matter systems, but rather are emergent symmetries. However, unlike ordinary symmetries, which when emergent have only approximate consequences, the consequences of emergent higher form symmetries are exact and can be used to make sharp distinctions between phases. In the third section, I will abruptly change perspective and develop an understanding of such generalized symmetries from the perspective of Entanglement Bootstrap; this will require some introduction. In the final section, I will discuss another class of symmetries that can emerge in condensed matter, namely conformal symmetry, from this same perspective.

If you have been to a condensed-matter talk in the past few decades, you have seen the beating that Landau has been taking. The speaker begins by saying that Landau told us that states of matter are classified by the symmetries they break. After showing a picture of a donut, the speaker explains that in *this* talk, in contrast, they will discuss a state of matter that goes beyond Landau's limited conception of the world.

Having given such talks myself, I think it is extremely interesting that, in fact, with modern generalizations of our understanding of symmetry, it may be possible to incorporate all known equilibrium phases of matter into a suitably extended version of the Landau Paradigm. Let me attempt to paraphrase the Landau Paradigm:

1. Phases of matter should be labelled by how they represent their symmetries, in particular whether they are spontaneously broken or not.

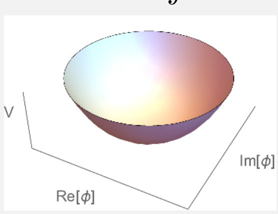
2. A further belief that comes with this point of view is that gapless degrees of freedom, or groundstate degeneracy, in a phase, should be swept out by a symmetry. That is, they should arise as Goldstone modes for some spontaneously broken symmetry.

3. The degrees of freedom at a critical point are the fluctuations of the order parameter.

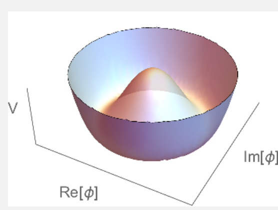
Beyond its conceptual utility, this perspective has a weaponization, in the form of Landau-Ginzburg(-Wilson-Fisher) theory, in terms of which we may find representative states, understand gross phase structure, and, when suitably augmented by the renormalization group (RG), even quantitatively describe phase transitions.

Landau-Ginzburg-Wilson implemenation.
It is worthwhile to review the logic that produces this weapon. If we take the Landau paradigm seriously, then the only low-energy modes we require are those swept out by the symmetry. The key idea is to introduce a degree of freedom $\phi(x)$ at each point in space that transforms *linearly* under the symmetry. ϕ should be regarded as a coarse-grained object, and this is an effective long-wavelength description. In the example of a magnet, $\phi(x)$ can be the magnetization averaged over a small cell at x. Now, because there are no other light degrees of freedom (by assertion 1), the effective action for ϕ should be given by an analytic functional of ϕ which is local in spacetime. This functional can therefore be expanded in a series consisting of all symmetric local functionals of ϕ, organized in a derivative expansion of terms of decreasing relevance. The length scale suppressing higher derivates is the short distance over which we averaged in constructing $\phi(x)$.
For example, in the case of a $\mathsf{U}(1)$ (0-form) symmetry, the order parameter transforms as $\phi(x) \mapsto e^{\mathbf{i}\alpha}\phi(x)$. All local, symmetric terms, organized by derivative expansion (what else could it be):

$$S_{\text{Landau-Ginzburg-Wilson}}[\phi] = \int d^D x \left(r|\phi|^2 + u|\phi|^4 + \cdots + |\partial\phi|^2 + \cdots \right) . \tag{1.1}$$



$r > 0$



$r < 0$

So this is an application of Effective Field Theory: once we know

(1) the symmetries (2) the degrees of freedom (3) the cutoff,

the dynamics is determined.

Indeed there are many apparent exceptions to the Landau Paradigm. Let us focus on apparent exceptions to item 1: phases of matter that are distinguished by something other than ordinary symmetry breaking. As a preview, exceptions that are only apparent include:

- **Topologically ordered states.** These are phases of matter distinguished from the trivial phase by something other than a local order parameter [1, 2]. Symptoms include a groundstate degeneracy that depends on the topology of space, and *anyons*, excitations that cannot be created by any local operator. Real examples found so far include fractional quantum Hall states, as well as gapped spin liquids.

- **Other deconfined states of gauge theory.** This category includes gapless spin liquids such as spinon Fermi surface or Dirac spin liquids (most candidate spin liquid materials are gapless). Another very visible manifestation of such a state is the photon phase of quantum electrodynamics in which our vacuum lives.

- **Fracton phases.** Gapped fracton phases are a special case of topological order, where there are excitations that not only cannot be *created* by any local operator, but cannot be *moved* by any local operator.

- **Topological insulators.** Here we can include both free-fermion states with topologically non-trivial bandstructure, as well as interacting symmetry-protected topological (SPT) phases.

Conventions. L is the linear system size. $D = d + 1$ is the number of spacetime dimensions. I'll denote the dimension of a manifold or the degree of a form by a subscript or superscript. I will use fancy upper-case letters (like $\mathcal{A}_\mu$) for background gauge fields and lower-case letters (like a_μ) for dynamical gauge fields. I will sometimes use $G^{(p)}$ to denote a p-form symmetry with group G.

2 A non-symmetry view of phases of matter

A useful definition of a gapped phase of matter is an equivalence class of gapped groundstates of local Hamiltonians, in the thermodynamic limit[1]. Two groundstates are considered equivalent if they are related by adiabatic evolution (for a time of order L^0) combined with inclusion or removal of product states. That is, there is a path between the two Hamiltonians along which the gap does not close (see Fig. 1, left).

[1]We should pause to comment on the meaning of 'gapped'. We allow for a stable groundstate subspace, which becomes degenerate in the thermodynamic limit. 'Stable' means that the degeneracy persists under arbitrary small perturbations of the Hamiltonian, and requires that the groundstates are not related by the action of local operators. In d spatial dimensions, the logarithm of the number of such states can grow as quickly as L^{d-1} [3] in fracton models.

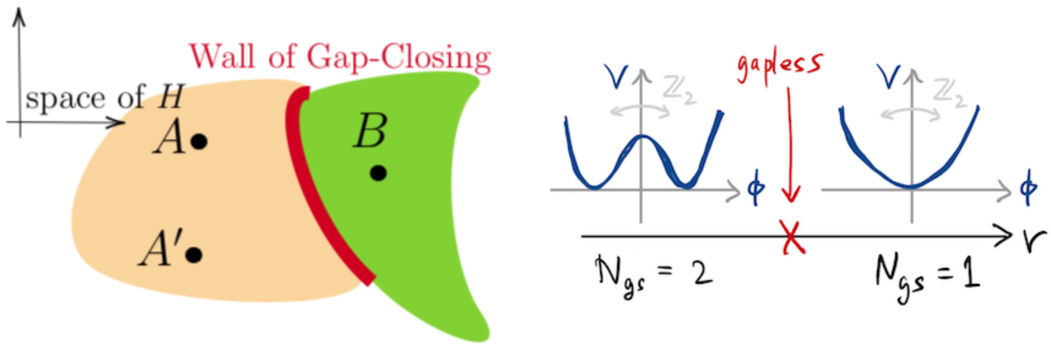


Figure 1: Left: A schematic illustration of the definition of gapped phases of matter. Two distinct phases are separated in the space of local Hamiltonians by a wall of gaplessness, the codimension-one locus where the gap closes. Here $H_A \simeq H_{A'}$. Right: The groundstate degeneracy, $N_{\rm gs}$, for example as swept out by a spontaneously broken global symmetry, is an example of a topological invariant that can label a phase.

This definition poses a difficulty for checking that two Hamiltonians represent distinct phases: we cannot check all possible paths between them. A crucial role is therefore played by universal properties of a phase – quantities, such as integers, that cannot change smoothly within a phase, and therefore can only vary across phase boundaries. A good example of such a topological invariant is the groundstate degeneracy, which is certainly an integer. A phase of matter that spontaneously breaks a discrete symmetry G has a groundstate degeneracy $|G|$, the order of the group (see Fig. 1, right). This is a topological distinction from the trivial paramagnetic phase, which has a unique groundstate and a representative that is a product state with no entanglement at all. In this sense, even spontaneous symmetry breaking (SSB) is a topological phenomenon.

Nevertheless, non-trivial phases of matter that don't break any ordinary symmetries are called topological phases. Non-trivial means distinct from the phase represented by a product state. Topological phases can be divided into two classes: those with topological order and those without. One way to define topological order [1] is a phase with localized excitations that cannot be created by any local operator. In 2+1 dimensions, such particle excitations are called *anyons*; they can be created in pairs by an open-string operator. A consequence of the fact that they can't be created locally is that these operators can have fractional spin and statistics. The quantum numbers of the anyons (their statistics and (if there are global symmetries) charges) can be used characterize the phase of matter.

Especially in $D = 2+1$, the theory of anyons (their statistics and fusion rules) is a highly-developed mathematical edifice, called unitary modular tensor category

(UMTC) theory. Perhaps now is a good time to mention the most elementary distinction, between abelian and non-abelian topological order. By fusion of anyons, I mean the following. An anyon is a particle whose presence can be detected from a distance, by circling some other excitation around it and measuring the change of the resulting state. Given two anyon types a and b, I can consider circling other excitations around both of them. If I have a complete basis of all the anyon types in the topological order under study, the result must look like one of them, but which one we get need not be uniquely determined:

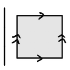
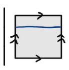
$$a \times b = c_1 + c_2 + \cdots . \tag{2.1}$$

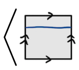
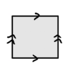
If the fusion rules look like

$$a \times b = c$$

for all the anyons, we say the topological order is abelian. Braiding such particles merely acts by a phase on the resulting unique state. In contrast, fusion rules like (2.1) require that the lowest-energy state in the presence of a and b is degenerate; in this case, braiding the two particles involves not only a phase, but a whole unitary matrix acting on this degenerate subspace.

If you have studied conformal field theory (CFT), you will notice a formal similarity between (2.1) and the operator product expansion. This is not a coincidence – a 2d CFT also defines a UMTC. In fact, the structure was defined first in that context, by Moore and Seiberg.

On a space with a non-contractible curve C such as a torus, new groundstates can be made by acting with the operator that transports an anyon around C. These groundstates are *locally indistinguishable*, in the following sense. If $\Big|$  $\Big\rangle$ and $\Big|$  $\Big\rangle$ are two such groundstates, then

$\Big\langle$  $\Big| \, \mathcal{O}_x \, \Big|$  $\Big\rangle = 0, \qquad (2.2)$

for all local operators $\mathcal{O}_x$. (The picture in the kets is a cartoon of two of the groundstates of the toric code on the 2-torus.) The fact that no local operator can mix these groundstates means that the degeneracy is robust to (at least small) changes of the (local!) Hamiltonian. So a second symptom of topological order, not independent of the first, is a robust groundstate degeneracy that depends on the topology of space. A final symptom is the existence of long-range entanglement in the groundstate; a review focussing on this aspect is [4].

Let's enshrine these symptoms of topological order in a list:

1. Fractionalization of quantum numbers.

2. Robust groundstate degeneracy that depends on the topology of space.

3. Long-ranged entanglement.

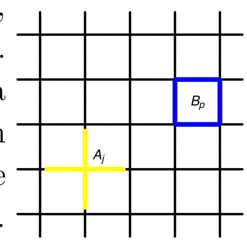
An interesting special case of topologically ordered states is fracton phases [5, 6]. A fracton phase has excitations that cannot be *moved* by any local operator (perhaps only in some directions of space). This is a strictly stronger condition than topological order, since an excitation can effectively be moved by annihilating it and creating it again elsewhere. Such phases (with a gap) exist in 3+1 dimensions (and higher). A consequence of the defining property is a groundstate degeneracy whose logarithm grows linearly with system size, and a subleading linear term in the scaling of the entanglement entropy of a region with the size of the region.

In the next two subsections I want to discuss in slightly more detail two paradigmatic examples of (families of) topologically ordered states.

2.1 Toric code

Here's the toric code [7]. It emerges $\mathbb{Z}_2$ gauge theory from a local Hilbert space. There is a sense in which it exists in certain forms of artificial condensed matter (cold atoms in optical lattices, trapped ions).

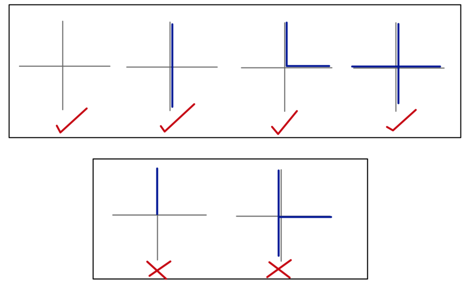
Consider a 2d cell complex. This means a graph (a set of vertices who know with whom they share an edge) with further information about plaquettes, who know which edges bound them). For example, consider the square lattice at right. Now place a qubit on each *edge*. Now let's make some stabilizers. Associate to each plaquette a a plaquette operator or 'flux operator', $B_p = \prod_{\ell \in p} Z_\ell$, and to each vertex a star operator or 'gauss law operator', $A_v = \prod_{\ell \in v} X_\ell$. (The former names just describe the support of the operators on the graph. The latter names are natural if we consider Z to be related to a gauge field by $Z \sim e^{\mathbf{i}A}$, and X is its electric flux. For more on the translation to gauge theory see §5.2 here.) These definitions are not special to the square lattice and work for any cell complex, in any dimension.



[Fig by D.Ben-Zion, after Kitaev]

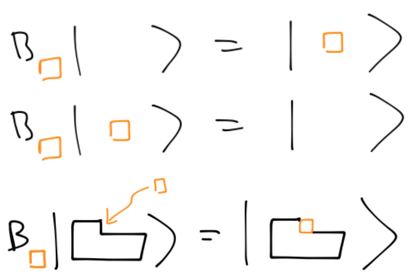
The hamiltonian is $\mathbf{H}_{\text{TC}} = -\Gamma_m \sum_p B_p - \Gamma_e \sum_v A_v$. These terms all commute with each other (since each vertex and plaquette share zero or two links), and they each square to one, so the Hamiltonian is easy to diagonalize. Let's find the groundstate(s).

Which states satisfy the 'gauss law condition' $A_v = 1$? In the X basis there is an extremely useful visualization: we say a link l of $\hat{\Gamma}$ is covered with a segment of string (an electric flux line) if $\mathbf{e}_l = 1$ (so $X_l = -1$) and is not covered if $\mathbf{e}_l = 0$ (so $X_l = +1$):  $\equiv X = -1$. In the figure at right, we enumerate the possibilities for a 4-valent vertex. $A_v = -1$ if a flux line ends at v.



So the subspace of $\mathcal{H}$ satisfying the gauss law condition is spanned by closed-string states (lines of electric flux which have no charge to end on), of the form $\sum_{\{C\}} \Psi(C) |C\rangle$.

Now we look at the action of B_p on this subspace of states:



$B_p = \prod_{\ell \in \partial p} Z_\ell$ creates and destroys strings around the boundary of the plaquette p:

$$B_p |C\rangle = |C + \partial p\rangle \ .$$

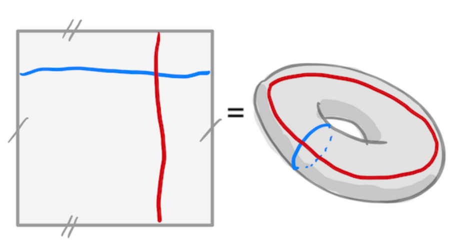
The argument of the ket is to be understood mod two. The condition that $B_p |\text{gs}\rangle = |\text{gs}\rangle$ is a homological equivalence. In words, the eigenvalue equation $\mathbf{B}_\square = 1$ says $\Psi(C) = \Psi(C')$ if C' and C can be continuously deformed into each other by attaching or removing plaquettes.

If the space is simply connected (like a sphere) – if all curves are the boundary of some region contained in the lattice – then this means the groundstate

$$|\text{gs}\rangle = \sum_C |C\rangle \tag{2.3}$$

is a uniform superposition of all loops.

Topological order. If the space has non-contractible loops, however, then the eigenvalue equation does not determine the relative coefficients of loops of different topology! The two-dimensional torus obtained by considering periodic boundary conditions in x and y is an example of such a space:


(2.4)

On a space with $2g$ independent non-contractible loops, there are 2^{2g} independent groundstates. (In fact, the above is the very definition of the simplicial homology of the space, with $\mathbb{Z}_2$ coefficients; more generally the number of independent groundstates is 2^{b_1} where $b_1 \equiv \dim H^1(M, \mathbb{Z}_2)$. For more on the connection with homology and algebraic topology in general, see these notes.)

No local operator mixes these groundstates. This makes the topological degeneracy stable to local perturbations of the Hamiltonian. The degenerate groundstates are instead connected by the action of (Wegner-Wilson) loop operators:

$$W_C = \prod_{\ell \in C} Z_\ell \quad V_{\check{C}} = \prod_{\ell \perp \check{C}} X_\ell \ .$$

(Notice that the loop operator for a single plaquette $W_{\partial\Box} = B_p$ is the plaquette operator.) V, W commute with $\mathbf{H}_{\text{TC}}$ and don't commute with each other (specifically W_C *anti*commutes with $V_{\check{C}}$ if C and $\check{C}$ intersect an odd number of times). This algebra must be represented on the groundstates, and it doesn't have any one-dimensional representations. In terms of our picture of strings, W_C *creates* a loop on C, and $V_{\check{C}}$ *detects* a loop intersecting $\check{C}$.

More generally, a system is said to have *topological order* if (approximately) degenerate (ground)states (in the thermodynamic limit) cannot be distinguished by any local operator:

$$\langle \psi_1 | \mathcal{O}_{\text{local}} | \psi_2 \rangle = 0 \tag{2.5}$$

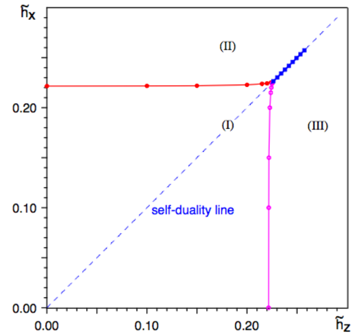
for all local operators.

(Here we encounter the connection between topological order and quantum error-correcting codes: the degenerate states are the codewords. The size of the operators that connect the degenerate states is then the analog of the code distance in an error-correcting code. For the toric code, these are string operators that wind around the whole system, so the code distance grows like L and blows up in the thermodynamic limit.)

The deconfined phase. So far everything I've said works on any graph (actually: cell complex, since we need to know where the plaquettes are). And so far I've described the solvable limit, where $H = H_{\text{TC}}$.

But the fact that the code distance goes like L (no local operator mixes the topological groundstates) is also the reason that the topological degeneracy is *robust*: adding local operators to the Hamiltonian will never split the degeneracy in perturbation theory. Therefore, this physics is characteristic of a phase of matter, and not just the special solvable Hamiltonian H_{TC}. The toric code is a (RG fixed point) representative of a phase of matter.

Perturbations $\Delta H = \sum_l (h_X X_l + h_Z Z_l)$ produce a nonzero correlation length. Let's focus on $D = 2+1$ for what follows. These couplings h_X and h_Z are respectively a string tension and a fugacity for the electric flux string endpoints: charges. Make these too big and the model is confined or higgsed, respectively. These are actually adiabatically connected [Fradkin-Shenker]: Both are connected to the trivial state where e.g. $H = \sum_l X_l$ whose groundstate is a product $\otimes_l |{\rightarrow_l}\rangle$.

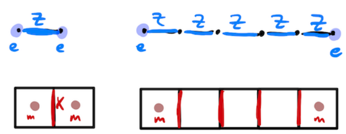


[from Tupitsyn-Kitaev-Prokof'ev-Stamp]

Anyons. There are two kinds of elementary excited states of the toric code: violations of $A_s = 1$ and violations of $B_p = 1$. [2]

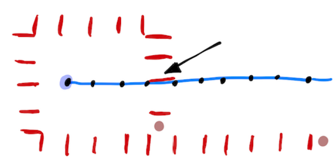
Here is how to make them The defects are created by the endpoints of open Wilson lines. Again there are two kinds:

$$W(C) = \prod_{\ell \in C} Z_\ell, \quad V(\check{C}) = \prod_{\ell \perp \check{C}} X_\ell. \tag{2.6}$$



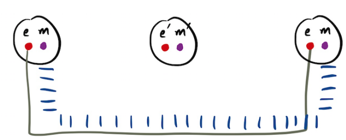
Here C is an open curve in the lattice, and $\check{C}$ is an open curve in the dual lattice. Endpoints of $W(C)$ violate A_s and endpoints of $V(\check{C})$ violate B_p.

These two kinds of particles have nontrivial mutual statistics, as you can see by moving one of them around the other and keep track of the strings trailing away from them. The process results in a net factor of (-1) on the state.



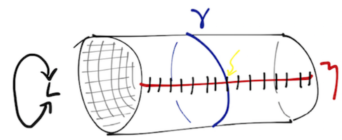
This has the further consequence that their bound state is a fermion, despite the fact that the model is entirely made from local, bosonic degrees of freedom.

To see this, observe that exchanging two particles can be accomplished by first rotating one around the other by a π rotation, and then translating both of them by their separation. As you can see in the figure, the first step requires the string creating the e particle to cross that creating the m particle on an odd number of links. (The second step is innocuous.)



[2] Cultural note: The limit where the coefficient of the star term A_s goes to infinity is called 'pure $\mathbb{Z}_2$ gauge theory', where he condition $A_s = 1$, the Gauss' law constraint, is imposed exactly. The e particle defects cost infinite energy and hence are strictly forbidden in this theory.

Consider the cylinder. There is one nontrivial class of loops; call a representative γ. Let η be a line running along the cylinder. The two groundstates are generated by the action of the Wilson loop operator



$$V(\eta) \equiv \prod_{\ell \text{ crossed by } \eta} X_\ell$$

in the sense that

$$|\text{gs}_2\rangle = V(\eta) |\text{gs}_1\rangle \ .$$

This is also a groundstate (at $h_X, h_Z = 0$) since there is no plaquette with $\mathbf{B}_p = -1$ (more simply: $[\mathbf{H}_{h_X=h_Z=0}, V_x(\eta)] = 0$). They are *distinguished* by $W(\gamma) \equiv \prod_{l\in\gamma} X_l$ in the sense that the two groundstates are eigenstates of this operator with distinct eigenvalues:

$$W(\gamma) |\text{gs}_\alpha\rangle = (-1)^\alpha |\text{gs}_\alpha\rangle \,, \quad \alpha = 1, 2.$$

This follows since $W(\eta)V(\gamma) = -V(\gamma)W(\eta)$ – the two curves share a single link (the one pointed to by the yellow arrow in the figure).

At finite h_X, h_Z (and in finite volume), there is tunneling between the topologically degenerate groundstates, since in that case

$$[\mathbf{H}, \prod_{l\in\gamma} X_l] \neq 0.$$

This means that for some n

$$\langle \text{gs}_2 | \mathbf{H}^n |\text{gs}_1\rangle \neq 0.$$

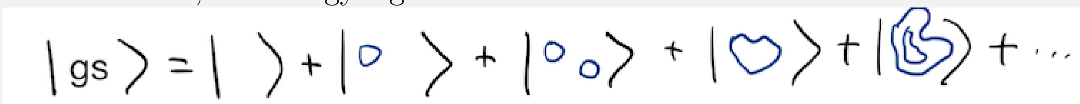
The process that mixes the groundstates requires the creation of magnetic flux on some plaquette (*i.e.* a plaquette P with $B_P = -1$, which costs energy $2\Gamma_m$), which then must hop (using the h_X term in $\mathbf{H}$) all the way along the path η, of length L, to cancel the action of $V(\eta)$. The amplitude for this process goes like

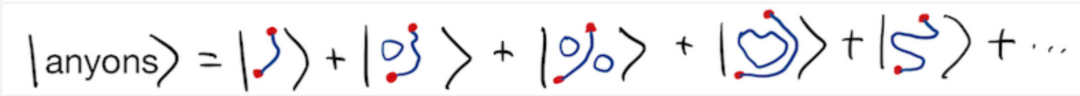
$$\Gamma \sim \frac{\langle \text{gs}_2 | (hX_1)(hX_2)\cdots(hX_L) |\text{gs}_1\rangle}{2\Gamma_m \cdot 2\Gamma_m \cdot \ldots 2\Gamma_m} \sim \left(\frac{h}{2\Gamma_m}\right)^L = e^{-L|\ln 2\Gamma_m/h|}$$

which is *extremely tiny* in the thermodynamic limit. The way to think about this is that the Hamiltonian is itself a local operator, and cannot distinguish the groundstates from each other. It takes a non-perturbative process, exponentially suppressed in system size, to create the splitting.

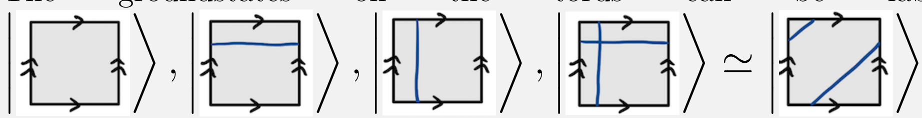
The toric code.

To summarize, the energy eigenstates of the toric code look like:

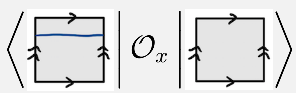




The groundstates on the torus can be labelled as:



These groundstates are *locally indistinguishable*:

 $= 0 \ \forall$ local operators $\mathcal{O}_x$.

- I've focussed on the case of two spatial dimensions, but the toric code is well-defined on an arbitrary cell complex, in particular on a lattice in any number of dimensions. It has various generalizations:

- For example instead of the putting the dofs on the links, we can put them on the p-cells. Instead of using qubits, we can use $\mathbb{Z}_N$ clock and shift variables.

- The version on the p-cells with $\mathbb{Z}_N$ variables computes $H_p(C, \mathbb{Z}_N)$, the pth homology of the cell complex, as its groundstate subspace.

- With the dofs on the links, the model can be generalized to any finite group G (in fact this step was already taken in Kitaev's original paper). This is usually called the quantum double model and was explained beautifully by Meng Cheng.

- I haven't emphasized the connection to gauge theory above. The toric code is (a limit of) G lattice gauge theory with the gauss law condition imposed energetically, meaning that the low energy states satisfy the gauss law condition. At low energies it is governed by the TQFT described by Greg Moore and Meng Cheng in their lectures.

- By attaching various phases to the plaquette operators, we can make twisted gauge theory, as introduced by Dijkgraaf and Witten [8] and mentioned in Greg Moore's and Meng Cheng's lectures. A better framework for making explicit solvable lattice models for such states is the string net models developed by Levin and Wen [9, 10].

2.2 Quantum Hall states (via effective field theory)

[Wen's book or this review; Zee, Quantum Hall Fluids; Zee's QFT book §VI.2] I want to explain another example of how properties 1 and 2 of topological order can be realized in a simple physical system, using the EFT (effective field theory) that describes the canonical examples of topologically-ordered states that have been realized in experiments: abelian *fractional* quantum Hall states in $D = 2 + 1$.

The low-energy effective field theory for abelian quantum Hall states [11] is Chern-Simons-Witten gauge theory, whose basic action is:

$$S_0[a_I] = \sum_{IJ}^{n} \frac{K_{IJ}}{4\pi} \int a_I \wedge \mathrm{d}a_J \tag{2.7}$$

The dynamical degrees of freedom here are a^I, a collection of abelian gauge fields.

Where did these gauge fields come from? One very simple way to motivate their introduction is as follows. By assumption, because of particle number conservation, our system has a conserved $\mathsf{U}(1)$ current, J^μ, satisfying $\partial_\mu J^\mu = 0$. In $D = 2 + 1$, we can *solve* this equation by introducing a field a and writing

$$J^\mu = \epsilon^{\mu\nu\rho} \partial_\nu a_\rho. \tag{2.8}$$

The continuity equation is automatic if J can be written this way (for nonsingular a) by symmetry of the mixed partials. (More generally, the equation could also be solved by a sum of such terms, $J = \sum_I \epsilon^{\mu\nu\rho} \partial_\nu a^I_\rho t^I$. This ambiguity reflects some of the enormous multiplicity of different quantum Hall states.) Then we must guess what dynamics should govern a. Here we just add all terms allowed by the symmetries, as usual. When it's not forbidden by time-reversal symmetry or parity, the Chern-Simons term is the most important term at low energies.

Notice that we wrote this action in a coordinate-invariant way without needing to mention a metric. This is a topological field theory. In the absence of charges, the equations of motion say simply that $0 = \frac{\delta S_0}{\delta a} \propto f = da$. Unlike Maxwell theory, there are no local, gauge invariant degrees of freedom. And, by Legendre transformation, the Hamiltonian is just zero. It is a theory of groundstates.

Consider the simplest case of (2.7) with a single such field a, $S_0[a] = \int \frac{k}{4\pi} a \wedge da$. As we'll see, this describes *e.g.* the Laughlin state of electrons at $\nu = 1/k$ for k an odd integer. (More general K describe the so-called hierarchy states, and give some understanding of the pattern of plateaux that appear in the Hall conductivity.)

When I say there are no local dofs, I am thinking of the limit where we totally ignore the Maxwell term. The Maxwell term is irrelevant: its effects go away at low energies. Let's add it back in and look at the spectrum of fluctuations with the action:

$$L = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{4M}f_{\mu\nu}f^{\mu\nu}$$

where M is some microscopic energy scale above which the Maxwell term matters. The equation of motion is

$$0 = \frac{\delta S}{\delta a_\lambda} = \frac{k}{2\pi}\epsilon^{\lambda\rho\nu}f_{\rho\nu} + \frac{\partial_\mu f^{\mu\lambda}}{M}. \tag{2.9}$$

In terms of $f^\lambda \equiv \epsilon^{\lambda\rho\sigma}f_{\rho\sigma}$ this is

$$\epsilon^{\mu\nu\rho}\partial_\nu f_\rho + \frac{Mk}{2\pi}f^\mu = 0. \tag{2.10}$$

Taking curl of the BHS ($\epsilon_{\mu\alpha\beta}\partial^\alpha$(BHS)) gives

$$\left(\partial_\mu\partial^\mu - \left(M\frac{k}{2\pi}\right)^2\right)f_\rho = 0. \tag{2.11}$$

This is the dispersion relation for an excitation of mass $\frac{Mk}{2\pi}$. As $M \to \infty$, the excitation goes off to infinite energy.

If we demand that (2.7) is invariant (or rather $e^{\mathbf{i}S_0}$ is invariant) under $\mathsf{U}(1)^n$ gauge transformations, including large gauge transformations, then k must be an integer[3]. From the point of view of (2.8), the demand that the gauge group is really $\mathsf{U}(1)$, and the concomitant quantization of flux of da, comes from demanding that the charge of the current J_μ is quantized (in units of the charge of the electron). It's pretty interesting that this seemingly-metaphysical microscopic information that all charges come in integer multiples of the electron charge has such strong consequences for the low-energy description of macroscopic quantum phases.

More generally, K must be a symmetric matrix (don't forget the sign from integration by parts) of integers.

Two more ingredients are required for this abelian CS theory to describe the low-energy EFT of a quantum Hall state:

(1) We must say how the stuff is coupled to the EM field. Notice that these gauge fields imply conserved currents $j^I_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu a^I_\rho$. This is automatically conserved by antisymmetry of $\epsilon_{\mu\nu\rho}$, as long as a is single-valued. In its realization as the EFT for a quantum Hall state, a linear combination of these currents is coupled to the external EM field $\mathcal{A}_\mu$:

$$S_{EM}[a_I, \mathcal{A}] = \int \mathcal{A}^\mu J_\mu = \int \mathcal{A}^\mu t_I j^I_\mu \ ,$$

[3]Strict invariance under large gauge transformations would require the stronger condition that k is an *even* integer. But the failure of gauge invariance of $e^{\mathbf{i}S_0}$ by a sign for $k = 1$ (and more generally k odd) is acceptable for systems containing fermions.

i.e. the actual EM current is $J_\mu = \sum_I t_I j^I_\mu$. The normalization is determined so that flux quantization implies quantization of charge.

(2) Finally, we must include information about the (gapped) quasiparticle excitations of the system. Creating a quasiparticle excitation costs some energy of order the energy gap, and their dynamics is not included in this ultra-low-energy description. As I described above, however, the quantum numbers of these excitations is a crucial part of the data specifying the topological order. This is encoded by adding (conserved) currents minimally coupled to the CS gauge fields:

$$S_{qp} = \int a_I j^I_{qp}.$$

Alternatively, we can think of this as inserting Wilson lines $e^{\mathbf{i}\oint_W a^I q_I}$ along the trajectories W of a (probe) anyon of charge q^I.

Now let's show item 1, fractional statistics, in the simplest case with a 1×1 K-matrix. In this case, the quasiparticles are anyons of charge e/k. The idea of how this is accomplished is called flux attachment. The CS equation of motion is $0 = \frac{\delta S}{\delta a} \sim -f_{\mu\nu}\frac{k}{2\pi} + j^{qp}_\mu$, where j^{qp} is a quasiparticle current, coupling minimally to the CS gauge field. The time component of this equation $\mu = t$ says $b = \frac{2\pi}{k}\rho$ – a charge gets $2\pi/k$ worth of magnetic flux attached to it. Then if we bring another quasiparticle in a loop C around it, the phase of its wavefunction changes by (the ordinary Bohm-Aharonov effect)

$$\Delta\varphi_{12} = q_1 \oint_C a = q_1 \int_{R, \partial R = C} b = q_1 \frac{2\pi}{k} q_2.$$

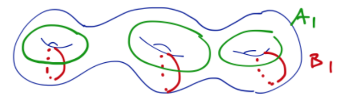
Hence, the quasiparticles have fractional braiding statistics[4].

Now topological order property 2: The number of of groundstates on a genus-g surface is $|\det(K)|^g$. Consider the simplest case, where $K = k$, and put the system on a torus $T^2 = S^1 \times S^1$, which has $g = 1$. The gauge-invariant operators acting on the Hilbert space of the CS theory on a torus are of the form $\mathcal{F}_x \equiv e^{\mathbf{i}\oint_{C_x} a}, \mathcal{F}_y \equiv e^{\mathbf{i}\oint_{C_y} a}$ and *integer* powers of these operators. These are the operators that transport the anyons around the cycles of the torus. The restriction to integers comes from the demand that they are invariant under large gauge transformations, which take $\oint_C a \to \oint_C a + 2\pi\mathbb{Z}$. According to the CS action, a_x is the canonical momentum of a_y. Canonical quantization then implies that

$$[a_x(r), a_y(r')] = \frac{2\pi\mathbf{i}}{k}\delta^2(r - r')$$

[4]The fractional statistics of the charge-$\frac{1}{3}$ quasiparticles of the $\nu = 1/3$ Laughlin state were finally observed experimentally just recently. Their charge had been measured using shot-noise measurements long ago.

and hence (by the BCH formula) that these flux-insertion operators satisfy a Heisenberg algebra: $\mathcal{F}_x\mathcal{F}_y = \mathcal{F}_y\mathcal{F}_x e^{2\pi i/k}$. The smallest irrep of this algebra is k dimensional, where $\mathcal{F}_x$ and $\mathcal{F}_y$ look like clock and shift matrices.

If space is a Riemann surface with g handles (like this: 

), then there are g pairs of such operators, so g independent Heisenberg algebras, all of which commute with the Hamiltonian, and hence k^g groundstates.

It is possible to show that CS theory also exhibits the third property of long-range entanglement. See [12, 13].

This description exhibits a quasiparticle with charge e/k: If we stick in a quasiparticle at the origin, the equations of motion become

$$0 = \frac{\delta S}{\delta a_0(x)} = \frac{k}{2\pi} f_{xy} - \delta^2(x). \tag{2.12}$$

From the relation $J^\mu = \frac{e}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$, the actual electric charge is then

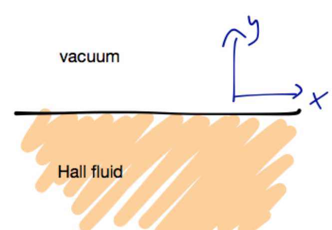
$$\rho = e\frac{1}{2\pi} f_{xy} = \frac{e}{k}\delta^2(x). \tag{2.13}$$

Finally, we can do the (gaussian!) path integral over a to produce an effective action for the background gauge field $\mathcal{A}$. (Complete the square.) We find a rational Hall conductivity

$$\sigma^{xy} = t_I \left(K^{-1}\right)^{IJ} t_J \frac{e^2}{h}. \tag{2.14}$$

In the simplest case of $K = k, t = 1$, this is $\sigma^{xy} = \frac{1}{k}\frac{e^2}{h}$. The fact that the Hall conductivity is not an integer is not a problem – $e^{\mathbf{i}S_{\text{eff}}[A]}$ does not need to be invariant under large gauge transformations, since there are k groundstates on the torus, which are permuted by flux-threading.

So far, we've shown that abelian CS theory reproduces the bulk phenomenology of some fractional quantum Hall states. Now here is a bonus: we can see what it does when the sample has a boundary in space (which actual samples in the laboratory tend to have).



Edge physics. Consider $\mathsf{U}(1)$ CS theory living on the lower-half plane.

$$S = \frac{k}{4\pi}\int_{\mathbb{R}\times\text{LHP}} a \wedge \mathrm{d}a$$

Let's work in $a_0 = 0$ gauge. We must still impose the equations of motion for a_0, which say $0 = f_{ij} = \epsilon_{ij}\partial_i a_j$. This is solved by $a = \mathbf{i}g^{-1}\mathrm{d}g = \mathrm{d}\phi$ ($g = e^{-\mathbf{i}\phi}, \phi \simeq \phi + 2\pi$), where d is the exterior derivative in just the spatial directions. This looks like a gauge transformation.

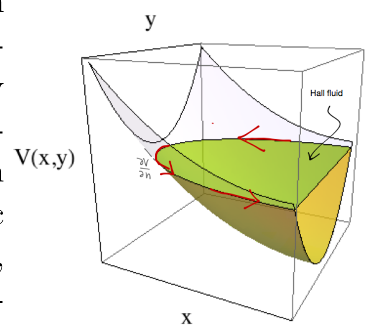
Only gauge transformations that approach $\mathbb{1}$ at the boundary preserve S_{CS}. This implies that the would-be-gauge-parameter ϕ is dynamical on the boundary. (Or equivalently, we must add a degree of freedom identical to ϕ to cancel the gauge variation of the action.)

A good choice of boundary condition is: $0 = a - v(\star_2 a)$ i.e. $a_t = va_x$. The velocity v is some non-universal UV data; it arises from a gauge invariant local boundary term, $\Delta S = \int_{\partial LHP} \frac{kv}{4\pi} a_x^2$. Plugging back into the CS action and adding the boundary term, we find[5]

$$S_{CS}[a = \mathrm{d}\phi] = \frac{k}{4\pi}\int dtdx \left(\partial_t\phi\partial_x\phi + v\left(\partial_x\phi\right)^2\right). \tag{2.18}$$

Conclusion: ϕ is a chiral boson. $kv > 0$ is required for stability. The sign of k determines the chirality.

For the case of IQHE ($k = 1$), the microscopic picture in terms of free fermions is at right. For free fermions in a magnetic field, the velocity of the edge states is determined by the slope of the potential which is holding the electrons together. (This can be understood by considering the motion of a classical charged particle in a large enough magnetic field that the inertial term can be ignored: $q\vec{v}\times\vec{B} = -\vec{\nabla}V$, solve for v.) The velocity is clearly not universal information.



The Hamiltonian H depends on the boundary conditions; the Hilbert space $\mathcal{H}$ does not.

I have to emphasize that a chiral theory like this cannot be realized from a local lattice model in $D = 1 + 1$ dimensions. There are more powerful arguments for this statement, but a viscerally appealing argument is simply to draw the bandstructure arising from any lattice Hamiltonian of free fermions. Each band is periodic in mo-

[5] In more detail, let $\tilde{d}$ denote the exterior derivative in just the spatial directions.

$$S_0[a = \tilde{d}\phi] = \frac{k}{4\pi}\int_{\mathbb{R}\times LHP} a \wedge \left(dt\partial_t + \tilde{d}\right) a = \frac{k}{4\pi}\int_{\mathbb{R}\times LHP} \tilde{d}\phi \wedge dt\partial_t\tilde{d}\phi \tag{2.15}$$

$$= \frac{k}{4\pi}\int_{\mathbb{R}\times LHP} \tilde{d}\left(\phi \wedge dt\partial_t\tilde{d}\phi\right) \overset{\text{Stokes}}{=} \frac{k}{4\pi}\int_{\mathbb{R}\times\partial LHP} \phi dt\partial_t\tilde{d}\phi \tag{2.16}$$

$$= \frac{k}{4\pi}\int_{\mathbb{R}\times\partial LHP} dxdt\phi\partial_t\partial_x\phi \overset{\text{IBP}}{=} -\int_{\mathbb{R}\times\partial LHP} dxdt\partial_x\phi\partial_t\phi. \tag{2.17}$$

mentum space. This means that an even number of bands cross the Fermi level, and moreover that each band that crosses with positive slope must cross again with negative slope to return to its starting point. This is the essence of the *Nielsen-Ninomiya fermion doubling theorem*. An analogous argument applies in any number of dimensions. In fact, interactions provide a real loophole in the case of $D = 3+1$. But in $D = 1+1$, a nonzero chiral central charge (which in the simple examples we've discussed is just the number of right-movers minus the number of left-movers) is associated with a *gravitational anomaly*. A lattice model has zero gravitational anomaly, and this is a scale-independent quantity that must agree between the microscopic description and the EFT. The real obstruction to making a local lattice model is the anomaly.[6]

In the case with general K matrix,

$$S = \frac{K^{IJ}}{4\pi} \int_{\mathbb{R}\times \mathrm{LHP}} a_I \wedge \mathrm{d}a_J$$

$$S_{CS}[a^I = \mathrm{d}\phi^I] = \frac{1}{4\pi} \int dt dx \left(K^{IJ} \partial_t \phi^I \partial_x \phi^J + v_{IJ} \partial_x \phi^I \partial_x \phi^J \right) .$$

(v is a positive matrix, non-universal.) This is a collection of chiral bosons. The number of left-/right-movers is the number of positive/negative eigenvalues of K.

[End of Lecture 1]

Abelian Chern-Simons theory of the toric code. Consider now the following theory of two gauge fields with a *mutual* Chern-Simons term:

$$S[a,b] = \frac{k}{4\pi} \int d^3x \, (a\partial b + b\partial a) \ . \tag{2.19}$$

So the K-matrix is $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$. The argument above suggests that a boundary of this model should have one left-mover and one right-mover, altogether an ordinary boson in $1+1$d. In this case, we can add local, single-valued, gauge-invariant terms to the boundary (such as $\cos\phi$) to kill the edge mode. Notice that unlike the generic abelian CS theory, this system has a time-reversal symmetry acting by $a \leftrightarrow b$.

So the TO described by this K matrix allows a gapped boundary. In fact it is an effective field theory of a familiar system. To see this, consider the anyon types: they can labelled by their electric charges under the two gauge fields (a, b). Because of the

[6] If, however, we break time translation symmetry, we can evade this outcome even in $D = 1+1$: for example, in a floquet system, where $H(t+T) = H(t)$, the set of energy eigenvalues is also periodic, so we can have a band that starts below the Fermi level and ends above it, separated by $2\pi/T$ from its starting energy.

CS term, the electric charge of a gets k units of magnetic flux of b attached to it, and vice versa. The well-defined operators (ferrying these anyons around) are

$$W_C = e^{\mathbf{i} \oint_C a}, \quad V_{\check{C}} = e^{\mathbf{i} \oint_{\check{C}} b}.$$

Because of the Aharonov-Bohm phase, if we place the curves in a fixed-time slice, they satisfy

$$W_C V_{\check{C}} = \omega^{\# \check{C} \cap C} V_{\check{C}} W_C.$$

These are the operators that ferry the e and m particles of the $\mathbb{Z}_k$ toric code.

Given a K-matrix theory with equal numbers of left-movers and right-movers, when can we gap out the boundary? The question is whether we can add local operators that give them a mass. For a chiral mode, $e^{\mathbf{i}\phi_R} + h.c. = \cos \phi_R$ is not a local operator because of the commutation relations of ϕ_L determined from (2.18). But $\cos(\phi_R + \phi_L)/2$ is local. A keyword for the answer is 'Lagrangian subalgebra'. In the case of the toric code, $a + b = \partial \phi_R, a - b = \partial \phi_L$, and both $\cos\left(\frac{1}{2}(\phi_R + \phi_L)(x)\right) = e^{\mathbf{i} \int^x a} + h.c.$ and $\cos\left(\frac{1}{2}(\phi_R - \phi_L)(x)\right) = e^{\mathbf{i} \int^x b} + h.c.$ are local. These two choices correspond to boundaries on the toric code where e and m are condensed, respectively.

Non-abelian CS theory. So far we've talked about CS theory with gauge group $\mathsf{U}(1)^n$. CS theory with more general gauge groups G, such as a non-abelian Lie group, can also arise as an EFT for states of matter. The non-abelian CS action looks like[7]

$$S_{\text{CS}}[a] = \frac{k}{4\pi} \int_M \text{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

where now a is a Lie-algebra-valued one-form, *i.e.* $a = \sum_{A=1}^{\dim G} a^A T^A$ where T^A are generators of the Lie algebra, say in the fundamental representation.

Again invariance under large gauge transformations, $g : M \to G$, requires that k is quantized. The variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \text{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

under $a \to gag^{-1} - \partial g g^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi} d \text{tr} dg g^{-1} \wedge a + \frac{k}{12\pi} \text{tr} \left(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right).$$

[7]Full disclosure: in treating a as a Lie-algebra-valued one-form I am assuming that it is a connection on a trivial G-bundle on M. More generally, M must be covered by patches between which a is related by a gauge transformation. One way to robustly define the CS action is to realize $M = \partial N$ as the boundary of some 4-manifold N and use the fact that $\frac{1}{8\pi^2} \text{tr} f \wedge f = d\omega_{\text{CS}}$. Therefore the integral $\int_N \frac{1}{8\pi^2} \text{tr} f \wedge f = \int_M \omega_{\text{CS}} = S_{CS}[a]$ is perfectly well-defined. One shortcoming of this method is that not every M is the boundary of some N.

The first term is a total derivative integrates to zero on a closed manifold. Over any closed surface, the second term integrates to the winding number of the map $g : M \to \mathsf{G}$, and therefore the integral of the second term is an integer. We conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

A similar story holds for the edge modes on $M = \mathbb{R} \times \Sigma$ with $\partial\Sigma \neq 0$. Again we work in $a_0 = 0$ gauge, and the constraint $0 = \frac{\delta S}{\delta a_0} \propto f = da + a \wedge a$ is solved by $a = g^{-1}\tilde{d}g$, where $\tilde{d}$ is the spatial exterior derivative. Only g that approach $\mathbb{1}$ at the boundary of Σ are gauge redundancies, and so the boundary value of g is a physical degree of freedom. Plugging into the action, and adding a local boundary term because you can't stop me,

$$S_{\mathrm{CS}}[a = g^{-1}\tilde{d}g] + \int_{\partial\Sigma\times\mathbb{R}} v\mathrm{tr}a_x^2 = \mathrm{tr}\left(\int_{\partial\Sigma\times\mathbb{R}} \left(kg^{-1}\partial_t g g^{-1}\partial_x g + vg^{-1}\partial_x g g^{-1}\partial_x g\right)\right.$$
$$\left. + \int_{\Sigma\times\mathbb{R}} \frac{1}{12\pi} g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right).$$

The first two terms are just like in the abelian case. The third term is still written as a 3d integral, but it only depends on the boundary value of g. It is called a WZW term. The resulting $1+1$d field theory is a conformal field theory (CFT) called a chiral G_k WZW model. The central charge for $G = \mathsf{SU}(N)$ at level k is

$$c = \frac{k \dim G}{k+N}.$$

For non-abelian G, G_k CS theory (at least for $k > 1$) realizes non-abelian topological order. For example, $\mathsf{SU}(2)_2$ seems to be a description of the (non-abelian) Moore-Read state (see *e.g.* p. 45 of this useful review).

2.3 Topological states without topological order

Even without topological order, there can be phases distinct from the trivial phase. One way in which they can be distinguished is by what happens if we put them on a space with boundary, so that there is a spatial interface with the trivial phase. A very rough (and not entirely correct) idea is that if the gap must close along the path to the trivial phase, then the coupling must pass through the wall of gap-closing at the edge of the sample. (We know that this argument is not entirely correct because there are nontrivial phases (like the toric code) that can have gapped interfaces with the trivial phase.) Phases that are distinguished in this way include integer quantum Hall states, topological insulators, and, more generally, symmetry-protected topological (SPT) phases such as the Haldane phase of the spin-one chain, or polyacetylene.

An example that is easy to see from what we've already done is integer quantum Hall states. Consider a system governed by the effective action (2.7) with $\det K = 1$. From our calculation of the ground state degeneracy we see that such a state has a unique groundstate on a closed surface of any genus. This is what is called an *invertible* theory. It is called this because it has an inverse under the operation of stacking.

2.4 Beyond Landau?

It seems that all of these examples transcend the Landau Paradigm. My goal here is not to use the Landau Paradigm as a straw man, but rather to pursue it in earnest. The idea is that by suitably refining and generalizing our notions of symmetry, we can incorporate all of these 'beyond-Landau' examples into a *Generalized Landau Paradigm*. There are two crucial ingredients, which work in concert: anomalies and generalized symmetries.

In these lectures, I am speaking of actual symmetries of physical systems, sometimes called 'global symmetries'. They act on the Hilbert space and take one state to another. In contrast, there is no such thing as 'gauge symmetry'. In a gauge theory, the gauge invariance is a redundancy of a particular description of the system, and is not preserved by relabelling degrees of freedom. For example, dualities (equivalences of physical observables at low energies) often relate a gauge theory with one gauge group to a gauge theory with a distinct gauge group. A familiar example in condensed matter physics is the duality between the XY model and the abelian Higgs model in 2+1 dimensions [14, 15], but there are many others, *e.g.* [16]. This complaint about terminology hides an abyss of human ignorance: if someone hands you a piece of rock and asks whether its low-energy physics is described by some phase of a gauge theory, how will you tell? It is certainly true that phases realizable by gauge theory go beyond other constructions with only short-ranged entanglement; this begs for a characterization of these phases that transcends a description in terms of redundancies. Higher-form symmetries offer such a characterization for some such phases.

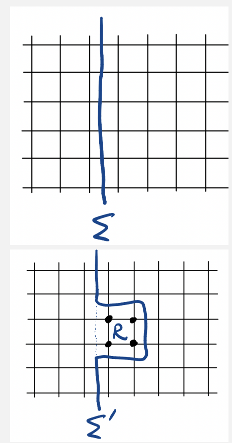
I want to highlight early attempts to understand topological order [17, 18], and the gaplessness of the photon [19] as a consequences of generalized symmetry, as well as early appearances of generalized symmetries in the string theory literature [20, 21, 22]. Other papers that have explicitly advocated for the utility of a generalized Landau Paradigm include [23, 24, 25, 26, 27].

3 The Generalized Landau Paradigm

3.1 What is a symmetry of a quantum many-body system?

An old-fashioned symmetry is an action on the degrees of freedom. Noether's theorem relates symmetries to **topological defect operators** $U_g(\Sigma)$ (or symmetry operators). The fact that the symmetry actions form a group implies that these operators enjoy the "fusion rule" $U_g(\Sigma)U_{g'}(\Sigma) = U_{gg'}(\Sigma)$ (up to possible interesting phases). Since the discussion of topological defect operators can be pretty abstract, let's discuss a very concrete example.

A very concrete example of a topological defect operator: Let's think about the nearest-neighbor Ising model on a Euclidean lattice of any dimension: $S[\sigma] = \sum_{\langle xy \rangle} J_{xy}\sigma_x\sigma_y$.



The topological defect operator $U_{-1}(\Sigma)$ is an instruction to flip the sign of J for any bond crossing Σ. (The -1 is the nontrivial element of the group $\mathbb{Z}_2 = \{1, -1\}$ in multiplicative notation.) That is:

$$\langle \cdots U_{-1}(\Sigma) \rangle = Z^{-1} \sum_{\{\sigma\}} e^{-S}|_{J_\ell \to -J_\ell \text{ if } \Sigma \text{ crosses } \ell} \ . \tag{3.1}$$

If $\Sigma' - \Sigma = \partial R$, $U_{-1}(\Sigma)$ and $U_{-1}(\Sigma')$ are related by redefining $\sigma_x \to -\sigma_x$ for $x \in R$, So U is topological. This implies that σ_x is charged:

$$\langle \cdots U_{-1}(\Sigma) \rangle = - \langle \cdots U_{-1}(\Sigma') \rangle \ . \tag{3.2}$$

The interesting realization of recent years is that it's useful to reverse our perspective on Noether's theorem. Topological defect operators are a sufficient condition for symmetry, and for most of our uses of symmetry we don't really care about the action on the degrees of freedom. This allows us to treat continuous and discrete symmetries on equal footing. Further, in your studies of field theory you may have noticed an awkward asymmetry between quantities that are conserved by Noether's theorem and quantities that are conserved because of topology of field space (like soliton numbers). This is awkward because we know that this distinction is not invariant under dualities, which can exchange field quanta and solitons. The new perspective is clearly better because it treats Noether symmetries and topological symmetries on equal footing. And most importantly it allows generalizations.

So from the old-fashioned point of view, a symmetry implies a collection of operators

$\{U_g\}$ with the following properties

1. $[H, U_g] = 0$.
2. U_g is supported on a whole constant-time slice.
3. U_g are fully topological.
4. $\{U_g\}$ form a group $U_{g_1} U_{g_2} = U_{g_1 g_2}$.

Of these properties, the only one we really can't give up is the first one[8]. Let's start by giving up the second.

3.2 Higher-form symmetries

Since these lectures are taking place in the last week of a month-long school on generalized symmetries, I will try to be brief here. The concept of higher-form symmetry that we review here was explained in [28, 23]. It is easiest to introduce using a relativistic notation, though I do not want to assume Lorentz invariance. Indices μ, ν run over space and time.

Let's begin by considering the familiar case of a continuous 0-form symmetry. Noether's theorem guarantees a conserved current J_μ satisfying $\partial^\mu J_\mu = 0$. In the useful language of differential forms, this is $d \star J = 0$, where $\star$ is the Hodge duality operation[9]. This continuity equation has the consequence that the *charge* $Q_\Sigma = \int_{\Sigma_{D-1}} \star J$ is independent of the choice of time-slice Σ. (Σ here is a closed d-dimensional surface, of codimension one in spacetime.) Notice that this is a topological condition. Q_Σ commutes with the Hamiltonian, the generator of time translations, and therefore so does the unitary operator $U_\alpha = e^{i\alpha Q}$, which we call the *symmetry operator*[10].

If the charge is carried by particles, Q_Σ counts the number of particle worldlines piercing the surface Σ (as in Fig. 2, left), and the conservation law $\dot{Q} = 0$ says that charged particle worldlines cannot end except on charged operators. If instead of a $\mathsf{U}(1)$ symmetry, we only had a discrete $\mathbb{Z}_p$ symmetry we could simply restrict $\alpha \in \{0, 2\pi/p, 4\pi/p ... (p-1)2\pi/p\}$ in the symmetry operator U_α. In that case, particles can disappear in groups of p.

[8]An additional condition we should impose is that U_g respects locality, in particular that it maps local operators to local operators. For example, this condition rules out projectors onto eigenstates of H.

[9]The Hodge dual of a p-form ω_p on a D-dimensional space with metric $g_{\mu\nu}$ has components $(\star\omega_p)_{\mu_1 \cdots \mu_{D-p}} = \sqrt{\det g}\epsilon_{\mu_1 \cdots \mu_D} \omega_p^{\mu_{D-p+1} \cdots \mu_D}$, where indices are raised with the inverse metric $g^{\mu\nu}$ and $\epsilon_{\mu_1 \cdots \mu_D}$ is the antisymmetric Levi-Civita symbol.

[10]Throughout I will assume that the normalization is such that $Q \in \mathbb{Z}$, so that $\alpha \equiv \alpha + 2\pi$.

Objects charged under a 0-form symmetry are created by local operators. Local operators transform under the symmetry by $\mathcal{O}(x) \to U_\alpha \mathcal{O}(x) U_\alpha^\dagger = e^{\mathbf{i}q\alpha}\mathcal{O}(x), d\alpha = 0$, where q is the charge of the operator. The infinitesimal version is: $\delta\mathcal{O}(x) = \mathbf{i}[Q, \mathcal{O}(x)] = \mathbf{i}q\mathcal{O}(x)$.

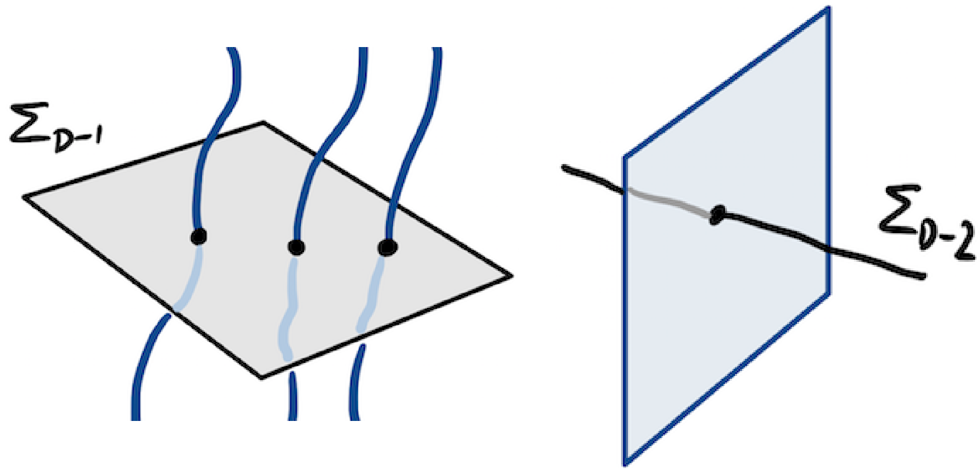


Figure 2: Left: In the case of an ordinary 0-form symmetry, the charge is integrated over a codimension-one slice of spacetime Σ_{D-1}, often a slice of constant time. All the particle worldlines (blue curves) must pass through this hypersurface. Right: The charge of a 1-form symmetry is integrated over a codimension-two locus of spacetime Σ_{D-2} (a string in the case of $D = 2+1$). This surface intersects the worldsheets of strings (blue sheet).

Now let us consider a continuous 1-form symmetry. This means that there is a conserved current which has two indices, and is completely antisymmetric:

$$J_{\mu\nu} = -J_{\nu\mu} \text{ with } \partial^\mu J_{\mu\nu} = 0. \tag{3.3}$$

We can regard J as a 2-form and write the conservation law (3.3) as $d \star J = 0$. As a consequence, for any closed codimension-*two* locus in spacetime Σ_{D-2}, the quantity $Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ. The analog of the symmetry operator is the unitary operator

$$U_\alpha(\Sigma) = e^{\mathbf{i}\alpha Q_\Sigma}. \tag{3.4}$$

Notice that reversing the orientation of Σ produces the adjoint of U: $U_\alpha(-\Sigma) = U_\alpha^\dagger(\Sigma)$.

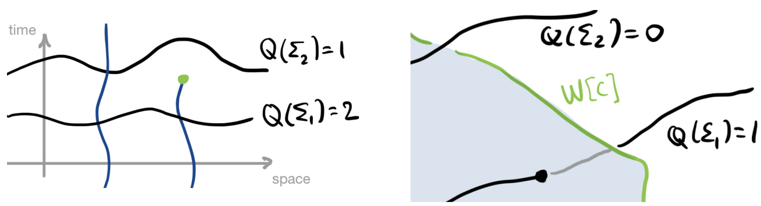
The charge Q_Σ in the 1-form case counts the number of charged string worldsheets intersecting the surface Σ (as in Fig. 2, right). The conservation law (3.3) then says that charged string worldsheets cannot end except on charged operators. The objects charged under a 1-form symmetry are loop operators, $W(C)$. Fixing a constant-time slice M_{D-1}, such a loop operator transforms as

$$W(C) \to U_\alpha(\Sigma) W(C) U_\alpha^\dagger(\Sigma) = e^{\mathbf{i}\alpha \oint_C \delta_\Sigma} W(C), \qquad d\delta_\Sigma = 0. \tag{3.5}$$

Here $\Sigma_{D-2} \subset M_{D-1}$ is any closed $(D-2)$-manifold, and δ_Σ is its Poincaré dual in M_{D-1}, in the sense that $\int_{M_{D-1}} \eta^{(D-2)} \wedge \delta_\Sigma = \int_{\Sigma_{D-2}} \eta^{(D-2)}$ for all η; $d\delta_\Sigma = 0$ because Σ has no boundary. The infinitesimal version of this transformation law is

$$\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i} q \# (\Sigma, C)\, W(C), \tag{3.6}$$

where $\#(\Sigma, C)$ is the intersection number in M.



In the case of a discrete 1-form symmetry, there is no current, but the symmetry operator $U_\alpha(\Sigma)$ is still topological. If the 1-form symmetry group is $\mathbb{Z}_p$, strings can disappear or end only in groups of p.

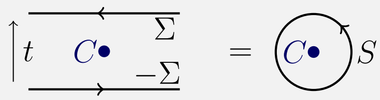
For general integer $p \geq -1$, a p-form symmetry $G^{(p)}$ means the existence of topological operators $U_\alpha(\Sigma_{D-p-1})$ labelled by a group element $\alpha \in G$ and a closed codimension-$(p+1)$ submanifold of spacetime[11]. For coincident submanifolds, these operators satisfy the "fusion rule" $U_\alpha(\Sigma) U_\beta(\Sigma) = U_{\alpha+\beta}(\Sigma)$. The operators charged under a p-form symmetry are supported on p-dimensional loci, and create p-brane excitations. The conservation law asserts that the $(p+1)$-dimensional worldvolume of these excitations will not have boundaries.

For $p \geq 1$, the symmetry operators commute with each other – higher-form symmetries are abelian [23]. To see this, consider a path integral representation of an expectation value with two symmetry operators $U(\Sigma_1)U(\Sigma_2)$ inserted on the same time slice t. The ordering of the operators can be specified in the path integral by shifting the left one to a slightly later time $t + \epsilon$. If $p \geq 1$, then $\Sigma_{1,2}$ have codimension larger than one, and their locations can be continuously deformed to reverse their order.

Action of higher form symmetry operators in Hamiltonian description. The relation between the spacetime point of view on higher-form symmetries and the Hamiltonian point of view common in the condensed matter literature

[11]For discussion of $p = -1$, see [29].

can be confusing. Above I have written the expression for the transformation as $U(\Sigma)W(C)U^\dagger(\Sigma)$. This operator ordering is obtained by placing the support of these operators on successive time slices. Since U is topological, from a spacetime point of view, the same result obtains if instead we deform the surfaces Σ and $-\Sigma$ to a single surface S in spacetime that *surrounds* the locus C, as illustrated here in cross-section:



(3.7)

The variation of the operator then depends on the *linking number* of S and C in spacetime.

Here is a recipe for thinking about this:

Choose a constant-time slice $\mathcal{M}_{D-1}$. For each $\Sigma_{D-p-1} \subset \mathcal{M}_{D-1}$,

$$W(C_p) \mapsto U_\alpha(\Sigma_{D-p-1})W(C_p)U_\alpha^\dagger(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha q \oint_{C_p} \delta_\Sigma} W(C_p) \tag{3.8}$$

where δ_Σ is the Poincaré dual of Σ_{D-p-1} in $\mathcal{M}_{D-1}$: $\int_{\mathcal{M}_{D-1}} \eta^{D-p-1} \wedge \delta_\Sigma = \int_{\Sigma_{D-p-1}} \eta^{D-p-1}, \forall \eta_{D-p-1}$. ($d\delta_\Sigma = 0$ since $\partial\Sigma = 0$.)

• $U_\alpha(-\Sigma) = U_{-\alpha}(\Sigma) = U_\alpha^\dagger(\Sigma)$.

• Infinitesimal version: $\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i}q\#_{\mathcal{M}_{D-1}}(\Sigma, C)W(C) = \mathbf{i}q\ell_{\mathcal{M}_{D-1}}(S, C)W(C)$

• If we assume Lorentz symmetry:

$$\mathcal{O}(C) \to U_\alpha(\Sigma)\mathcal{O}(C)U_\alpha^\dagger(\Sigma) = U_\alpha(S)\mathcal{O}(C) \tag{3.9}$$

in the euclidean path integral.

3.3 Physics examples of higher-form symmetries

A higher-form symmetry can be exact:

- Maxwell theory in $D = 3+1$ with electric charges but no magnetic charges has a continuous 1-form symmetry with current $J^{\mu\nu}_{(m)} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \equiv \frac{1}{2\pi}(d\tilde{A})^{\mu\nu}$. The statement that this current is conserved $\nabla_\mu J^{\mu\nu}_{(m)} = 0$ is the Bianchi identity expressing the absence of magnetic charge. The symmetry operator is $U^{(m)}_\alpha(\Sigma) = e^{\frac{\mathbf{i}\alpha}{2\pi}\int_\Sigma F}$. The fact that the charge operator $\int_\Sigma F$ depends only on the topological class of Σ is the magnetic gauss law – when Σ is contractible, it counts the number of magnetic monopoles inside. This symmetry shifts the dual gauge field $\tilde{A}$ by a flat connection; the charged line operator is the 't Hooft line, $W^{(m)}[C] = e^{\mathbf{i}\oint_C \tilde{A}}$.

In free Maxwell theory without electric charges, there is a second 1-form current, $J_{(e)} = F$ whose charged operator is the Wegner-Wilson line $W^{(e)}[C] = e^{\mathbf{i}\oint_C A}$. The symmetry operator for this 'electric' 1-form symmetry is $U_\alpha^{(e)}(\Sigma_2) = e^{\mathbf{i}\frac{2\alpha}{g^2}\int_{\Sigma_2} \star F}$, which (by canonical commutators) shifts the gauge field A by a flat connection.

- Pure $\mathsf{SU}(N)$ gauge theory or $\mathbb{Z}_N$ gauge theory or $\mathsf{U}(1)$ gauge theory with charge-N matter has a $\mathbb{Z}_N$ 1-form symmetry, called the 'center symmetry'. The charged line operator is the Wegner-Wilson line in the minimal irrep, $W[C] = \mathrm{tr} P e^{\mathbf{i}\oint_C A}$.

... or it can be emergent.

- When we spontaneously break a 0-form U(1) symmetry in $d = 2$, there is an emergent 1-form U(1) symmetry whose charge counts the winding number of the phase variable φ around an arbitrary closed loop C, $Q[C] = \oint_C d\varphi$. It is conserved because $d(d\varphi) = 0$ if φ is single-valued. In d spatial dimensions, this produces a $(d-1)$-form symmetry. The charged operator creates a vortex (in $d = 2$, or a vortex line or sheet in $d > 2$), which makes a codimension-two locus where φ is not single-valued, so $d^2\varphi \neq 0$. Unlike the examples above, this symmetry is generally not an exact symmetry of a microscopic Hamiltonian for a superfluid; it is explicitly broken by the presence of vortex configurations. More on this example and its consequences for superfluid physics in §3.12.

- There is a sense in which the 3d Ising model has a $\mathbb{Z}_2$ 1-form symmetry reflecting the integrity of domain walls between regions of up spins and regions of down spins. The charged line operator is the Kadanoff-Ceva disorder line [30] – the boundary of a locus along which the sign of the Ising interaction is reversed (for a review, see [31]). But because a domain wall is always the boundary of some region, no states are charged; relatedly, the disorder line is not a local string operator. If we gauge the $\mathbb{Z}_2$ symmetry of the Ising model, the disorder line becomes the Wegner-Wilson line of the resulting $\mathbb{Z}_2$ gauge theory, and this theory has a genuine 1-form symmetry.

- Spin liquids, FQHE... as we'll see.

3.4 Spontaneous symmetry breaking

Anything we can do with ordinary (0-form) symmetries, we can do with higher-form symmetries. In particular, they can be spontaneously broken. A symmetry is spontaneously broken if a ground state (more generally an equilibrium state) is not invariant.

SSB $\Leftrightarrow$ LRO.
There exists a charged operator O with $\langle\psi|\, O\,|\psi\rangle \neq 0$ (long-range order) if and only if $|\psi\rangle$ is not invariant under the symmetry (SSB).
Proof:
$\boxed{\Rightarrow}$ We'll prove the contrapositive (the state is symmetric means that no charged operator has an expectation value). Assume by way of contradiction that the state is a stationary state of the symmetry operator, $S\,|\psi\rangle = e^{\mathbf{i}\alpha}\,|\psi\rangle$. Then for any charged operator $SO = e^{\mathbf{i}\gamma}OS$, $\gamma \notin 2\pi\mathbb{Z}$, *i.e.* $O = S^\dagger O S e^{\mathbf{i}\gamma}$,

$$\langle\psi|\, O\,|\psi\rangle = e^{\mathbf{i}\gamma}\,\langle\psi|\, S^\dagger O S\,|\psi\rangle = e^{\mathbf{i}\gamma}\,\langle\psi|\, O\,|\psi\rangle \tag{3.10}$$

which says $\langle\psi|\, O\,|\psi\rangle = 0$.
$\boxed{\Leftarrow}$ [a] For any region X, we can write its reduced density matrix as

$$\rho_X = \mathrm{tr}_{\bar{X}}|\psi\rangle\langle\psi| = \sum_I \langle O_I\rangle\, O_I \tag{3.11}$$

where $\bar{X}$ is the complement of X and $\{O_I\}$ is a basis of hermitian operators on X orthonormal under the Hilbert-Schmidt norm $\mathrm{tr}O_IO_J = \delta_{IJ}$. If no charged operator has a nonzero expectation value, then the sum only contains neutral operators. But then $S\rho_X S^\dagger = \rho_X$, meaning that the state is invariant.

[a] This argument was explained to me by Tarun Grover.

A comment about the preceding result: This argument makes no assumptions about the support of the symmetry operators. The argument for LRO $\implies$ SSB also says nothing about the support of the charged operators – they actually needn't be operators of the correct dimension indicated above[12].

To appreciate the consequences of SSB for higher-form symmetries, let's spend a little time reviewing the story for 0-form symmetries. One way to characterize the unbroken phase of a 0-form symmetry is that correlations of charged operators are short-ranged, meaning that they decay exponentially with the separation between the operators

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0)\right\rangle \overset{x\to\infty}{\sim} e^{-m|x|}. \tag{3.12}$$

A language that will generalize is to regard the two points at which we insert a charged operator and its conjugate as an S^0, a zero-dimensional sphere, and the separation between the points as the size of the sphere. The broken phase for 0-form symmetry

[12] Thanks to Sal Pace for raising this question.

can be diagnosed by long-range correlations:

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \right\rangle \overset{x\to\infty}{\sim} \left\langle \mathcal{O}^\dagger(x) \right\rangle \left\langle \mathcal{O}(0) \right\rangle + \cdots \ , \tag{3.13}$$

independent of the size of the S^0.

For a p-form symmetry, the unbroken phase is also when correlations of charged operators are short-ranged, and decay when the charged object grows. For a 1-form symmetry, this is when the charged loop operator exhibits an area law:

$$\langle W(C) \rangle \sim e^{-T_{p+1}\mathrm{Area}(C)}, \tag{3.14}$$

where Area(C) is the area of the minimal surface bounded by the curve C. In the case of electricity and magnetism, an area law for $\left\langle W^E(C) \right\rangle$ is the superconducting phase.

The broken phase for a p-form symmetry is signalled by a failure of the expectation value of the charged operator to decay with size. For a 1-form symmetry, this is when the charged loop operator exhibits a perimeter law:

$$\langle W(C) \rangle = e^{-T_p \mathrm{Perimeter}(C)} + \cdots \ . \tag{3.15}$$

The coefficient T_p can be set to 0 by modifying the definition of $W(C)$ by counterterms local to C (for $p = 1$, this is mass renormalization of the probe particle), so (3.15) says that a large loop has an expectation value.

SSB of higher-form symmetry has been a fruitful idea. The fact that charged operators have long-range correlations means that the generators of the symmetry act nontrivially on the groundstate – the argument in the box above was not special to 0-form symmetry. In the next two subsections, I'll illustrate the consequences in the case of discrete and continuous higher-form symmetries, respectively.

3.5 Topological order as SSB

Suppose we spontaneously break a discrete higher-form symmetry. The generators of the broken part of the higher-form symmetry commute with the Hamiltonian and take a groundstate to a different groundstate. These groundstates are therefore related to each other by the action of an extended operator, rather than by a local operator. But this is precisely a definition of topological order: the presence of a groundstate subspace of *locally indistinguishable* states, as in (2.2) [23, 32].

Let's think about the example of $\mathbb{Z}_p$ gauge theory (whose solvable limit is the toric code [7]) in D spacetime dimensions. This is a system with $\mathbb{Z}_p$ 1-form symmetry with symmetry operators $U(M_{D-2})$, supported on a $(D-2)$-dimensional manifold, and charged operators $V(C_1)$, supported on a curve. In terms of the toric code variables, we can be completely explicit. On each link we have a p-state system on which act the Pauli operators $Z = \sum_{k=0}^{p} \omega^k |k\rangle\langle k|$ and $X = \sum_{k=0}^{p} |k+1\rangle\langle k|$ (where $\omega \equiv e^{2\pi \mathbf{i}/p}$ and the arguments of the kets are understood mod p). Then $V(C) = \prod_{\ell \in C} Z_\ell$, $U(M) = \prod_{\ell \perp M} X_\ell$, where we regard M as a surface in the dual lattice, and $\ell \perp M$ indicates all links crossed by the surface M. Their algebra is

$$U^m(M)V^n(C) = e^{2\pi \mathbf{i} \frac{mn}{p} \#(C,M)} V^n(C)U^m(M) \tag{3.16}$$

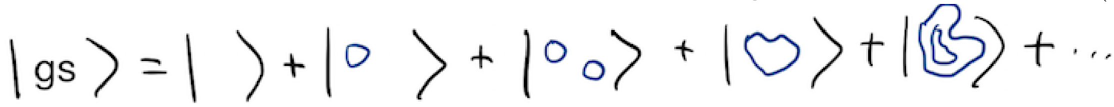
where $\#(C, M)$ is the intersection number of the curve C with the surface M. This is the algebra of electric strings and magnetic flux surfaces in $\mathbb{Z}_p$ gauge theory. Deep in this gapped phase, $H = 0$, and there is a description in terms of topological field theory. A simple realization is BF theory of a 1-form potential a and $(D-2)$-form potential b (generalizing (2.19)) with action

$$S[b, a] = \frac{p}{2\pi} \int_D b \wedge da = \frac{p}{2\pi} \int d^D x \epsilon^{\mu_1 \cdots \mu_D} b_{\mu_1 \cdots \mu_{D-2}} \partial_{\mu_{D-1}} a_{\mu_D} \tag{3.17}$$

in terms of which

$$U^n(M) = e^{\mathbf{i}n \oint_M b_{D-2}}, \quad V^m(C) = e^{\mathbf{i}m \oint_C a}. \tag{3.18}$$

The algebra (3.16) follows from canonical commutation relations in this gaussian theory. Since $V(C)$ has a perimeter law in the deconfined phase, the charged objects whose condensation breaks the 1-form symmetry are the loops of electric flux. (Recall that an excitation is condensed if the operator that creates it has an expectation value.) This is consistent with the explicit form of the toric code groundstate wavefunction(s)



Another example is the Laughlin fractional quantum Hall states. So far in our discussion the symmetry operators for a 1-form symmetry with group A form a representation of A on the 1-cycles of space, Z, *i.e.* a linear map $U : Z \to U(1)$, where the representation operators commute $U(M)U(M') = U(M')U(M)$. This relation can be generalized to allow for phases – *i.e.* a projective representation. Consider a system in $D = 2+1$ with a $\mathbb{Z}_k$ 1-form symmetry that is realized projectively in the following sense:

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn \#(C,C')}{k}} U^n(C')U^m(C) \tag{3.19}$$

where $\#(C, C')$ is the intersection number of the two curves C, C' in space. Regarding $U(C)$ as the holonomy of a charged particle along the loop C, this is the statement that flux carries charge. Representing this algebra nontrivially gives k groundstates on T^2. This algebra, too, has a simple realization via abelian Chern-Simons theory, $S[a] = \frac{k}{4\pi} \int a \wedge da$, with $U^m(C) = e^{\mathbf{i}m \oint_C a}$.

The algebra in (3.19) is a further generalization of 1-form symmetry, in that the group law is only satisfied up to a phase. As we will discuss in §3.12, it is an example of a 1-form symmetry anomaly.

The preceding discussion applies to *abelian* topological orders. In this context, abelian means that the algebra of the line operators transporting the anyons forms a group, which must be abelian by the argument above. In §3.14 we discuss the further generalization that incorporates non-abelian topological orders.

3.6 Photon as Goldstone boson

What protects the masslessness of the photon? The case of quantum electrodynamics (QED) is the most visible version of this question; the same question arises in condensed matter as: why are there $\mathsf{U}(1)$ spin liquid phases, with an emergent photon mode?

Higher-form symmetries provide a satisfying answer to this question (unlike appeals to gauge invariance, which is an artifact of a particular description): the gaplessness of the photon can be understood as required by spontaneously-broken $\mathsf{U}(1)$ 1-form symmetry [19, 23, 24, 33], as a generalization of the Goldstone phenomenon.

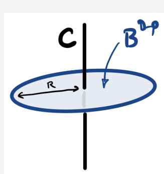
Here is the proof of the p-form Goldstone theorem from [24]. A continuous p-form symmetry implies a $p+1$-form current J such that

$$U_\alpha(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha \int_{\Sigma_{D-p-1}} \star J}. \tag{3.20}$$

A covariant form of the statement that the $p+1$-form current J is conserved and that $W[C]$ is a charged operator is the Ward identity

$$(d \star J(x)) W[C] = \mathbf{i} q \delta_C(x) W[C] \tag{3.21}$$

where the $D-p$-form delta function (Poincaré dual) $\delta_C(x)$ satisfies $\int b_p \wedge \delta_C(x) = \int_C b_p$ for any p-form b_p, and J is the $p+1$-form current. Let's consider the broken phase and choose $W[C]$ to have the multiplicative normalization where $\langle W[C] \rangle = c$, so no perimeter law.
Take C to be an infinite flat p-plane and integrate the BHS of (3.21) with respect to x over a $(D-p)$-ball B^{D-p} of radius R that intersects C at a single point:



$$W[C] \int_{\partial B^{D-p}} \star J = \mathbf{i} q W[C] \ . \tag{3.22}$$

The boundary of the ball ∂B^{D-p} is a $D-p-1$ sphere linked with C in spacetime, a Gaussian surface. Taking expectation values of the BHS we have

$$\langle J(R)W[C]\rangle \sim \frac{\mathbf{i}qc}{R^{D-p-1}}, \tag{3.23}$$

a power-law correlation, implying the presence of a gapless mode.

Here is a perspective on the zero-form version of the Goldstone theorem. Given a continuous zero-form symmetry with current j_μ, we can couple to a background gauge field $\mathcal{A}$ by adding to the Lagrangian $\Delta L \ni j_\mu \mathcal{A}^\mu$. If the symmetry is spontaneously broken, the effective Lagrangian will contain a Meissner term proportional to $\mathcal{A}^2$. But the effective action must be gauge invariant, and this requires the presence of a field that transforms nonlinearly under the $\mathsf{U}(1)$ symmetry: $\varphi \to \varphi + \lambda, \mathcal{A} \to \mathcal{A} - d\lambda$; this is a global symmetry if $d\lambda = 0$. Altogether, the effective Lagrangian must contain a term of the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4\pi g}\left(d\varphi + \mathcal{A}\right)^2 \tag{3.24}$$

(where by $(\omega)^2$ I mean $\omega_p \wedge \star\omega_p = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\omega^{\mu_1\cdots\mu_p}$). The coefficient $\frac{1}{4\pi g}$ is the superfluid stiffness.

The analog for a continuous 1-form symmetry works as follows. The current is now a two-form, so the background field must be a two-form gauge field $\mathcal{B}_{\mu\nu}$ and the coupling is $\Delta L \ni J_{\mu\nu}\mathcal{B}^{\mu\nu}$. The same logic implies that the effective action for the broken phase must contain a term

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2g^2}\left(da + \mathcal{B}\right)^2 \tag{3.25}$$

where the Goldstone mode a is a 1-form that transforms nonlinearly $a \to a + \lambda, \mathcal{B} \to \mathcal{B} - d\lambda$; this is a global symmetry if $d\lambda = 0$. Setting the background field $\mathcal{B} = 0$, we recognize this as a Maxwell term for a. The coupling strength g is determined by the analog of the superfluid stiffness.

For p-form $\mathsf{U}(1)$ symmetry, we conclude by the same logic that there is a massless p-form field a with canonical kinetic term

$$S_{\text{Max}}[a] = -\frac{1}{2g^2}\int da \wedge \star da. \tag{3.26}$$

Returning to QED, we see that the familiar Coulomb phase is the SSB phase for the $\mathsf{U}(1)$ 1-form symmetry. The unbroken phase is the superconducting phase, where the photon has short-ranged correlations. (In an ordinary superconductor, where the Cooper pair has charge two, a $\mathbb{Z}_2$ subgroup of the 1-form symmetry remains broken.)

As in the case of 0-form SSB, the broken phase can be understood via the condensation of charged objects; in this case the charged objects are the strings of electric flux [9, 34]. Notice that the presence of charged matter, on which these strings can end, and which therefore explicitly breaks this symmetry, does not necessarily destroy the phase. We'll comment on this robustness more in §3.8. In fact, because of electromagnetic duality, the Coulomb phase is the broken phase for *either* the electric 1-form symmetry or the magnetic 1-form symmetry [23].

[End of Lecture 2]

3.7 Effects of IR fluctuations

The analog of the Hohenberg-Coleman-Mermin-Wagner (HCMW) theorem for higher-form symmetries [17, 18, 23, 33] is interesting. As in the proof of the HCMW theorem, we suppose that a p-form $\mathsf{U}(1)$ symmetry in D spacetime dimensions is spontaneously broken and that there is therefore a Goldstone mode, a massless p-form field a. Then we ask if indeed the symmetry is broken by evaluating the expectation value of a charged operator W_C, including the fluctuations of the would-be-Goldstone mode a. We can choose C to be a copy of $\mathbb{R}^p \subset \mathbb{R}^D$ so that we can do the integrals, and the result is (see [33] for a discussion of a convenient gauge choice)

$$\langle W_C \rangle = Z^{-1} \int [Da] e^{-S_{\text{Max}}[a] + \mathbf{i} \int_C a} \simeq \exp\left(-\frac{1}{2} g^2 L^p \int \frac{đ^{D-p}k}{k_\perp^2} \right) \tag{3.27}$$

where $đk \equiv \frac{dk}{2\pi}$ and $k_\perp$ is the momentum transverse to C. The integral in the exponent of (3.27) is IR divergent when $D - p \leq 2$. As in the $p = 0$ case, we interpret this as the statement that the long-wavelength fluctuations of the would-be-Goldstone mode necessarily destroy the order. (For $D - p \geq 2$, the integral is UV divergent. This divergence can be absorbed in a counterterm locally redefining the operator $W_C \to W_C e^{-\delta T \int_C d^p x}$, which can be interpreted as a renormalization of the tension T of the charged brane.) In the marginal case of $p = D - 2$, the long-range order is destroyed, but $\langle W_C \rangle$ decays as a power-law in the loop size, rather than an exponential; this is a higher-form analog of algebraic long-range order in $D = 2$.

The calculation above is independent of compactness properties of the Goldstone form field, in the sense that in (3.27) we just did a Gaussian integral over the topologically-trivial fluctuations of a. In the marginal case $D = 2{+}1, p = 1$, if we treat a as a compact $\mathsf{U}(1)$ gauge field, SSB of the 1-form symmetry is avoided instead because monopole instantons generate a potential for the dual photon $d\sigma = \star da/2\pi$ [35]. This mechanism generalizes to any case with $D - p = 2$ [33].

[17, 18] interpret such results as a generalization of Elitzur's theorem on the unbreakability of local gauge invariance [36].

3.8 Robustness of higher-form symmetries

We are used to the idea that consequences of emergent (aka accidental) symmetries are only approximate: explicitly breaking a spontaneously-broken continuous 0-form symmetry gives a mass to the Goldstone boson.

This raises a natural question. The existence of magnetic monopoles with $m = M_{\text{monopole}}$ explicitly breaks the 1-form symmetry of electrodynamics:

$$\partial^\mu J^E_{\mu\nu} = j^{\text{monopole}}_\nu \ .$$

If the photon is a Goldstone for this symmetry, does this mean the photon gets a mass? Perhaps surprisingly, the answer is 'no' (early discussions of the robustness of broken higher-form symmetries using different words include [37, 2, 38]). This is a way in which zero-form and higher-form symmetries are quite distinct.

A cheap way to see that 'no' is the right answer is by dimensional analysis. How does the mass of the photon m_γ depend on the mass of the magnetic monopole, M_{monopole}? Suppose all the electrically charged matter (such as the electron) is very heavy or massless. We must have $m_\gamma \to 0$ when $M_{\text{monopole}} \to \infty$. But there is no other mass in the problem to make up the dimensions.

A slightly less cheap way to arrive at this answer is by dimensional reduction. If we put quantum electrodynamics (QED) on a circle of radius R, we arrive at low energies at abelian gauge theory in $D = 2+1$, which is confined by monopole instantons [35]. The monopole instantons arise from euclidean worldlines of magnetic monopoles wrapping the circle, and so their contribution to the mass of the (2+1)d photon is

$$m_\gamma(R) \sim e^{-RM_{\text{monopole}}}. \tag{3.28}$$

The polarization of the photon along the circle gets a mass from euclidean worldlines of charged matter (like the electron) wrapping the circle, so its mass is

$$m_4(R) \sim e^{-Rm_e}. \tag{3.29}$$

But now the point is simply that when $R \to \infty$, both of these effects go away and the (3+1)d photon is massless.

A third argument is that operators charged under a 1-form symmetry are loop operators – they are not local. We can't add non-local operators to the action at all. This argument is not entirely satisfying, since on the lattice even the action for pure gauge theory is a sum over (small) loop operators. The question is whether the dominant contributors in this ensemble of charged loop operators grow under the RG. [39] describes a toy calculation to address this question: begin in a phase with a perimeter law $\langle W[C]\rangle \sim t^{\text{length}[C]}$ and consider adding to the action $g\int [dC] W[C] +$ $h.c.$ in perturbation theory in g. Regularizing on the lattice and neglecting collisions of loops, the result is the same as integrating out a charged particle whose mass is determined by the parameter t:

$$\left\langle \sum_C W[C] \right\rangle = -\frac{L^D}{2}\int đ^D q \log\left(1 - 2t\sum_{\mu=1}^{D}\cos q_\mu\right) . \tag{3.30}$$

Thus, for small enough $t \leq \frac{1}{2D}$ there is an IR divergence indicating a transition to a phase where the charged particle is condensed. Until that happens, the SSB phase survives. A useful slogan extracted from this calculation is that a loop operator becoming relevant (changing the IR physics) indicates the onset of a Higgs transition.

The discrete analog of this phenomenon is instructive. In the solvable toric code model, the discrete 1-form symmetries are exact. But in the rest of the deconfined (spontaneously broken) phase, they are *emergent*, but still spontaneously broken, and still imply a topology-dependent groundstate degeneracy that becomes exact in the thermodynamic limit. A rigorous proof of this [38] constructs (slightly thickened) string operators by quasi-adiabatic continuation.

Known forms of topological order in $D \leq 3+1$ have the property that at any $T > 0$ they are smoothly connected to $T = \infty$ (a trivial product state). If the 1-form symmetry is emergent, then as soon as $T > 0$, a mass *is* generated for the photon (by the argument above, with the circle regarded as the thermal circle, so that $R = 1/T$), and the state is smoothly connected to $T = \infty$.

We do know an example of a topologically ordered phase that is stable at $T > 0$, namely the two-form toric code in $D = 4+1$ [40]. In the $\mathsf{U}(1)$ version of this theory, the masslessness of the two-form gauge field should survive explicit short-distance breaking of the $\mathsf{U}(1)$ two-form symmetry, even at finite temperature. The reason is that a theory with a two-form symmetry on a circle still has a 1-form symmetry.

We conclude that the consequences of higher-form symmetries are more robust to explicit breaking than zero-form symmetries [39, 41, 42, 43]. This is a double-edged sword. One the one hand, it means that even though higher-form symmetries are rarely microscopically exact, they can be generic. On the other hand, it means that the Generalized Landau Paradigm is not as simple as the old-fashioned one. In classifying phases of matter, we can't just worry about the exact symmetries of the microscopic Hamiltonian. We also have to worry about symmetries that may emerge.

3.9 Mean field theory

Landau-Ginzburg mean field theory is our zeroth-order tool for understanding symmetry-breaking phases and their neighbors. It is therefore natural to ask whether it has an analog for higher-form symmetries [39]. We focus on the simplest nontrivial case of a $\mathsf{U}(1)$ 1-form symmetry.

Earlier we reviewed the logic that produces this weapon in the zero-form case. The 1-form analog of the order parameter field $\phi(x)$ (which is a function from the space of points into a linear representation of G), is a functional $\psi[C]$ from the space of loops into a linear representation of G, a 'string field'. While $\phi(x)$ transforms under the zero-form symmetry as $\phi(x) \to \phi(x)e^{\mathbf{i}\alpha}$, with $d\alpha = 0$, the 1-form analog transforms like $\psi[C] \to \psi[C]e^{\mathbf{i}\oint_C \Gamma}$, with $d\Gamma = 0$.

To write an action for such a field requires the analog of a derivative, which compares its values on nearby loops. Such an 'area derivative' was discovered in the study of loop-space formulations of gauge theory [44] (see Fig. 3, left). The analog of integrating the action over spacetime $\int d^D x$ is integrating over the space of loops $\int [dC]$. The most general action consistent with the symmetries then takes the form

$$S[\psi] = \int [dC] \left(V(|\psi[C]|^2) + \frac{1}{2L[C]} \oint ds \frac{\delta\psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi] \ . \quad (3.31)$$

The last 'recombination term'

$$S_r[\psi] = \int [dC_{1,2,3}] \delta[C_1 - (C_2 + C_3)] \left(\lambda\psi[C_1]\psi^\star[C_2]\psi^\star[C_3] + h.c. \right) + \cdots \quad (3.32)$$

is not local in loop space, but is local in real space since it involves only a single integral over the center-of-mass of the loops. Here the delta function imposes the equality of loops regarded as integration domains (see Fig. 3, right). The $\cdots$ denote terms with more derivatives or more powers of ψ. Models similar to this Mean String Field Theory

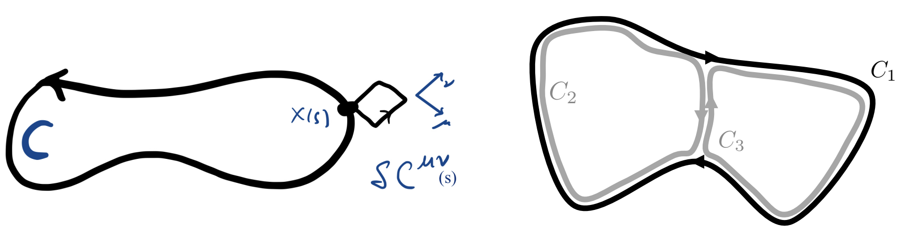


Figure 3: Left: a sketch of the definition of the area derivative $\frac{\delta}{\delta C_{\mu\nu}(s)}$. Right: The arrangement of loops involved in the topology-changing term S_r in the MSFT action.

(MSFT) have been considered before in various specific contexts [45, 46, 47, 48, 49].

The potential term $V(|\psi[C]|^2) = r|\psi[C]|^2 + u|\psi[C]|^4 + \cdots$ controls the low-energy behavior. If $r > 0$, we find an unbroken phase where $\psi[C] \simeq e^{-\sqrt{r}A[C]}$. When $r < 0$, the strings want to condense. The fluctuations around nonzero $|\psi|$ are all massive, except for the geometric mode $\psi[C] = ve^{\mathbf{i}\oint_C ds a_\mu(x(s))\dot{x}^\mu(s)}$, which describes a slowly-varying 1-form symmetry transformation, and in terms of which the action (3.31) reduces to the Maxwell action for a, with coupling $g^2 = \frac{1}{2v^2}$.

As in the zero-form case, another application of this mean field theory is to classify topological defects of the resulting ordered media. The conclusion for $G = \mathsf{U}(1)$ is that the only topological defect is the codimension-three magnetic monopole.

So far, we have discussed the case of a $\mathsf{U}(1)$ 1-form symmetry. The case of discrete symmetries can be approached by explicitly breaking the $\mathsf{U}(1)$ 1-form symmetry to a discrete subgroup. A term of the form

$$h \int [dC]\psi^k[C] + \text{h.c.} \tag{3.33}$$

breaks it down to $\mathbb{Z}_k$. In the broken phase, the effective action reduces to a continuum (BF) description of $\mathbb{Z}_k$ gauge theory.

The action (3.31) can be given a lattice definition and contact can be made with microscopic Hamiltonians in the following way. Zero-form mean field theory arises from a variational using a product state ansatz $|\Psi_\phi\rangle = \otimes_x |\phi(x)\rangle$; given a microscopic Hamiltonian, the variational energy $\langle\Psi_\phi| \mathbf{H} |\Psi_\phi\rangle = H[\phi]$ takes the form of the Landau-Ginzburg Hamiltonian.

Consider for definiteness a $\mathbb{Z}_2$ lattice gauge theory Hamiltonian, in the form

$$H_{\text{TC}} = -\infty \sum_{\text{sites},s} A_s - \Gamma \sum_{\text{plaquettes},p} B_p - g \sum_{\text{links},\ell} Z_\ell. \tag{3.34}$$

This acts on a Hilbert space that is a tensor product of qubits on the links of a cell complex; $A_s = \prod_{\ell\in s} Z_\ell$ and $B_p = \prod_{\ell\in p} X_\ell$. X and Z denote the Pauli operators. In the Z eigenbasis, we regard a link as covered by a segment of string if $Z = -1$. We take the coefficient of the star term A_s to infinity so that the loops are closed and there is an exact (electric) 1-form symmetry generated by $U(C) = \prod_{\ell\in C} X_\ell$. When $g = 0$, the groundstate is the uniform superposition over all collections of closed loops. g represents a tension for the electric strings; for large enough g/Γ, there is a transition to a confined phase.

The analog of a product state for the 1-form case is a many-body wavefunction on collections of loops determined by a function $\psi[C]$ on a single loop:

$$|\Psi_\psi\rangle =: e^{\sum_{c,\text{connected}} \psi[c]U[c]} : |0\rangle \tag{3.35}$$

where $U[c]\,|0\rangle = |c\rangle$ creates the loop c, and the normal-ordering symbol $: \cdots :$ is a prescription for dealing with overlapping loops. The variational energy for this state is a lattice Hamiltonian for the action (3.31) plus (3.33).

Brief comments on phase transitions. As in the 0-form case, we expect the mean-field description to break down near critical points, below the upper critical dimension. (The extension of the renormalization group to MSFT has not yet been attempted.) Dimensional analysis says that the string field ψ has dimension $(D-4)/2$ and hence u has mass dimension $8-D$, and λ has dimension $6-D/2$, which puts the upper critical dimension at 8 or 12, depending on which coupling matters. More significantly, the recombination term is a symmetric term *cubic* in the order parameter field, and we expect that it generically renders the transition first order. This is consistent with numerical work on deconfinement transitions in gauge theory in $D > 3$ (see *e.g.* [50] and references therein).

Notice that the string field has engineering dimension zero in $D = 4$. There are two possible notions of lower critical dimension, which coincide at $D = 2$ in the case of 0-form symmetries. One is the largest dimension where the HCMW theorem forbids symmetry breaking, which is $D = 3$ for 1-form symmetry. The other is the dimension in which the linearly-transforming field is classically dimensionless, which is $D = 4$ for 1-form symmetries. In the case of 0-form symmetry, this allows for the rich physics of the Berezinskii-Kosterlitz-Thouless transition, where there is a line of (free) fixed points (parameterized by g in (3.24)) that terminates when a symmetry-allowed operator becomes relevant. A universal prediction is the value of the stiffness at the transition, since as in (3.24), the stiffness determines the coupling.

For the special case of 1-form symmetry-breaking in $D = 4$, there is again a line of (free) fixed points, parameterized by the Maxwell coupling, as in (3.25). Consider the application of this picture to $\mathbb{Z}_k$ gauge theory, described by perturbing the MSFT action by (3.33). In the free theory, this operator can be argued [51] to have an anomalous dimension $\Delta_k(g) = \frac{g^2k^2}{32\pi^2}$; for large enough k, $\Delta_p(g)$ passes through 4 at some $g_c < \sqrt{4\pi}$, and we can interpret this as the location of a transition in the low-energy physics from a Coulomb phase to a phase with $\mathbb{Z}_k$ topological order. The prediction is again a universal jump in the 'superfluid stiffness', namely the value of the gauge coupling at the transition. Many of these ideas were anticipated by Cardy [52] without the benefit of the language higher-form symmetry.

There is a catch: this transition is observed in Monte Carlo simulations to actually be weakly first order (see [53] and references therein). Does that mean there is nothing universal to say? There is a reason the transition is *weakly* first order. The magnetic charge whose condensation drives the transition has a good dual description via the Abelian Higgs model. In this model, fluctuations drive the transition first order [54].

If the coupling is weak at the transition, this description is good and the transition is weakly first order. But, using the slogan of §3.8 (a loop operator becoming relevant means a Higgs transition), the mechanism of the previous paragraph determines the critical coupling and shows that it should be small at large k.

3.10 Anomalies

My motivation for including a discussion of anomalies here is twofold. One is that anomalies are a necessary ingredient in a suitably-generalized Landau Paradigm that incorporates all phases, in particular topological insulators and SPT phases. A second motivation is that, as I will review, the existence of anomalies makes symmetries much more useful for constraining the dynamics of a physical system, and their generalization to higher-form symmetries is therefore an essential step.

The historical, high-energy-physics perspective on anomalies starts from specifying a quantum field theory by a path integral

$$Z = \int [D(\text{fields})] e^{\mathbf{i}S[\text{fields}]}. \tag{3.36}$$

An anomaly is a symmetry of the action S that is not a symmetry of the path-integral measure. The first example found was the chiral anomaly, the violation of the axial current of a charged Dirac field (the symmetry that rotates left-handed and right-handed fermions with opposite phases)

$$\partial_\mu j^\mu_A = N \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \tag{3.37}$$

which controls the decay of the neutral pion into two photons.

A more concrete perspective arises if we consider the same kind of system on the lattice, in one dimension for simplicity: consider a tight-binding model of fermions hopping on a chain, at some small filling as in Fig. 4. In this case, there is no chiral symmetry at all at the lattice scale. It is an emergent symmetry, violated by the UV physics in a definite way. At low energies, the system is approximately described by the neighborhood of the two boundaries of the Fermi sea, giving a 1d massless Dirac fermion, with a chiral symmetry. But if we adiabatically apply an electric field E_x, every fermion increases its momentum and the chiral charge changes by

$$\Delta Q_A = \Delta(N_R - N_L) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi} e \int dt E_x(t) = \frac{e}{2\pi} \int \epsilon_{\mu\nu} F^{\mu\nu}. \tag{3.38}$$

The left hand side is $\Delta Q_A = \int \partial^\mu j^A_\mu$, and so (3.38) is the 2d version of the chiral anomaly:

$$\partial_\mu j^\mu_A = \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}. \tag{3.39}$$

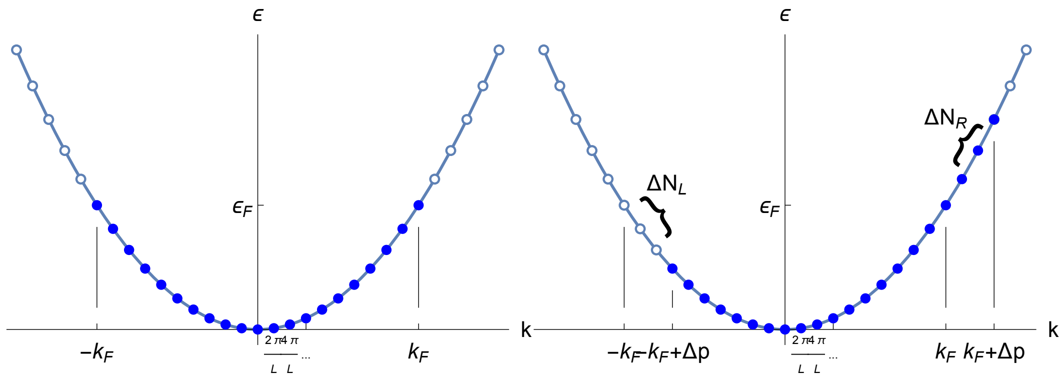


Figure 4: Left: Spectrum of a free-fermion tight-binding model in one dimension, near the bottom of the band at some small filling. Green circles indicate filled states. Right: The result of adiabatically applying an electric field. $N_{L/R}$ indicate the number of left-moving and right-moving excitations.

A reason for excitement about this phenomenon is that the coefficient N in (3.37) is an integer. This is the first hint that an anomaly is a topological phenomenon, a quantity that is RG invariant [55]. Much of physics is about trying to match microscopic (UV) and long-wavelength (IR) descriptions. That is, we are often faced with questions of the form "what could be a microscopic Hamiltonian that produces these phenomena?" and "what does this microscopic Hamiltonian do at long wavelengths?". Anomalies are precious to us, because they are RG-invariant information: any anomaly in the UV description must be realized somehow in the IR description.

The idea is that the existence of the anomaly means that the partition function varies by some particular phase under the anomalous symmetry, but an RG transformation must preserve the partition function. Any symmetric theory may be coupled to background gauge fields associated with the symmetry by a minimal coupling procedure, very sketchily by the replacement $\partial \rightsquigarrow \partial + A$. The anomaly is the phase acquired by the partition function $Z \to e^{\mathbf{i}\int \lambda \mathcal{A}} Z$ under a gauge transformation $A \to A + d\lambda$. (The minimal coupling procedure is ambiguous up to adding gauge-invariant terms (like F^2); this ambiguity does not affect the anomaly.)

Another useful perspective on anomaly is therefore as an obstruction to gauging the symmetry. Gauging a symmetry means creating a new system where the symmetry is a redundancy of the description, by coupling to gauge fields and then making them dynamical. If the symmetry is not conserved in the presence of background gauge fields, the resulting theory would be inconsistent.

Above I've described an example of an anomaly of a continuous symmetry. Discrete symmetries can also be anomalous.

Anomaly is actually a more basic notion than phase of matter: The anomaly is a property of the degrees of freedom (of the Hilbert space) and how the symmetry acts on them, independent of a choice of Hamiltonian. Multiple phases of matter can carry the same anomaly.

3.11 SPT phases and anomalies

The definition of gapped phases can be refined by studying only the space of Hamiltonians preserving some particular symmetry group G. Two phases that are distinct in this smaller space may nevertheless be connected by a gapped path in the larger space of non-symmetric Hamiltonians.

One way to define [56] a Symmetry-Protected Topological (SPT) phase is as a nontrivial phase of matter (with some symmetry G) without SSB or topological order (for a review, see [57]). SPT phases can be characterized by their edge states. The idea is that the edge theory has to represent an anomaly of the symmetry G. It is really this anomaly that labels the bulk phase. This phenomenon is called *anomaly inflow*.

As a simple example, consider an effective field theory for the integer quantum Hall effect, regarded as an SPT for charge conservation symmetry[13]. The charge conservation symmetry is associated, by Noether's theorem, with a conserved current j^μ, with $\partial_\mu j^\mu = 0$. In $D = 2+1$, this equation can be solved by writing $j^\mu = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho/(2\pi)$, in terms of a 1-form gauge field a_μ, with redundancy $a \to a + d\alpha$. As we discussed above, the leading effective action for such a field, in the absence of parity symmetry, is a Chern-Simons term [11, 58]:

$$S_{\text{IQH}}[a, \mathcal{A}] = \frac{1}{4\pi}\int_M \epsilon^{\mu\nu\rho}\left(a_\mu\partial_\nu a_\rho + 2\mathcal{A}_\mu\partial_\nu a_\rho\right) \tag{3.40}$$

where $\mathcal{A}$ is a background field for the charge conservation symmetry. Under $\mathcal{A} \to \mathcal{A} + d\lambda$, $\delta S_{\text{IQH}} = \int_{\partial M} \frac{\epsilon^{ij}}{4\pi} f_{ij}\lambda$. This is the contribution to the chiral anomaly from a single right-moving edge mode.

In terms of the definition of the anomaly as a variation of the partition function of the edge theory in the presence of background fields, the variation of the bulk action cancels the anomaly of the edge theory, so that the whole system is G symmetric. The edge theory cannot be trivial, since it has to cancel the variation of the bulk under the symmetry transformation: it has to be either [59]

[13] Actually, the integer quantum Hall phase is more robust, and survives explicit breaking of the charge conservation symmetry. It is protected by the gravitational anomaly manifested in the nonzero chiral central charge.

- gapless
- symmetry-broken
- or topologically ordered.

In particular, there cannot be a trivial gapped groundstate. These are the same conditions arising from the Lieb-Schultz-Mattis-Oshikawa-Hastings (LSMOH) theorem [60, 61] (for more recent developments, see *e.g.* [62]), and we can call this an LSMOH constraint.

A perhaps-simpler example is the free fermion topological insulator in $D = 3 + 1$, protected by charge conservation and time-reversal symmetry. In this case, the bulk effective action governs a single massive Dirac fermion; a boundary is an interface where the mass changes sign, at which a single Dirac cone arises. A single Dirac cone in $D = 2 + 1$ realizes the so-called parity anomaly. The fact that anomaly transcends a phase of matter is illustrated by the fact that, in the presence of interactions or disorder, there are other possible edge theories for the topological insulator.

There is by now a sophisticated (still conjectural) mathematical classification of SPTs for various G in various dimensions [63, 64] about which I will not say more here. My point is that we are still using their realization of symmetries to label these phases!

3.12 Anomalies of higher-form symmetries

Let's return to the example from §3.3 of the $(d - 1)$-form symmetry that arises in any superfluid phase [24, 25, 65]. The new current can be written as $(\star J)_\mu = \partial_\mu \varphi$. However, in the presence of a background gauge field $\mathcal{A}$ for the $\mathsf{U}(1)$ symmetry, the gauge-invariant current is instead

$$(\star J)_\mu = D_\mu \varphi \tag{3.41}$$

where $D_\mu \varphi = \partial_\mu \varphi - q\mathcal{A}_\mu$ is the covariant derivative. But this current is not conserved:

$$d \star J = -q\mathcal{F} \tag{3.42}$$

with $\mathcal{F} \equiv d\mathcal{A}$. This equation has a simple interpretation: applying an electric field leads to a supercurrent that increases linearly in time.

The symmetry violation in (3.42) is an example of a mixed anomaly between a 0-form symmetry and a $(d - 1)$-form symmetry, that arises automatically from SSB. Reference [25] shows a converse statement: any system with $\mathsf{U}(1)^{(0)} \times \mathsf{U}(1)^{(D-2)}$ symmetry with anomaly (3.42) contains a Goldstone boson in its spectrum. Since no long-range order is assumed, this is a more general statement than Goldstone's theorem – it applies even in $D = 2$.

This perspective can be used to demonstrate the existence of equilibrium states with non-dissipating current [65].

A direct 1-form generalization of Oshikawa's argument [60] appears in [66]. This is an example of a mixed anomaly between a 1-form symmetry and lattice translation symmetry.

We should give an example of an anomaly of a higher-form symmetry that does not involve zero-form symmetries. An example is provided by the theory of abelian anyons in $D = 2 + 1$, and is best understood by regarding an anomaly as an obstruction to gauging. Gauging a continuous 1-form symmetry means coupling the conserved current $J^{\mu\nu}$ to a *dynamical* two-form gauge field $b_{\mu\nu}$ by a term like $b_{\mu\nu}J^{\mu\nu}$. That is, gauging a symmetry means summing over all possible background fields. In the discrete case, this is the same as summing over the insertions of all possible symmetry operators. (In the continuous case, it also requires summing over connections that are not flat.)

Thus, gauging a 1-form symmetry in 2+1 dimensions means proliferating the worldlines of the associated anyons [23, 67]; this is 'anyon condensation' [68]. But it only makes sense to condense particles with bosonic self-statistics: condensation means essentially that the many-particle wavefunction is a constant, which manifestly has bosonic statistics. Therefore, a subgroup of a 1-form symmetry generated by line operators with nontrivial statistics cannot be gauged. We conclude that, in 2+1 dimensions, the 't Hooft anomaly of a 1-form symmetry is encoded in the self-statistics of the line operators, *i.e.* of the anyons. Thus, the algebra (3.19) is an example of a 1-form symmetry with an 't Hooft anomaly. Notice that from this point of view, nontrivial mutual statistics of a pair of anyon types a and b is a mixed 't Hooft anomaly: it does not stop us from gauging (*i.e.* condensing) a, but we cannot condense both simultaneously, since in the presence of the a condensate, b is confined. The algebra for discrete gauge theory (3.16) can also be regarded an example of an anomaly for higher-form symmetry because the charged operators V_n are also topological; so this is a 1-form symmetry and a $(D - 2)$-form symmetry with a mixed anomaly.

3.13 Subsystem symmetries and fracton phases

Above we have discussed p-form symmetries, described by symmetry operators acting on codimension-$(p + 1)$ submanifolds of spacetime. These operators were *flexible*, in the sense that their correlations only depend on their deformation class in spacetime (avoiding any charged operator insertions).

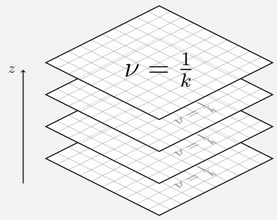
A distinct generalization of the notion of symmetry arises by defining symmetry operators acting independently on *rigid* subspaces of the space on which the system

Properties of symmetry operator	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	>1	>1	≥ 1
How topological is it?	fully	fully	not completely	fully
Fusion rule	group $g_1 \cdot g_2 = g_3$	group $g_1 \cdot g_2 = g_3$	group $g_1 \cdot g_2 = g_3$	category $\mathcal{D} \cdot \mathcal{D}^\dagger \neq 1$

Table 1: This table (from Shu-Heng Shao) gives a nice overview of further possible generalizations of the notion of symmetry.

is defined. That is, we can imagine that there is a different symmetry operator for each subspace, even in the same homology class, so that the symmetry operators are not topological, but still commute with the Hamiltonian. This is sometimes called a "faithful" symmetry [69] or *subsystem symmetry*. This generalization is not compatible with Lorentz invariance, since the operators are still topological in time.

An object charged under such a subsystem symmetry cannot leave the locus on which the symmetry is defined. This sort of restricted mobility of excitations is a defining property of fracton phases [5, 6]. A fracton phase with topological order can be identified as one that spontaneously breaks such a "faithful" higher-form symmetry [69, 70, 71]. Foliated fracton phases [72] like the X-cube model [73] spontaneously break a 'foliated 1-form symmetry' acting independently on each plane of a lattice [69].



A trivial example of a fracton phase can be made by stacking 2+1d topological states. For example, let's stack a bunch of copies of abelian quantum Hall states each extended in the xy plane, but separated in the z direction at locations $z = Ia$, $I = 1..L$. Each layer is described at low energies by abelian CS theory and the whole thing has the action

$$S[a_I] = \sum_I \int_{x,y} \frac{k}{4\pi} a_I da_I. \tag{3.43}$$

The anyons in each layer are fractons in the sense that they cannot escape their layer (they are specific kind of fracton called 'planeons'). Notice that with periodic boundary conditions, the GSD goes like k^L: the log GSD is linear in the system size. This is characteristic of 3+1D fracton phases.

We can make a more interesting class of fracton models simply by coupling the layers to each other. That is, consider instead the following action [74]:

$$S[a_I] = \sum_{IJ} \frac{K_{IJ}}{4\pi} \int_{x,y} a_I da_J. \tag{3.44}$$

If K is a quasi-diagonal integer matrix, this can arise as an effective description of coupled layers of quantum Hall states, and sometimes is gapped. Now the log GSD as a function of L is more interesting, but still has a linear envelope. The excitations still move in planes, but they can have interesting braiding statistics (encoded in inverse of the matrix K) that approach irrational numbers as $L \to \infty$ and are not ultralocal in $I - J$. Actually this construction may even have a realization in experiments on quantum Hall layers [75].

Fracton phases are interesting for many reasons. One is that gapped fracton phases are a huge class of counterexamples to the lore that the low energy physics of gapped phases is always described by TQFT. These phases can arise from ordinary-looking lattice models, like the layered quantum Hall system described above, or the X-cube model, but even the GSD depends on the geometry of the lattice and therefore they are definitely not described by ordinary TQFT in any regime. A second reason is the bad news I mentioned in §3.8 about the lack of robustness of known topological order in 3+1d to finite temperature. One of the routes [76] to the discovery of fractons was the quest for finite-temperature passive quantum memory. (They have not quite provided such a thing as of yet.) A third reason is that they problematize our notions of what is a phase of matter. A phase of matter is a sharp notion in the thermodynamic limit,

$L \to \infty$. In fracton phases, the GSD depends on L and so makes it difficult even to define such a limit. A good idea for how to get around this is to strengthen the equivalence relation defining a phase to allow the addition not only of decoupled qubits, but also decoupled 2d layers [77].

A closely-related concept to subsystem symmetries is that of multipole symmetries (*e.g.* [78, 79, 80, 81, 82, 83]). A multipole symmetry is one where the continuity equation involves extra derivatives, like $\partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$ (a dipole symmetry). Such a conservation law produces conserved charges that need not be integrated over all of space, and act independently of each other. (For example [80], consider the continuity equation $\partial_0 J^0 + \partial_x \partial_y J = 0$ in $D = 2 + 1$; then $Q_x(x) = \int dy J^0(x, y)$ is conserved for each x.) The simplest example is that conservation of dipole moment implies that charges are immobile [78].

Models with such symmetries have been studied for a long time in the condensed matter literature [84]. Efforts to understand how the rules of ordinary field theory must be relaxed to accommodate such systems and their symmetries have been vigorous (see *e.g.* [85, 72, 86, 87, 88, 80, 81, 82, 89] and references therein and thereto). Attempts have been made to classify subsystem-symmetry-protected topological phases [90] and their anomalies [91], and to understand subsystem-symmetry-enriched topological order [92]. A subsystem-symmetry-based understanding of Haah's code [76], the most interesting gapped fracton model, appears in [93].

An important issue is the robustness of such phases, especially in the gapless case, upon breaking the large symmetry group. At least in examples, the scaling dimensions of operators charged under the subsystem symmetry is large, and in fact diverges in the continuum limit [84, 88, 80, 81, 82] (see in particular Eq. (121) of the first reference). This shows that there is at least a small open set in the space of subsystem-symmetry-breaking couplings in which such phases persist.

Fractal symmetry. The subsystem on which a symmetry acts can be more interesting than just a line or a plane. For example, it can be a fractal [94, 95]. The Newman-Moore model [96] is a simple example of a model with a symmetry operator supported on a fractal subset of space. Put qubits on the sites i of the triangular lattice and consider

$$H = \sum_{ijk \in \Delta} Z_i Z_j Z_k + g \sum_i X_i, \tag{3.45}$$

where the sum is only over up-pointing triangles. To see that this has a fractal symmetry, pick a spin to flip, say the circled spin in Fig. 5. Moving outward from that starting point and demanding that each up-triangle contains an even number of flipped spins, there are many possible self-similar subsets of the lattice we can choose to flip. In fact, there is an extensive number.

This transverse-field Newman-Moore model (3.45) has a number of interesting properties. It has a self-duality mapping $g \to 1/g$, obtained by defining dual spins $\tilde{X}_\Delta \equiv \prod_{i\in\Delta} Z_i Z_j Z_k$ on a new lattice with sites corresponding to the up-pointing triangles. A phase transition at $g = 1$ separates a gapped paramagnetic phase from a gapless phase in which the fractal $\mathbb{Z}_2$ symmetry is spontaneously broken. There is some controversy about the nature of the critical point in the literature: though [97] sees evidence of interesting critical behavior, earlier work [94, 98] found indications of a first-order transition, which seems to be confirmed in more recent simulations [99, 100]. Perhaps some deformation of this lattice model does have a critical point. Such critical points were claimed [97] to be 'beyond renormalization'; rather, what is broken is the connection between short distances and high energies [89]. Other models with such fractal symmetry have been studied in [101].

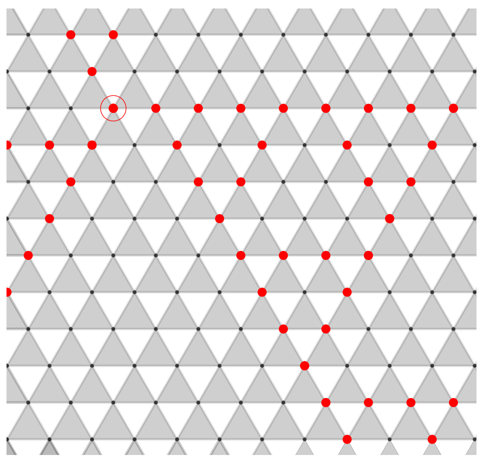


Figure 5: An example of the support of a fractal symmetry operator in the Newman-Moore model. If we flip only the red spins, it preserves the Hamiltonian (3.45). That is, every up-triangle has an even number of red dots. There are many ways to accomplish this.

3.14 Non-invertible symmetries

The preceding discussion suggests a further generalization, which we will need in order to describe *non-abelian* topological order as SSB: if the worldlines of abelian anyons are generalized symmetry operators, what about the worldlines of non-abelian anyons? This is a dramatic step because the algebra of topological operators T_a that transport non-abelian anyons is no longer a group. Rather, they satisfy the fusion algebra:

$$T_a T_b = \sum_c N_{ab}^c T_c. \tag{3.46}$$

By definition, a topological order is non-abelian if there is more than one term on the RHS of this equation for some choice of a, b. Whereas multiplication of two elements of a group always produces a unique third element, here we produce a superposition of elements, weighted by *fusion multiplicities* N_{ab}^c. Further, there is some tension between the fusion algebra (3.46) and unitarity of the operators T_c. The trivial anyon corresponds to the identity operator, $T_1 = 1$. Each type of anyon a has an antiparticle $\bar{a}$. Since $T_{\bar{a}}$ corresponds to transporting a in the opposite direction, we expect that $T_{\bar{a}} = T_a^\dagger$, and therefore (3.46) says in particular

$$T_a T_a^\dagger = \sum_c N_{a\bar{a}}^c T_c. \tag{3.47}$$

If the RHS here has a term other than $N_{a\bar{a}}^1$, then T_a is not unitary. As an example, consider the Ising topological order, with three anyon types $\{1, \psi, \sigma\}$ and the fusion rules

$$T_\sigma T_\sigma = 1 + T_\psi, \quad T_\sigma T_\psi = T_\psi T_\sigma = T_\sigma, \quad T_\psi T_\psi = T_1. \tag{3.48}$$

Note that σ is its own antiparticle. (3.48) implies that the topological line operator T_σ cannot be unitary, and moreover does not have an inverse. Such symmetries are called *non-invertible symmetries* (or sometimes *categorical symmetries* or *fusion category symmetries*).

More generally, any algebra of topological operators acting on a physical system can be regarded as encoding some kind of generalized symmetry.

At the moment, condensed matter applications of the idea of fusion category symmetries remain in the realm of relatively formal developments, as opposed to active phenomenology of real materials. But one application is to understand non-abelian topological order as spontaneous symmetry breaking[14]. A concrete example of a (2+1)d model with non-invertible symmetries is G_k Chern-Simons (CS) theory, with non-Abelian gauge group G at level $k > 1$. The non-invertible symmetry operators are the Wegner-Wilson lines. The specific example of $\mathsf{SU}(2)_2$ CS theory can describe the Ising topological order, and is possibly realized as part of the effective low-energy description of $\nu = \frac{5}{2}$ quantum Hall states.

More generally, any topological field theory for non-Abelian topological order enjoys such a non-invertible symmetry. A nice example of the application of this perspective on anyon worldlines as symmetry operators is [104] which provides a condition on the anyon data required for a general 2+1D topological order to admit a gapped boundary condition, beyond vanishing chiral central charge.

[14] A related perspective appears via the 'pulling-though' operators in the tensor network description of topological orders reviewed in [102]. For a study of categorical symmetries realized as matrix product operators, see [103].

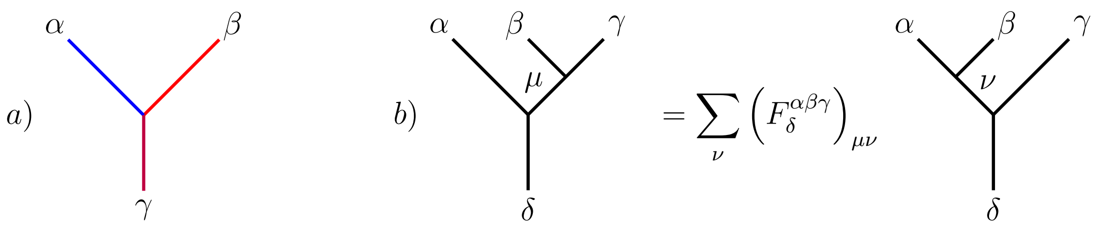


Figure 6: a) Fusion of symmetry operators: this junction is allowed if $N_{\alpha\beta}^{\gamma} \neq 0$. b) Associativity data of fusion of symmetry operators (in the simpler case where the fusion coefficients $N_{\alpha\beta}^{\gamma}$ are only 0 or 1).

Part of the reason for the nomenclature 'categorical symmetry' is that such a collection of symmetry operators comes with some additional data. Besides putting two symmetry operators right on top of each other, we can also consider symmetry operators associated with branched manifolds, as in Fig. 6a. Once we allow such objects, we must also consider more complicated objects related to the associativity of the product, as in Fig. 6b, which relates the two ways of resolving a 4-valent junction of topological operators into two 3-valent junctions. This associativity information (creatively called F-symbols) is part of the specification of the categorical symmetry, and must satisfy the pentagon identities (see *e.g.* Fig. 1 of [105]). In the case of 1-form symmetry in (2+1)-D, there is further information associated with braiding.

3.15 How general is the Generalized Landau Paradigm?

Above we have captured some previously beyond-Landau phases in terms of how they represent their symmetries. Some frontiers worth mentioning are:

- Landau's own Fermi liquid is a gapless phase, that is in some sense protected by the size of the Fermi surface. Recent work [106] describes a large emergent symmetry of a certain class of states generalizing the Fermi liquid, and its anomalies. Perhaps this is a good starting point.

- I said that SPTs are distinguished by the anomaly of the edge theory. What about invertible phases which don't have topological order, but don't need symmetry to distinguish them from the trivial phase. An example is the integer quantum Hall state, which above I described as merely an SPT for particle number symmetry. Such states do have an anomaly, but it is an anomaly involving the coupling to spacetime curvature and I don't know how to think about this as involving a symmetry (diffeomorphisms are a redundancy, not a symmetry).

- Crystalline solids are distinguished from liquid and gas by the fact that they spontaneously break translation symmetry down to the symmetry of the lattice. What about amorphous solids?[15] Well, one thing that distinguishes them from fluids is that the particles are frozen in place, rather than mobile. This means that the Edwards-Anderson order parameter will be nonzero. This is obtained by measuring the density excess $\rho(x, 0) - \rho_0$ and the density at the same point in space later $\rho(x, t) - \rho_0$ and averaging over space and/or configurations. It is zero in a fluid. It is a sign of a breakdown of ergodicity – that there is more than one equilibrium state. How do we think of this as symmetry breaking? One way to detect it is to consider n copies of the system. There is a S_n symmetry that permutes the copies, called replica symmetry. If ergodicity is broken, the different copies can go into different equilibria and spontaneously break the replica symmetry.

[15] Thanks to DaChuan Lu and Roman Geiko for raising this question.

4 Generalized symmetries from Entanglement Bootstrap

Now I will make a dramatic change of perspective, and try to give a very different point of view on symmetries of quantum many body systems. This point of view comes from a program called Entanglement Bootstrap. The underlying **Principle of Entanglement Bootstrap** is:

> All of the universal information about a state of matter is encoded in the local reduced density matrix of a representative wavefunction.

By local reduced density matrix, I mean the reduced density matrix of a large-enough ball. There is quite a bit of evidence for this principle.

- A zeroth order question that the groundstate can answer is whether it is gapped or not. We can answer this by computing equal-time correlators of local operators at large separation. In a gapped state, these will all decay exponentially at a rate that we call the correlation length. Even better is to compute the mutual information of well-separated regions, since this bounds the correlations of any operators supported in those regions (see (4.3) below). Its decay rate therefore gives a basis-independent notion of correlation length.

  This claim that the correlations in any gapped state decay with distance is something that mathematical physicists would like to prove but there is an enormous amount of evidence for it.

- In 1+1d CFT, the entanglement entropy (EE) of a single interval extracts the central charge [107, 108]: $S(\ell) = \frac{c}{6} \log \frac{\ell}{a}$. Similarly, in higher dimensions, the EE of a round ball extracts the corresponding universal entanglement monotone F or a [109].

- In liquid topological order, the entanglement entropy of any region with disk topology is $S(A) = \frac{|\partial A|}{a} - \gamma$ where [110, 111] $\gamma = \log \sqrt{\sum_a d_a^2}$, the log of the total quantum dimension[16].

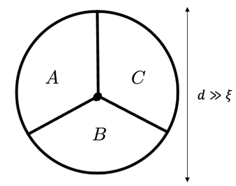
- For any three bulk regions,

$$J(A, B, C) \equiv \mathbf{i} \langle\psi| [K_{AB}, K_{BC}] |\psi\rangle \tag{4.1}$$

[16] A refinement of this statement is here: [112]

is the modular commutator [113, 114, 115]. Notice that this quantity probes the operator structure of the reduced density matrix, in contrast with the EE or even the spectrum of the entanglement Hamiltonian.

For three regions in 2+1d that meet in a triple junction (as at right) and do not touch the boundary, $J(A, B, C) = \frac{\pi c_-}{3}$ where $c_- = c_L - c_R$ is the chiral central charge.



Some evidence for this relation is: Bulk A1 implies that it is insensitive to deformations of the regions. It is odd under parity in that $J(A, B, C) = -J(A, C, B)$. Finally, it is additive under stacking:

$$J(A, B, C)_{|\psi_1\rangle \otimes |\psi_2\rangle} = J(A, B, C)_{|\psi_1\rangle} + J(A, B, C)_{|\psi_2\rangle}. \tag{4.2}$$

If there is a $\mathsf{U}(1)$ symmetry, there is a similar formula that replaces one of the Ks with the current, which extracts the Hall conductivity [116].

[End of Lecture 3]

One excuse for talking about this subject here is that included amongst "all of the universal information" is all the symmetries of the state of matter. My goal for the discussion below is to learn to extract generalized symmetry operators from this rather austere starting point, in the case of liquid topological states, and to extract generators of the conformal symmetry in cases where they should be present.

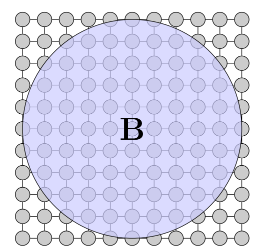
A consequence of the above Principle is that there should be *local* conditions on a wavefunction that tell us which category of state it represents: Is it topologically ordered? Is it a CFT groundstate? Is it a fracton state?

In the case of liquid topological states ('liquid' here means 'not fracton'), the right conditions were described by Bowen Shi, Kohtaro Kato, and Isaac Kim in [117]. They are two simple conditions on the entanglement entropies of partitions of a ball. From a state satisfying these conditions, much of the structure of topological quantum field theory can be (rigorously!) extracted. This is a long story, but I want to tell some of it in order to explain how this perspective can be used to construct the generalized symmetry operators that are spontaneously broken by the reference state. Also, there have been a number of lectures about TQFT and its application as a framework for thinking about topological states of matter. TQFT is great, but it has two serious shortcomings (besides its general mathematical forbiddingness): First, it is an empty shell waiting to be filled with data from somewhere else. Second, the whole wonderful structure relies on a set of assumptions chosen by mathematicians. Are they the right assumptions? Can they be proven to apply in some physical circumstances? The

Entanglement Bootstrap offers a route to address both of these holes: starting from a single wavefunction, we can extract the TQFT data. And more importantly, we can understand whether TQFT is indeed the correct framework. I emphasize this second point especially in $D > 2+1$ where the correct framework of extended TQFT is much less clear. Meng Cheng mentioned various rigorous classification theorems about 3+1d TQFT, and these are indeed rigorous theorems that prove a conclusion from a particular set of assumptions. Entanglement Bootstrap is a framework from which we can test those assumptions, and it gives a very different point of view on the subject.

Then in the final section, I will apply this perspective to CFT groundstates and 2+1d gapped states with gapless boundary. In that case, we will see conformal symmetry emerge from a single wavefunction.

4.1 Basics of Entanglement Bootstrap for liquid topological states

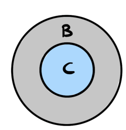


Suppose we are given a density matrix σ on the Hilbert space of a big ball **B**. The Hilbert space is a tensor product over sites of a lattice. Let's ask how can we guarantee that our density matrix σ on a ball comes from a liquid topologically-ordered groundstate?

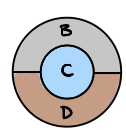
At least we should somehow impose that it satisfies an area law for the EE.

The following two simple conditions [117] on the von Neumann entropies of a decomposition of a smaller ball inside **B** are sufficient:

A0: $0 = \Delta(B, C) \equiv (S_{BC} + S_C - S_B)_\sigma$

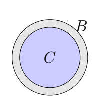


A1: $0 = \Delta(B, C, D) \equiv (S_{BC} + S_{CD} - S_B - S_D)_\sigma$



Note that $\Delta(B, C) = \Delta(B, C, \emptyset)$. The area law bits cancel in both of these combinations. We'd like these conditions to be true on any disk of radius a few lattice sites. Why? [Foundational references: [Shi-Kato-Kim 1906.09376, Shi 1911.01470, Shi-Kim 2008.11793]]

A0 implies an area law for round balls.

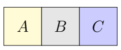


$$S_C = S_B - S_{BC} \leq S_B.$$

A0 $\implies$ $0 = I(A : C)_\sigma \equiv S_A + S_C - S_{AC}$. $\quad A =$ any region outside BC. (The proof is simple: purify the state and let $\tilde{A}$ be the complement of the ball BC; then $0 = \Delta(B, C) = I(\tilde{A} : C) \geq I(A : C)$. The first step uses the fact that $S_A = S_{\bar{A}}$ in a pure state, and the inequality is an avatar of strong subadditivity (SSA).) The mutual information between A and C bounds above connected correlation functions between any operators supported in A and C:

$$I(A:C)_\rho \geq \frac{1}{2} \frac{\langle \mathcal{O}_A \mathcal{O}_C \rangle_{\rho,c}^2}{\| \mathcal{O}_A \|^2 \| \mathcal{O}_C \|^2} \tag{4.3}$$

where $\langle AB \rangle_c \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$, and $\| \mathcal{O} \|^2$ is the square of the operator norm, the largest eigenvalue of $\mathcal{O}^\dagger \mathcal{O}$. This builds in the idea that in a gapped groundstate, correlations should be short-ranged.

A1 $\implies 0 = I(A:C|B)_\sigma \equiv S_{AB} + S_{BC} - S_B - S_{ABC}$, that is, the state is a quantum Markov chain. 'B' is for buffer: this statement holds for any A and C separated by B: 

This leads to lots of useful tools. Note that assumption A1 is better than directly assuming that the state is a quantum Markov chain because it is a *local* condition.

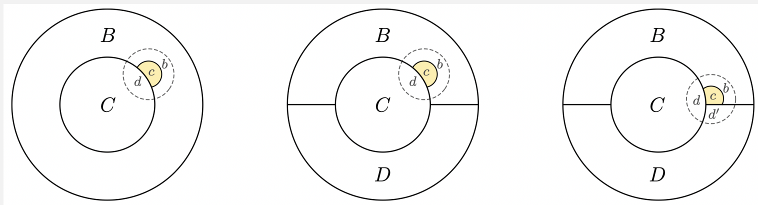
The axioms are implied by the strict area law $S_A = |\partial A|/a - b_0(\partial A)\gamma$ with γ independent of the shape of A, and $b_0(\partial A)$ the number of components of the boundary of A.

RG picture. The axioms are sufficient but not necessary conditions for the state to represent a liquid TO phase. They are exactly true at zero-correlation-length fixed points of the RG, such as the groundstates of commuting projector Hamiltonians like the toric code. In fact, for chiral states, those with $c_- \neq 0$, the axioms cannot hold exactly with finite-dimensional local Hilbert space. However, we expect that approximately-true axioms are enough because of the renormalization group. That is, given a state with some small violation of the axioms on small disks, the violation for larger disks will be even smaller. [118] makes a fascinating conjecture for a threshold in the size of the violation of A1 below which the axioms are expected to heal themselves as we scale up.

In fact these axioms are self-reproducing, in the sense that if they hold on small disks, they hold for any (large-enough) region of the given topology.
Here's the idea: The idea is to show that adding little bits to the regions in the axioms does not change Δ. This works because of strong subadditivity (SSA) of the von Neumann entropy, which comes in many forms. The crucial one here is

$$\Delta(B, C, D) \geq \Delta(BB', C, DD'). \tag{4.4}$$

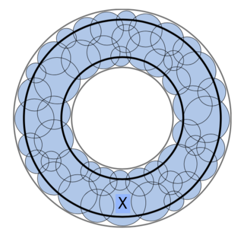
(Taking DD' to be empty this also implies $\Delta(B,C) \geq \Delta(BB',C)$.) But $\Delta(B,C)_\rho \geq 0$ for any state (this is called subadditivity) and $\Delta(B,C,D)_\rho \geq 0$ for any state (this is called weak monotonicity, which is equivalent to SSA). So this shows that the axioms continue to hold if we enlarge B or D leaving C fixed. To show that we can also deform C keeping B and D fixed uses a similar strategy involving these figures:



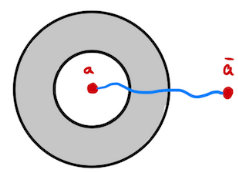
For example, in the leftmost figure,

$$\Delta(B \setminus c, Cc) \overset{S_{BC}=S_B-S_C}{=} \Delta(B \setminus c, c, C) \overset{\text{SSA}}{\leq} \Delta(b, c, d) \overset{\text{A1}}{=} 0. \tag{4.5}$$

Information convex set. The information convex set is a machine that associates to any subset of the ball a convex set of density matrices $\Sigma(X)$: those density matrices that are locally indistinguishable from the reference state σ in (a slight thickening of) X.

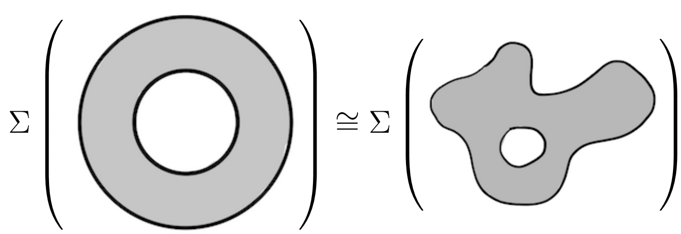


This is an interesting thing because it allows for the possibility of topological excitations *outside* X:



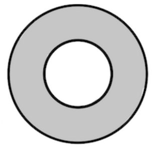
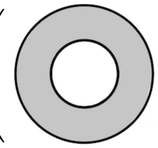
The axioms imply that $\Sigma(X)$ is a topological invariant:

- It is unchanged by smooth deformations of X ('Isomorphism Theorem')

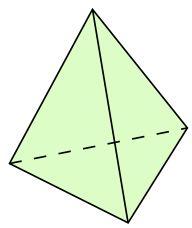
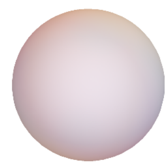


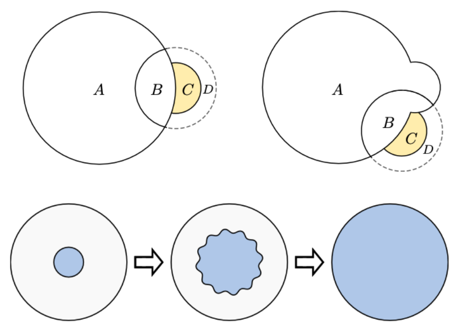
  where $\cong$ is not only equivalence as convex sets, but preserves entropy differences and fidelity.

- It should be insensitive to small changes of the reference state within the phase (by the argument of [110]):

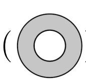
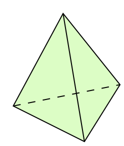
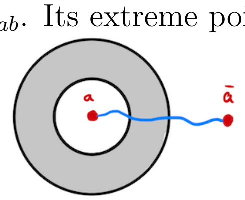
Σ_σ () $\cong \Sigma_{\sigma'}$ ()

The structure of $\Sigma(X)$ for various X extracts universal data. If the local Hilbert space is finite dimensional, the $\Sigma(X)$ is for sure a finite-dimensional compact convex set. Such a set is characterized by its extreme points. They come in two varieties: either the extreme points are isolated, like in a simplex, or they come in families like in the Bloch ball, whose extreme points are the Bloch sphere of pure states:

 or 



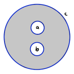
A basic result is that $\Sigma(\text{ball}) = \{\sigma_{\text{ball}}\}$. The idea is to apply A1 to extend the state on the given ball to the whole reference state.

Let me mention some key results in 2d: $\Sigma($  $)$ is a simplex:  Its extreme points are isolated and orthogonal $\text{tr}\rho_a\rho_b \propto \delta_{ab}$. Its extreme points ρ_a can be associated with anyon types, because of this picture: 

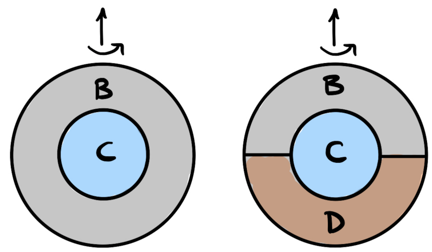
An important property of an anyon is its quantum dimension d_a. N of them together has a Hilbert space $\mathcal{H}_N$ with dimension that approaches d_a^N for large N. An anyon is called abelian if $d_a = 1$ so $\mathcal{H}_N$ is just one dimensional for all N. One way to extract the quantum dimension from $\Sigma(X)$:

$$d_a^2 = e^{S(\rho_a) - S(\rho_1)} \tag{4.6}$$

, where ρ_1 is associated with the trivial excitation – the reduced density matrix of the reference state.

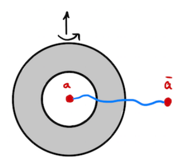
$\Sigma($$)$ produces fusion spaces of dimension N_{ab}^c. A theorem of [117] shows that the expression (4.6) above of d_a agrees with the positive solution of $d_a d_b = N_{ab}^c d_c$, which implements the definition above.

Entanglement Bootstrap in 3d (and higher). Essentially the same axioms work in any dimension [119]. By work I mean that all the basic results generalize: The axioms are self-reproducing, $\Sigma(X)$ is still invariant to deformations of X or of the groundstate.

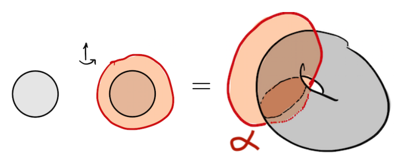


What generalizes the annulus for classifying topological excitations in 3d?

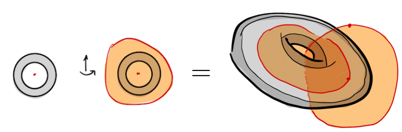
Particle excitations are detected by Σ(sphere shell).



(Pure) loop excitations are detected by Σ(solid torus).

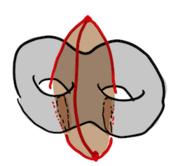


General loop excitations are detected by Σ(torus shell)

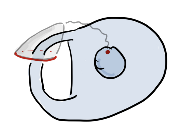


There are new possibilities beyond these!

Σ(genus-2 solid) detects **graph excitations** (also a simplex).

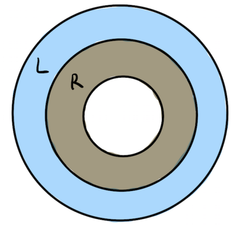


Correlated particle-loop excitations ('shrinkable loops') are detected by Σ(torus minus ball)

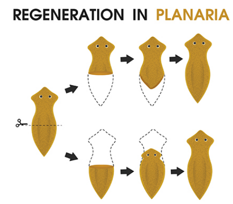


Excitations along knots and links are detected by Σ(knot or link complement).

Classical and Quantum Information in $\Sigma(\Omega)$. To organize this rich set of data, we can divide it into two classes. There are two kinds of data in $\Sigma(\Omega)$ for general Ω: classical and quantum. the difference between them is that classical information can be copied.

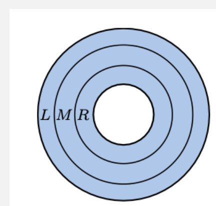


Which kind of information is carried by Ω is determined by a simple topological property. A region Ω is said to be *sectorizable* if it contains two disjoint subsets, each of which can be extended to the whole thing, as in the example at right. It is a bit like the animals (such as planaria) that you can cut in half and each half grows back to the whole animal (figure from here). In both cases, it means all the information about the system is contained in each copy.



For a sectorizable region Ω, $\Sigma(\Omega)$ is a simplex (extreme points are isolated and orthogonal $\mathrm{tr}\rho_a\rho_b \propto \delta_{ab}$).

The general idea is visible in the definition of sectorizable: if we can cut Ω into two parts each of which deforms to the whole Ω, then a copy of the information in the state is contained in *both* parts. Here is a formalization of this idea:



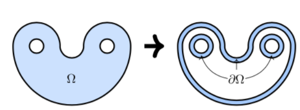
Proof sketch: X is sectorizable implies $X = LMR$ where X, L, R can be deformed into each other. The Isomorphism Theorem says $\Sigma(X) = \Sigma(L) = \Sigma(R)$ and $F(\rho,\sigma) = F_L = F_R = F_{LMR}$, for all $\rho, \sigma \in \Sigma(X)$. $F_{LMR} \leq F_{LR}$ by monotonicity of fidelity.
For ρ, σ extreme points, $F_{LR} = F_L F_R$. This step is nontrivial and is proved in [117].
Therefore $F \leq F^2$. But $F \in [0,1]$ so $F = 0$ or 1.

For example, the thickened boundary $\partial\Omega$ of any region Ω is sectorizable. This fact plays a crucial role in the following **Structure theorem:**

$$\Sigma(\Omega) = \bigoplus_A \Sigma_A(\Omega) \tag{4.7}$$

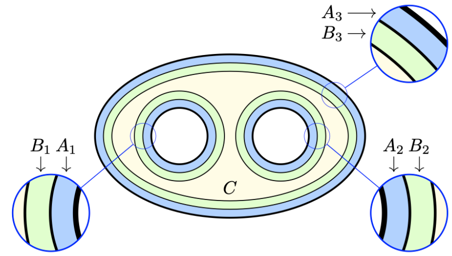
where by '$\oplus$' of convex sets I mean convex hull $\bigoplus_A \Sigma_A \equiv \{\sum_A p_A \rho_A, \rho_A \in \Sigma_A\}$

(1) A labels extreme points of $\Sigma(\partial\Omega)$, which is a simplex. $\Sigma_A(\Omega)$ the subset of $\Sigma(\Omega)$ where $\rho \in \Sigma_A$ if $\mathrm{tr}_{\Omega\setminus\partial\Omega}\rho = \rho_A$, an extreme point of $\Sigma(\partial\Omega)$ (some sectors could be empty).



(2) $\Sigma_A(\Omega) = \mathcal{S}(\mathbb{V}_A(\Omega))$ space of density matrices on a Hilbert space $\mathbb{V}_A(\Omega)$. Therefore $\dim \mathbb{V}_A \in \mathbb{Z}_{\geq 0}$ is an invariant (of Ω and of the topological order).

The key point behind this theorem is the form of a quantum Markov chain. Let ABC be a decomposition of a region where A is a thickened boundary and AB is an even thicker boundary, as in the figure at right. Then first we can label states by their sector on the thickened boundary; the subset with label I is called $\Sigma_I(\Omega)$.



Now any state in $\Sigma_I(\Omega)$ has the form (see Appendix D of [120]):

$$\rho^I = \rho^I_{AB_L} \otimes \rho^{ij} |Ii\rangle\langle Ij|_{B_R C} \tag{4.8}$$

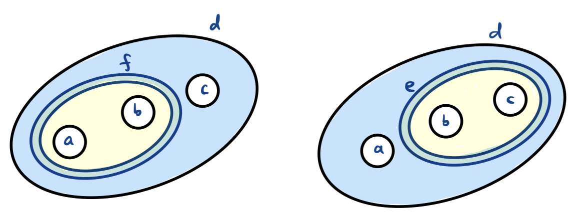
where $\mathcal{H}_B = \mathcal{H}_{B_L} \oplus \mathcal{H}_{B_R} \oplus \cdots$. Here $|Ii\rangle_{B_R C}$ is a basis of states on $\mathbb{V}_I(\Omega) = \mathcal{H}_{B_R C}$, which we regard as the Hilbert space of a 'fuzzy region' contained in BC but containing C. It is not really associated with a definite region. These states $|Ii\rangle$ can be regarded as 'low-energy states'.

Merging. In general the quantum marginal problem is Hard. But for quantum Markov chains it has a unique solution. Moreover, it plays nicely with the information convex set [117]: For $ABC \subset \mathbf{B}$, states in $\Sigma(AB)$ and $\Sigma(BC)$ that agree on B can be *merged* to a unique state in $\Sigma(ABC)$, which is a quantum Markov chain, $I(A:C|B) = 0$. The output of the (Petz) reconstruction map is the max entropy state $\rho^\star \in \Sigma(ABC)$ consistent with the marginals (since SSA implies $S_{ABC} \leq S_{AB} + S_{BC} - S_B$, but the merged state saturates SSA). The merging process preserves $S(\rho) - S(\rho^1)$.

Associativity Theorem: if $AB = \Omega_L$ and $BC = \Omega_R$ are merged along a whole boundary component B, the fusion dimensions are related by:

$$\dim \mathbb{V}^{a_R}_{a_L}(\Omega) = \sum_{i \in \mathcal{C}_B} \dim \mathbb{V}^{i}_{a_L}(\Omega_L) \cdot \dim \mathbb{V}^{a_R}_{i}(\Omega_R),$$

For example, for particle excitations in 2d, $N^d_{abc} = \sum_{e \in \mathcal{C}_{\text{point}}} N^d_{ae} N^e_{bc} = \sum_{f \in \mathcal{C}_{\text{point}}} N^f_{ab} N^d_{fc}$. This is exactly the associativity condition discussed from the axiomatic TQFT point of view by Meng. More holes don't give new information.



(Note that it is somewhat of an open question to find a nice way to extract the associator, *i.e.* the F-symbols, purely within this approach. The best we know how to do at the moment is following the discussion in [121].)

These ideas were also pushed through in the case of gapped boundaries and gapped interfaces in [120].

4.2 Algebras of flexible operators

[This section is based on work with Bowen Shi and Jin-Long Huang that we are trying to write up.] Given a reference state, we can define a notion of equivalence of (bounded) operators on X:

$$A \overset{\Omega}{\approx} A' \quad \text{if} \quad A\rho_\Omega = A'\rho_\Omega \quad \forall\ \rho_\Omega \in \Sigma(\Omega) \tag{4.9}$$

(and a similar condition on $A^\dagger$). The intuition is that $A \approx A'$ means that they act the same way on the low-energy states defined above.

The *algebra of flexible operators* on Ω is [122]:

$$K(\Omega) = \{\text{operators on } \Omega \text{ such that } \star\} \tag{4.10}$$

$\star$: For any subregion $\zeta \subset \Omega$ obtained by removing interior balls and deformation retraction, $\exists$ an operator A_ζ such that $A_\zeta \otimes 1\!\!1_{\Omega\setminus\zeta} \approx A$.

Prop: this is an algebra, *i.e.* $W_1 \in K(\Omega), W_2 \in K(\Omega) \implies W_1W_2 \in K(\Omega)$, and the product respects the equivalence relation $\approx$.

In fact it's quite a simple algebra, a multi-matrix algebra. For example, if Ω is sectorizable, a basis of $K(\Omega)$ is labelled by extreme points of $\Sigma(\Omega)$. More explicitly and more generally, recall that for each sector of the thickened boundary $\Sigma_I(\Omega) = \mathcal{S}(\mathbb{V})_I(\Omega)$ is the state space of a certain Hilbert space.

So if we pick an orthonormal basis

$$\mathbb{V}_I(\Omega) = \text{span}\{|I, j\rangle\} \tag{4.11}$$

we can construct a subspace $K_I(\Omega) \subset K(\Omega)$ of operators of the form

$$Q^I_{ij} = |I, i\rangle\langle I, j|\ . \tag{4.12}$$

These operators are in $K(\Omega)$ because the states $|I, j\rangle$ are locally indistinguishable, meaning indistinguishable on interior balls; because of this, their support may be deformed freely: if $\{A_i\}$ are collection of balls in ζ, the support of the state, then

$$|I, j\rangle_\zeta = \left(\otimes_{i=1}^k |1_{A_i}\rangle\right) \otimes |I, j\rangle_{\tilde{\zeta}}\,, \tag{4.13}$$

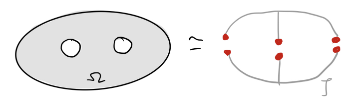
where $|1_{A_i}\rangle$ is the same for all the states.

These operators generate all of $K(\Omega)$. The idea is that any flexible operator on Ω has a representative that acts only on the interior of Ω. The flexible property says it acts directly on the fuzzy pure states:

$$\mathcal{O}\rho_\Omega = \sum_I p_I \sum_{ijk} \rho^I_{ij} O_{ik} |Ik\rangle\langle Ij|_{CB_R} \otimes \rho^I_{AB_L}. \tag{4.14}$$

which is just the action of the operators Q^I_{ij}.

Note that these operators need not be invertible and need not be supported on manifolds.

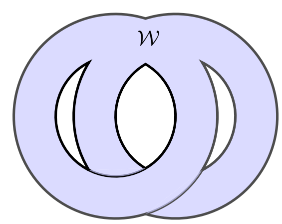


4.3 Pairing manifold and S-matrix

To see what we can do with these operators, let me introduce a bit more technology. In trying to use a single density matrix on a ball to extract all the universal data of a topological state, you might be worried about the following thing. A topological field theory is a machine that eats nontrivial manifolds and spits out invariants. But here we seem to have only the one, very trivial, manifold, namely the ball. Here is a way around this [123], which can be called the Kirby torus trick.

We can make a reference state on a closed manifold by the following two-step procedure.

Step 1: immersions. The first step is to realize that we can define the information convex set not just for regions embedded in $\mathbf{B}$, but also for regions *immersed* in $\mathbf{B}$. The key point is that in the definition of the information convex set, we just need to compare a given density matrix on small balls to the reference state on small balls. This means we can re-use parts of the reference state for making such comparisons.

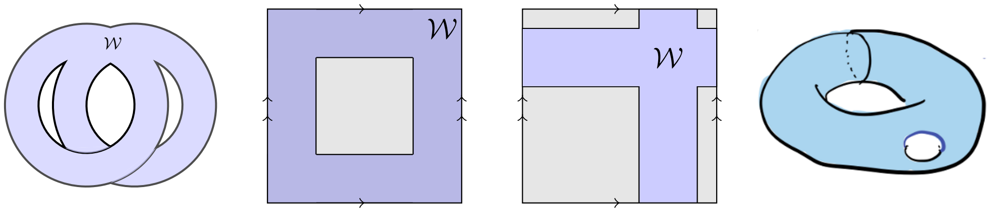


Kirby torus trick: given an immersion $i : \mathcal{W}^d \looparrowright \mathbf{B}^d$, we can pull back structure from $\mathbf{B}$ to $\mathcal{W}$:
given a function f on $\mathbf{B}$, $f_\star(w) \equiv f(i(w))$. Kirby (1969) [124] pulls back smooth/PL structure. Hastings [125] pulls back the Hamiltonian for an invertible phase or a QCA [126]. We pull back the reference state σ on balls.

Generalized isomorphism theorem: axioms imply regions related by **regular homotopy** have isomorphic information convex sets. This means that we can deform through immersions. For example, it means that the information convex set of any thickened knot is the same as that of the unknot. (However Σ of a knot *complement* depends on the knot.)

For a general immersion, it is not obvious that $\Sigma(\mathcal{W})$ is non-empty or that $\Sigma(\mathcal{W})$ contains a vacuum state.

Step 2: heal the punctures. Here are some pictures of a torus minus a ball.

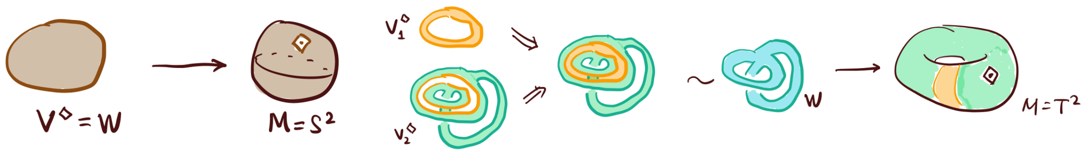


The first one is immersed in the plane.

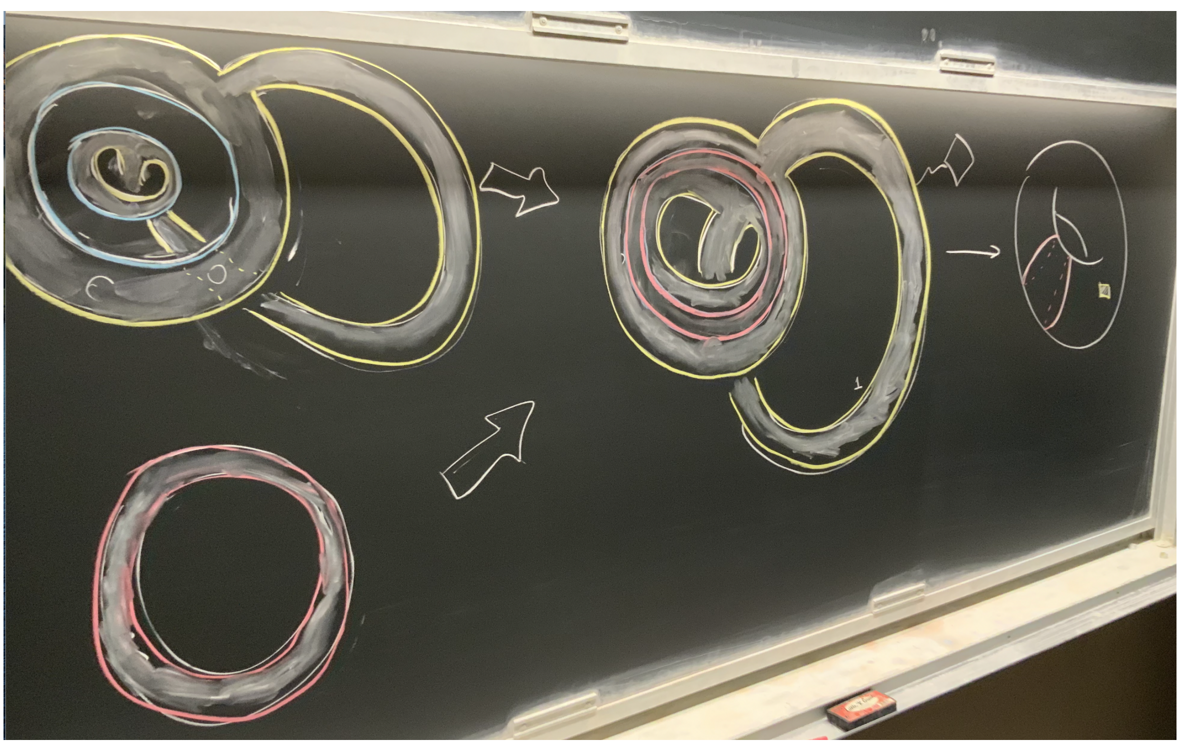
Every closed manifold with $w_n(T\mathcal{M}) = 0$ can be made this way [127].

Given a state on an immersed manifold minus a ball, if the boundary is in an *abelian* sector ($d(\partial\mathcal{W}) = 1$), we can heal the holes in $\mathcal{W}$ to make a reference state on a closed manifold ('vacuum completion'). The idea is simply to purify the state by adding some extra degrees of freedom. Then identify these extra degrees of freedom with a point, a potential puncture in the closed manifold. We can show that if $d(\partial\mathcal{W}) = 1$ the axioms are satisfied in the neighborhood of this point.

With some assumptions, we can guarantee a vacuum sector $\hat{1}$ on $\partial\mathcal{W}$. For example:



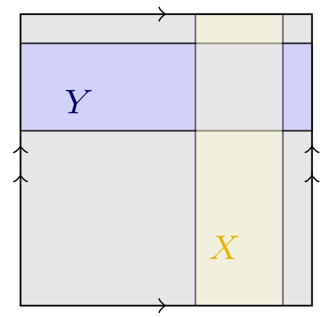
Here was my attempt to depict this on the blackboard:



Application: Regular homotopy on S^n is more powerful than on $\mathbf{B}^n$.

Pairing manifolds.

Definition (rough sketch): A *pairing manifold* $\mathcal{M}$ is a closed manifold $\mathcal{M} = X\bar{X} = Y\bar{Y}$ such that all intersections of $X, \bar{X}, Y, \bar{Y}$ are balls, and are transverse (plus a few other conditions on the state).
The idea is that the information of the state on X is invisible to Y: a min entropy state of X is a max entropy state of Y.



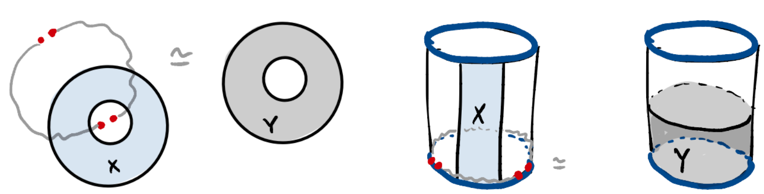
The full definition seems quite restrictive, but there are many examples! The simplest example is the torus $\mathcal{M} = T^2$ depicted above, where X and Y are both annuli.

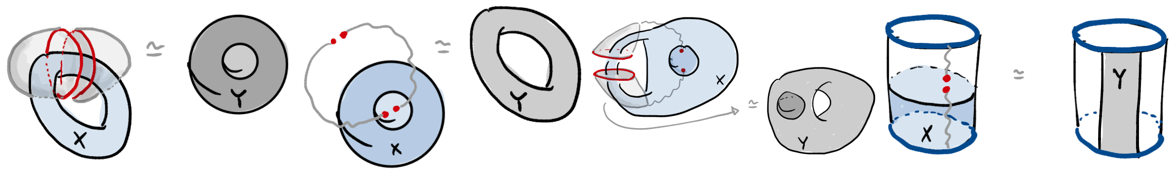
A key point is that we can make two bases of $\mathbb{V}(\mathcal{M})$, one from states on X, and one from states on Y. The unitary matrix that takes one basis to the other is the *pairing matrix*. In the case of $\mathcal{M} = T^2$, this is the familiar S-matrix describing braiding of anyons. Here we are using the construction of [128, 129] that relates the S-matrix to the overlaps of minimally-entangled states.

One outcome of this construction is therefore a proof of remote detectability of topological excitations. This is an axiom of the TQFT approach described by Meng Cheng in his lectures. Given a representative state satisfying the axioms A0 and A1 it is a theorem: the pairing matrix is unitary, and it encodes the braiding of each class of topological excitations with its partner class of excitations.

Another outcome is an independent proof that in 3+1d TO the number of particle types is the same as the number of pure flux types. (Meng talked about this same conclusion in terms of line and surface operators.) This follows because they each give a basis of the fusion space of the pairing manifold $S^2 \times S^1$.

A natural question at this point is: given X (a region whose information convex set encodes a certain set of topological excitations), what is Y, with which forms a pairing manifold? The answer is the following. The excitations detected by X are created by a certain class of extended operators, at the boundary of their support. Take two such operators with the same boundary, so that the union of their support ζ is closed. That locus ζ is a deformation retraction of Y. Here are some examples:





4.4 Verlinde-like formulae

If X participates in a pairing manifold, there is a second basis of the flexible operator algebra. There are flexible operators W_Y^α on Y such that $|\alpha_X\rangle = W_Y^\alpha |1_X\rangle$ for $W_Y^\alpha \in K(Y)$. In fact they are constructed from the pairing matrix: $W_Y^\alpha = \sum_a S_{aij}^\alpha Q_{aij}$, where $Q_{aij} = |ai\rangle\langle aj|$ are the generators of $K_a(Y)$. (I've written this formula for the case where Y is not sectorizable, but X is; if Y is sectorizable you can ignore the ij indices.) These operators also generate $K(Y)$. We can write their algebra in terms of structure constants F:

$$W^\alpha W^\beta \approx \sum_\gamma F_\gamma^{\alpha\beta} W^\gamma. \tag{4.15}$$

By studying $\langle \alpha_X| (W_Y^\beta)^\dagger |a_{ijY}\rangle$ in two different ways, we can show:

$$F_\gamma^{\alpha\beta\star} = \sum_a \frac{\mathrm{tr} S_{\alpha a} S_{\beta a} S_{\gamma a}^\star}{S_{1,a_{11}}}. \tag{4.16}$$

A precedent for this kind of Verlinde formula with extra indices was written down in [130] for the case of gapped boundary excitations.

If Y has fusion multiplicity, $K(Y)$ is not abelian: $F_\gamma^{\alpha\beta} \neq F_\gamma^{\beta\alpha}$.

Similarly, there are flexible operators on X such that $|a_{ijY}\rangle = W_X^{aij} |1_Y\rangle$. (The notation assumes X is sectorizable but Y is not.) This leads to a second Verlinde formula

$$F_{c_{i''j''}}^{a_{ij} b_{i'j'}\star} = \sum_\alpha \frac{S_{\alpha a_{ij}} S_{\alpha b_{i'j'}} S_{\alpha c_{i''j''}}^\star}{S_{\alpha,1}}. \tag{4.17}$$

In some cases (in particular when α, β, γ label particle excitations, we can show $F_\gamma^{\alpha\beta} = N_\gamma^{\alpha\beta} = \dim \mathbb{V}_\gamma^{\alpha\beta}(\Omega) \in \mathbb{Z}$ for some fusion region Ω. When they label loops, it may not be true.

See also recent work from a very mathy TQFT perspective by Johnson-Freyd, who also discovers some generalized Verlinde formulae.

5 Emergence of conformal symmetry from Entanglement Bootstrap

5.1 Entanglement Bootstrap for 1+1d CFT

What is the local condition on a 1d wavefunction for it to be the groundstate of a CFT? We can answer this question [131] using our knowledge of the entanglement structure of 1+1d CFT groundstates.

The behavior of the entanglement entropy of intervals, $S(\ell) = \frac{c}{6}\log\frac{\ell}{a}$ [107, 108], is not quite enough. But we know more than just the entropy. For the groundstate of CFT we actually know [132] the form of the entanglement Hamiltonian K_A, $\rho_A \propto e^{-K_A}$ for a single interval[17]:

$$K_{[x_1,x_2]} = 2\pi \int_{x_1}^{x_2} dx \beta_{[x_1,x_2]}(x) h(x) + S_{[x_1,x_2]} \mathbb{1}, \quad \beta_{[x_1,x_2]}(x) = \frac{(x-x_1)(x_2-x)}{x_2-x_1} \Theta(x \in [x_1,x_2]) \tag{5.1}$$

where h is the hamiltonian density of the CFT, and $\Theta(\mathtt{x})$ is 1 if $\mathtt{x}$ is true and zero otherwise.

[17] At this school I've been experiencing a bit of culture shock from all the people worrying about the lack of factorization of the Hilbert space in QFT. I don't at all dispute the value and promise of algebraic approaches to QFT, but in my experience this is a fake problem. When QFT arises as the low-energy limit of a quantum lattice model, the factorization of the lattice Hilbert space provides an excellent definition. Moreover, formulae derived in the continuum like the ones I am using here are very well verified by even fairly small lattice systems.

In any case, if you choose not to believe in the reduced density matrix in QFT, you can think of what I am about to tell you in the same (horrifying to me as a vegetarian) way that Gell-Mann described his use of QFT in the 1960s [133], which I think is quoted in the book by Coleman named after this TASI school (this seems to have been a false memory; I think I learned about this quote from David Gross' Nobel lecture): *We may compare this process to a method sometimes employed in French cuisine: a piece of pheasant meat is cooked between two slices of veal, which are then discarded.* We all know how this one turned out.

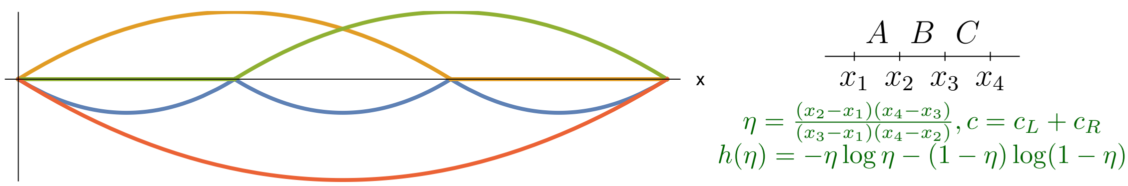
By the way, there are two other prominent places where the non-factorization of the Hilbert space arises. One is in gauge theory, where the physical Hilbert space satisfies a gauss Law constraint and therefore is not a tensor product. I also think this is a fake problem because of the possibility described earlier of emerging gauge theory from a tensor product Hilbert space by imposing the Gauss law energetically. It's true that this is not a universal choice. The final example is in quantum gravity. This one is not fake! The fact that the Hilbert space of a theory of gravity is not a tensor product over space is the essential lesson of holography, and all the of the various paradoxes such as 'firewalls' only arise by forgetting this lesson.

In any case, in fact, all of the UV dangerous stuff in (5.1) is in the identity term.

How do we know (5.1)? Recall the Bisognano-Wichmann theorem. For the groundstate of any relativistic QFT in flat space, the entanglement hamiltonian for half the space is $K_{y>0} = 2\pi \int d^{d-1}x_\perp dy y T_{00}(y, x_\perp)$. This follows from examining the path integral for $\rho_{y>0}$ and viewing the angular coordinate around $y = 0$ as time. But a Weyl transformation relates the half-space to an interval.

A certain linear combination of these locally-quadratic functions is identically zero. Therefore the following combination of entanglement Hamiltonians is just a c-number:

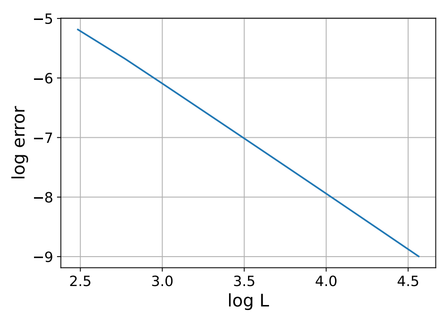
$$\boxed{K_\Delta \equiv K_{AB} + K_{BC} - \eta(K_A + K_C) - (1-\eta)(K_B + K_{ABC}) = \frac{c}{3} h(\eta) \mathbb{1} \ .} \tag{5.2}$$



In a lattice model, the vector form of the equation

$$K_\Delta |\psi\rangle = \frac{c}{3} h(\eta) |\psi\rangle \tag{5.3}$$

is a more robust statement. This is because the density matrix can have a kernel, on which its log is not defined. But when acting on $|\psi\rangle$ this part goes away. Here is a plot of the variance of K_Δ as a function of system size for the critical Ising model.



So this might be a locally-checkable condition for a state to be the groundstate of a CFT. Two questions:

1. How do we know it is a sufficient condition?

2. I claimed above that all universal data encoded in local density matrix of a ground state. How could we prove this? What is the full set of universal data? Here is what David Lin describes as a hack to avoid this question. Reconstruct the full **groundstate** and a parent **Hamiltonian** from the local reduced density matrix.

This business of reconstructing the whole state from its parts is an instance of the quantum marginal problem. This is a Hard Problem in general. One way to think about the goal of the local conditions is that they should help solve the marginal problem.

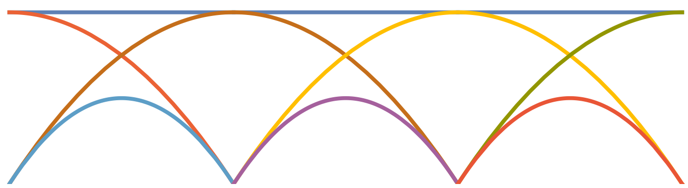
Reconstruct the **groundstate** (merging): solve (5.2) for K_{ABC}:

$$\tilde{K}_{ABC} = (\tilde{K}_{AB} - \tilde{K}_B + \tilde{K}_{BC}) - \frac{\eta}{1-\eta}\left(\tilde{K}_{AB} + \tilde{K}_{BC} - \tilde{K}_A - \tilde{K}_C\right), \qquad \tilde{K} \equiv K - \langle K \rangle . \tag{5.4}$$

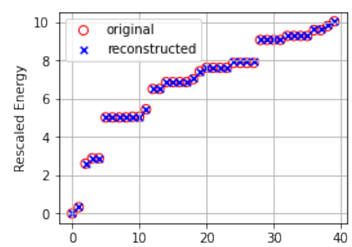
$\implies$ We can reconstruct ρ_{ABC} given ρ_{AB} and ρ_{BC}.
This equation reduces to Markov reconstruction when $\eta = 0$.

What about H? Divide up the circle into equally-spaced intervals; regard each as a 'site' (with infinite-dimensional $\mathcal{H}$). The density matrix on ABC can be used to reconstruct a lattice **Hamiltonian** for the CFT

$$H_{\text{rec}} = \sum_{i=-\infty}^{\infty} \left(K_{[i,i+2]} - K_{[i,i+1]}\right) \overset{(5.1)}{=} \int dx h(x). \tag{5.5}$$



This works even when the local Hilbert space is small.



To answer question 1, we can construct the rest of the global conformal generators! I will not write down the formula here, see [131].

At this point, we encounter a wonderful surprise: (5.2) is the condition to extremize the function

$$\begin{aligned} S_\Delta(|\psi\rangle) &\equiv S_{AB} + S_{BC} - \eta(S_A + S_C) - (1-\eta)(S_B + S_{ABC}) \\ &= \eta\Delta(A,B,C) + (1-\eta)I(A:C|B) \geq 0 \,. \end{aligned} \tag{5.6}$$

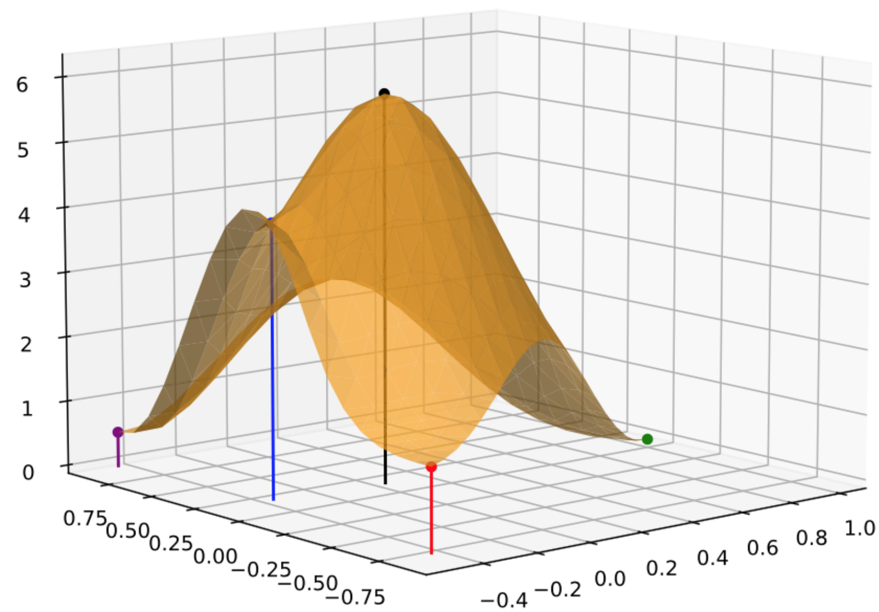
The key to showing this is that

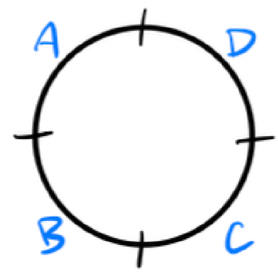
$$dS = \langle d\psi| K |\psi\rangle + \langle\psi| K |d\psi\rangle \,; \tag{5.7}$$

the terms with δK vanish when acting on $|\psi\rangle$ because the variation preserves $\text{tr}\rho = 1$.

The value of this function at the critical point is the central charge (times $h(\eta)/3$).

Conjecture: critical points of this function (for all η) $\leftrightarrow$ RG fixed points





Moreover, it is numerically effective even for tiny systems! In the table, A, B, C (and their complement on S^1) are each a single, measly qubit. In blue, we recognize approximations to certain well-known CFT groundstates with small central charge. As we make the local Hilbert space dimension larger, we see more CFTs with larger central charge. In addition to the CFTs, we can identify other, *gapped* fixed points of the RG. If we start from these states and build the reconstructed Hamiltonian (5.5), we find a degeneracy below a gap. For example, the state with the maximum value of S_Δ is a perfect tensor, which has entanglement Hamiltonian zero for any subsystem. In such examples, the choice of η is arbitrary. The state with the second-biggest c in the table produces a Hamiltonian whose degeneracy on L sites is the Lth Lucas number; this set of groundstates is therefore Fibonacci chain.

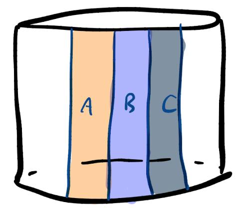
c	Description	Explicit form up to on-site unitary
0	cat states	$a\lvert 0000\rangle + b\lvert 1111\rangle$
0.526	Ising CFT	
1.132	W state	$\frac{1}{2}(\lvert 0001\rangle + \lvert 0010\rangle + \lvert 0100\rangle + \lvert 1000\rangle)$
1.211	XX model	$-\frac{1}{2}(\lvert 0101\rangle + \lvert 1010\rangle)+$ $\frac{1}{2\sqrt{2}}(\lvert 0011\rangle + \lvert 0110\rangle + \lvert 1100\rangle + \lvert 1001\rangle)$
1.245	Heisenberg	$-\frac{1}{\sqrt{3}}(\lvert 0101\rangle + \lvert 1010\rangle)+$ $\frac{1}{2\sqrt{3}}(\lvert 0011\rangle + \lvert 0110\rangle + \lvert 1100\rangle + \lvert 1001\rangle)$
3.510	ferromagnet	$\frac{1}{\sqrt{3}}\lvert 0000\rangle+$ $\frac{1}{\sqrt{6}}(\lvert 0111\rangle + \lvert 1011\rangle + \lvert 1101\rangle + \lvert 1110\rangle)$
4.165	Fibonacci chain	$\frac{1}{\sqrt[4]{5}}\left(\lvert 0000\rangle + \sqrt{\frac{\sqrt{5}-1}{2}}(\lvert 0101\rangle + \lvert 1010\rangle)\right)$
6.000	maximum point	$\frac{1}{2}(\lvert 0000\rangle + \lvert 1111\rangle + \lvert 0101\rangle + \lvert 1010\rangle)$

[End of Lecture 4]

5.2 Entanglement Bootstrap for 2+1d chiral states

This subsection and the next are about work in progress with Bowen Shi, Ting-Chun David Lin, Xiang Li and Isaac Kim.

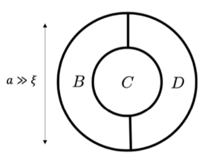
Consider a 2+1d topological state (satisfying A1), possibly chiral, on a thin cylinder. This is a 1+1d non-chiral CFT. That is, if we dimensionally reduce along the direction normal to the boundaries, we obtain a 1+1d field theory with conformal symmetry. Eq. (5.2) then implies



$$\left(\eta\hat{\Delta}(A,B,C) + (1-\eta)\hat{I}(A:C|B)\right)|\psi\rangle = \frac{c}{3}h(\eta)\,|\psi\rangle \tag{5.8}$$

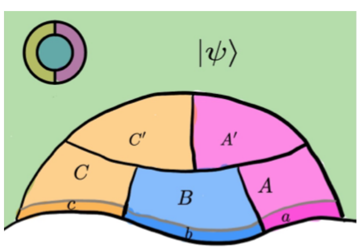
for the regions shown. Here

$$\hat{\Delta}(A,B,C) \equiv K_{AB}+K_{BC}-K_A-K_C, \quad \hat{I}(A:C|B) \equiv K_{AB}+K_{BC}-K_B-K_{ABC} \tag{5.9}$$



A1: $(S_{BC} + S_{CD} - S_B - S_D)_{|\psi\rangle} = 0$

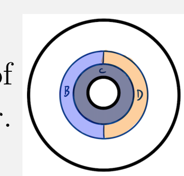
Bulk A1 plus some weak assumptions imply the following fixed point equation:



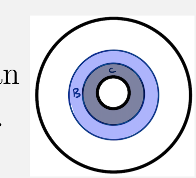
$$\boxed{\underbrace{\left(\eta\hat{\Delta}(AA',B,CC') + (1-\eta)\hat{I}(A:C|B)\right)}_{\equiv K_D}|\psi\rangle = \frac{c}{3}h(\eta)\,|\psi\rangle\,.} \tag{5.10}$$

Here is a bit more detail. The weak assumptions are:

- Full-boundary A1: $\Delta(B,C,D)_\psi = 0$ for BCD a partition of an annulus surrounding one of the boundaries of the cylinder.

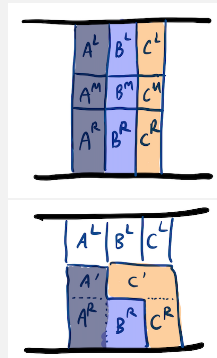


- Full-boundary A0: $\Delta(B,C)_\psi = 0$ for BC a partition of an annulus surrounding one of the boundaries of the cylinder.



These guarantee that no nefarious person has distributed a Bell pair between the two boundaries.

We wish to show:



$$K_D^L |\psi\rangle = \Gamma^L |\psi\rangle \tag{5.11}$$

and

$$K_D^R |\psi\rangle = \Gamma^R |\psi\rangle \tag{5.12}$$

where, for $\alpha = L, R$, $K_D^\alpha \equiv \big(\eta\Delta(A_\uparrow^\alpha, B^\alpha, C_\uparrow^\alpha) + (1-\eta)I(A^\alpha : C^\alpha|B^\alpha)\big)$, with $\eta = \eta_G(a,b,c)$ is the geometric cross-ratio for the regions abc on the boundary, and $C_\uparrow^L \equiv C^L C'$, $A_\uparrow^L \equiv A^L A'$, $C_\uparrow^R \equiv C^R C'$, $A_\uparrow^R \equiv A^R A'$. Furthermore $\Gamma^L + \Gamma^R = \frac{ch(\eta)}{3}$. Here is the argument, in two steps:

1. We can disconnect the two boundaries in the following sense. The LHS of the equation (5.10) is

$$K_D |\psi\rangle = (K_D^L(\eta) + K_D^R(\eta)) |\psi\rangle$$

where $K_D^{L/R}$ is made from entanglement hamiltonians of regions that only touch the L/R boundary.

More specifically, we use full-boundary A1 to conclude that $I(A^L : A^R|A^M)_\psi = 0$ (these regions are defined in the figure to the right of (5.11)), and therefore

$$K_A |\psi\rangle = (K_{A^{ML}} + K_{A^{MR}} - K_{A^M}) |\psi\rangle\,, \tag{5.13}$$

and similarly for the other vertical strips. Therefore

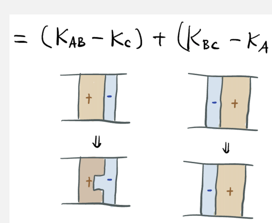
$$\hat{I}(A:C|B)|\psi\rangle = \Big(\hat{I}(A^{LM} : C^{LM}|B^{LM}) + \hat{I}(A^{RM} : C^{RM}|B^{RM}) - \hat{I}(A^M : C^M|B^M)\Big)|\psi\rangle\ . \tag{5.14}$$

But bulk A1 says $\hat{I}(A^M : C^M|B^M)|\psi\rangle = 0$. Therefore, using bulk A1 to deform the regions,

$$\hat{I}(A:C|B)|\psi\rangle = \Big(\hat{I}(A^L : C^L|B^L) + \hat{I}(A^R : C^R|B^R)\Big)|\psi\rangle \tag{5.15}$$

Now we use bulk A1 to deform the regions in

$$\hat{\Delta}(A,B,C) = (K_{AB} - K_C) + (K_{BC} - K_A) \tag{5.16}$$



as follows:

Then using (5.13) to decompose each of these pieces vertically, we have

$$\hat{\Delta}(A,B,C)\,|\psi\rangle = \left(\hat{\Delta}(A^L_\uparrow, B^L, C^L_\uparrow) + \hat{\Delta}(A^R_\uparrow, B^R, C^R_\uparrow)\right)|\psi\rangle\,. \tag{5.17}$$

Combining (5.15) with (5.17) gives the sum of the criticality conditions for the two boundaries:

$$\left(K^L_D + K^R_D\right)|\psi\rangle = \frac{ch(\eta)}{3}\,|\psi\rangle\,. \tag{5.18}$$

2. (5.18) implies

$$\left(K^L_D + K^R_D\right)\rho^{LR} = \frac{ch(\eta)}{3}\rho^{LR}, \tag{5.19}$$

where ρ^{LR} is the reduced density matrix for ψ reduced to the union of the regions L, R of support of K^L_D and K^R_D respectively.

Using full-boundary A0, we conclude that $I(L:R)=0$ and therefore $\rho^{LR} = \rho^L \otimes \rho^R$. Since the state factorizes, the contributions of K^L_D and K^R_D to (5.18) are independent. More precisely, we can take the trace of (5.19) over R to get

$$K^L_D\rho^L + \mathrm{tr}_R(K^R_D\rho^R)\rho^L = \frac{ch(\eta)}{3}\rho^L, \tag{5.20}$$

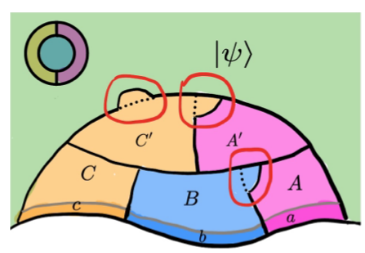
which says

$$K^L_D\rho^L \propto \rho^L. \tag{5.21}$$

Therefore, each of them acting on $|\psi\rangle$ gives a c-number. The two c-numbers add up to $\frac{ch(\eta)}{3}$.

■

Bulk A1 implies that the LHS of (5.10) is independent of deformations in the bulk.

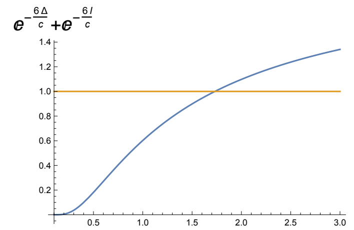


A useful perspective:

- given any three intervals, compute $\Delta(AA', B, CC') \equiv S_{AA'B} + S_{BCC'} - S_{AA'} - S_{CC'}$ and $I(A:C|B)$.

- Determine $c(\psi)$ as the solution to

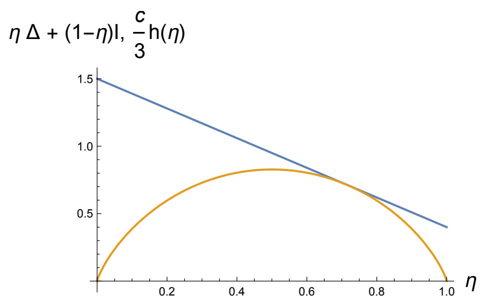
$$1 = e^{-6\Delta/c} + e^{-6I/c}. \tag{5.22}$$



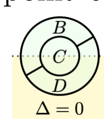
- Determine η by

$$\frac{\Delta}{I} = \frac{\ln \eta}{\ln 1 - \eta} \tag{5.23}$$

(or by $\eta = e^{-6\Delta/c}$).



Examples: (1) No bulk, 1+1d CFT. Then these equations follow from $S(\ell) = \frac{c}{3}\log\frac{\ell}{a}$. That is, η is the ordinary geometric cross-ratio, in terms of which $\Delta = -6c\log\eta, I = -6c\log(1-\eta)$.

(2) Gapped boundary: $c = 0$. In this case, the fixed-point equation reduces to boundary A1 of [120] for gapped boundaries [Kim-Shi, 2008.11793] 

(3) CFT on the boundary of a bulk topological state.

(4) Rough edge of chiral state.

With this definition, which applies also in the case of 1+1d CFT, we can show the following statement:
Theorem: A state satisfies the vector fixed point equation $K_D(\eta)|\psi\rangle \propto |\psi\rangle$ IFF $\delta c(\psi) = 0$, that is, $c(\psi)$ extracted as above from the state is stationary under arbitrary variation of the state (preserving its norm).

The proof follows from

$$\delta c(\psi) \propto c(\psi)\left(\langle d\psi| K_D |\psi\rangle + \langle\psi| K_D |d\psi\rangle\right) . \tag{5.24}$$

The term $\langle\psi| dK_D |\psi\rangle$ vanishes because the variation preserves $\mathrm{tr}\rho = 1$ as in (5.7).

If $c = 0$, then both statements are true; below we assume $c \neq 0$.

$\boxed{\Leftarrow}$ Let the variation of the state be generated by some hermitian operator G. Then $\delta c \propto \langle\psi| [K_D, G] |\psi\rangle = 0$.

$\boxed{\Rightarrow}$ If $K_D\psi \not\propto |\psi\rangle$, we can vary $|\psi\rangle$ in this direction and get $\delta c \neq 0$.

■

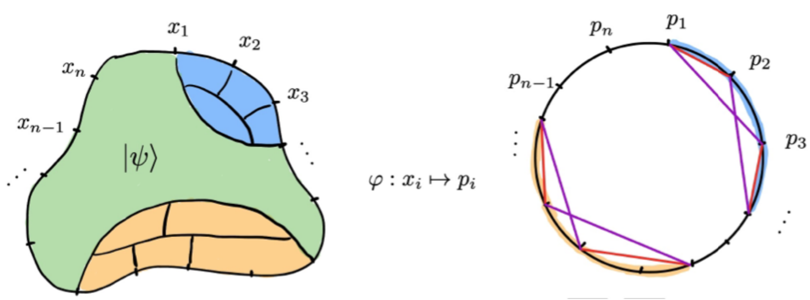
One perspective on this theorem is that the vector fixed-point equation is a way to check the stationarity locally.

5.3 Emergent conformal symmetry and geometry in 2+1d chiral states.

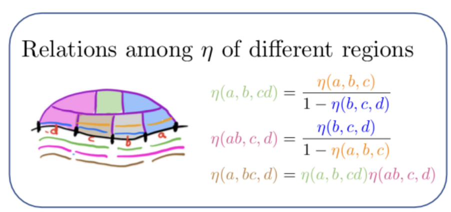
$$(\Delta, I) \to (c, \eta): \ 1 = e^{-6\Delta/c} + e^{-6I/c}, \quad \frac{\Delta}{I} = \frac{\ln \eta}{\ln(1-\eta)}.$$

Theorem [Quantum cross ratios are geometric]: The ηs computed in this way from a fixed point state are the cross-ratios of intervals on a circle. Furthermore, c is the same for any choice of region.

i.e. $\exists$ a map $\varphi : \partial\Sigma \to S^1$ with $\eta(x_i, x_j, x_k, x_l) = \frac{\overline{p_i p_j}\,\overline{p_k p_l}}{\overline{p_i p_k}\,\overline{p_j p_l}}$.



Idea of proof: Sufficient conditions for such a map to define a cross-ratio [134, 135] [Labourie] are the relations for any four consecutive intervals:



$$\eta(ab, c, d) = \frac{\eta(b,c,d)}{1-\eta(a,b,c)}, \ \eta(a, bc, d) = \frac{\eta(a,b,c)\eta(b,c,d)}{(1-\eta(a,b,c))(1-\eta(b,c,d))}, \ \eta(a, b, cd) = \frac{\eta(a,b,c)}{1-\eta(b,c,d)} \ . \tag{5.25}$$

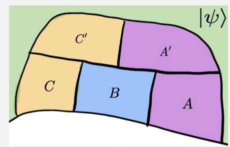
There are two ways to prove these relations. One uses only the vector form of the fixed point equation (5.3). The idea is simply that the decomposition of K_{abcd} using (5.4) can be done in multiple ways, and the answer must be independent of the path – the decomposition is associative. To get (5.25) from this requires one extra assumption that various entanglement hamiltonians acting on $|\psi\rangle$ produce linearly independent states. This should be true for a generic state, but can fails for certain exotic fixed point states like a perfect tensor state.

The other method only works if the state is chiral in the sense that $c_- \neq 0$.

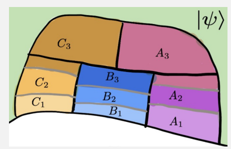
Here is the proof of (5.25) using $c_- \neq 0$. It uses the modular commutator of [113, 114, 115] and its relation to the chiral central charge c_-.

1. For regions that do touch the boundary, bulk A1 implies

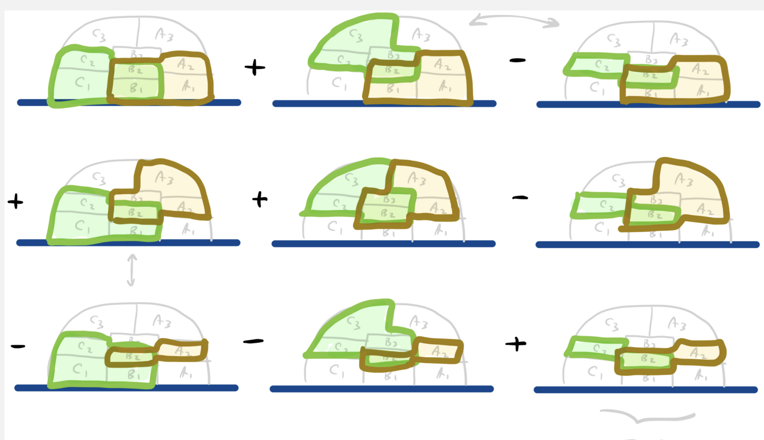
$$J(AA', B, CC') = J(A, B, C) - \frac{\pi c_-}{3}. \qquad (5.26)$$



where I remind you that the modular commutator is $J(A, B, C) \equiv \mathbf{i} \langle\psi| [K_{AB}, K_{BC}] |\psi\rangle$.
The idea is to chop up the regions into pieces and use the Markov property of states satisfying A1: $I(A_1B_1 : A_3B_3|A_2B_2) = 0$ implies the corresponding property of operators, which we can solve for for $K_{AA'B}$, and similarly for $K_{BCC'}$.



Plug these expressions into the definition of $J(AA', B, CC')$ and use the fact that the modular commutator of bulk regions vanishes if A does not touch C. The calculation is illustrated here:



2. The modular commutator for regions touching the boundary also depends on the cross-ratio of the boundary regions: $J(A, B, C) = \frac{\pi c_-}{3}\eta_J$. (5.26) can then be written as $J(AA', B, CC') = \frac{\pi c_-}{3}(\eta_J - 1)$. I've called it η_J to distinguish it for a moment from the geometrical cross ratio and from η_I defined by $\frac{\Delta}{I} = \frac{\ln \eta_I}{\ln 1-\eta_I}$. These are a priori two different ways to extract a cross ratio from the state. But they are in fact the same for states satisfying the vector fixed point equation $K_D(\eta_I) |\psi\rangle \propto |\psi\rangle$.

The idea is very direct: $K_D(\eta_I)\,|\psi\rangle \propto |\psi\rangle$ implies that

$$
\begin{aligned}
0 &= \mathbf{i}\,\langle\psi|\,[K_D, K_{BC}]\,|\psi\rangle && (5.27)\\
&= \eta_I \mathbf{i}\,\langle\psi|\,[(K_{AA'B} - K_{CC'}), K_{BC}]\,|\psi\rangle + (1-\eta_I)\mathbf{i}\,\langle\psi|\,[K_{AB}, K_{BC}]\,|\psi\rangle && (5.28)\\
&= \frac{\pi c_-}{3}\left(\eta_I(\eta_J - 1) + (1-\eta_I)\eta_J\right) && (5.29)
\end{aligned}
$$

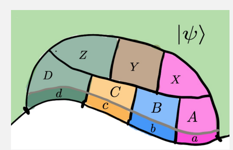
which says $\eta_I = \eta_J$.

3. Finally, to show (5.25), if $c_- \neq 0$, we can use

$$\eta(ab, c, d) = \frac{3}{c_-\pi} J(AB, C, D) = \frac{3}{c_-\pi}\mathbf{i}\,\langle\psi|\,[K_{ABC}, K_{CD}]\,|\psi\rangle \qquad (5.30)$$

and now use the fixed point equation for ABC to solve for K_{ABC}:

$$
\begin{aligned}
&K_{ABC}\,|\psi\rangle\\
&= \frac{1}{\eta(a,b,c)}\left(\eta(a,b,c)(K_{AXB} + K_{CYB} - K_{AX} - K_{CY})\right.\\
&\left.+(1-\eta(a,b,c))(K_{AB} + K_{BC} - K_B) + \text{const}\mathbb{1}\right)|\psi\rangle\,. \qquad (5.31)
\end{aligned}
$$



Using this in (5.30) gives the first equation of (5.25). The same strategy works for the other two.

Therefore, we can drop all the subscripts on η – the geometric cross ratio is the same as the 'quantum' cross-ratio computed via either J or I and Δ.

Here is the proof that c is the same for every choice of regions. Divide the boundary circle into five regions labelled consecutively $i = 1..5$. Denote by c_i and η_i the central charge and quantum cross-ratio computed using regions $(i-1, i, i+1)$ via (5.22) and (5.23), where the labels are understood mod five.

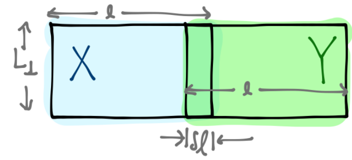
1. First an identity that follows from the definition $\Delta(B, C, D) \equiv S_{BC} + S_{CD} - S_B - S_D$:
$$\Delta(1, 23, 4) = \Delta(1, 2, 34) + \Delta(12, 3, 4). \tag{5.32}$$
2. Using the fact that the full state is pure, (5.32) can be rewritten as
$$\Delta(1, 5, 4) = I(5:2|1) + I(3:5|4). \tag{5.33}$$
3. Using the definitions of c, η, we can rewrite (5.33) as
$$c_5 \ln \frac{1}{\eta_5} = c_1 \ln \frac{1}{1-\eta_1} + c_4 \ln \frac{1}{1-\eta_4}. \tag{5.34}$$
4. The fact that the ηs are geometric implies that $\eta_5 = (1-\eta_1)(1-\eta_4)$. Since the labels on the regions are arbitrary, we have
$$c_i = c_{i+1} p_i + c_{i-1}(1 - p_i) \tag{5.35}$$
where
$$p_i = \frac{\ln \frac{1}{1-\eta_{i+1}}}{\ln \frac{1}{1-\eta_{i+1}} \frac{1}{1-\eta_{i-1}}} \in (0, 1)\ . \tag{5.36}$$
5. (5.35) implies that $c_{i+1} - c_i$ and $c_i - c_{i-1}$ have the same sign. But this means they all have to be the same.

Virasoro generators from modular hamiltonians. We've shown that a state satisfying the axioms produces a map to the circle. Furthermore, certain modular flows,
$$|\psi\rangle \to e^{\mathbf{i}tK} |\psi\rangle \tag{5.37}$$
where K is a linear combination of entanglement hamiltonians of regions, produce a new state that still satisfies the axioms. After such a good modular flow we get a new map to the circle. $\phi_1 \phi_2^{-1}$ is then an element of Diff(S^1). In this way we can build a representation of the Virasoro algebra!

5.4 c-functions

The fact that critical points of our function (S_Δ or $c(\psi)$) seem to be fixed points of the RG suggests that they may behave like a c-function, an RG monotone.

In the case of a 1+1d relativistic QFT groundstate, for a certain choice of intervals, this $c(\psi)$ is the same as the entropic c-function of Casini and Huerta [hep-th/0405111]:



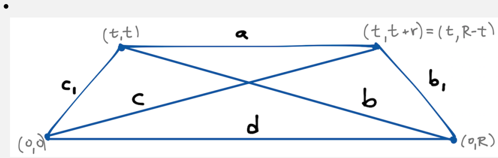
$$c(\psi) = 6\frac{\Delta}{\ln 1/\eta} = 6\frac{2S(\ell) - 2S(\ell - \delta\ell)}{2\delta\ell} = 6\ell\partial_\ell S(\ell) \equiv c_{CH}.$$

(Here I used $|AB| = |BC| = \ell, |A| = |C| = \ell - \delta\ell$ and take $\delta\ell$ infinitesimal. Then $\eta = (\ell - \delta\ell)^2/\ell^2 = 1 - 2\delta\ell/\ell$, $\ln 1/\eta = 2\delta\ell/\ell$.)

Casini and Huerta showed, using SSA and the assumption of Lorentz invariance, that their function is monotonic under rescaling, *i.e.* it decreases under zooming out.

Proof of entropic c-theorem in $d = 1 + 1$.

First a Minkowski space geometry exercise. Consider an interval d, and the inward light rays from its endpoints b_1, c_1. Extend these for a time t. The interval connecting the endpoints of the lightrays is a. Let b and c be the spacelike surfaces in the figure. Lorentzian geometry then implies $bc = ad$, where the letters indicate the proper lengths of the associated segments. This says $c = \lambda a, d = \lambda b, \lambda = \frac{c}{a} = \frac{d}{b} \geq 1$.



As David Lin explained to me, the relation $bc = ad$ is a Minkowski space version of a Euclidean geometry relation due to Ptolemy.

More explicitly, if $d \equiv |d^\mu d_\mu| = R, a = r$, then $c^\mu = (t, t + r)^\mu = (t, R - t)^\mu$ has $c^2 = |c^\mu c_\mu| = r(r + 2t) = rR = |b^2| = bc$.

SSA says

$$\underbrace{S(c_1 a)}_{=S(c)} + \underbrace{S(ab_1)}_{=S(b)} \geq S(a) + S(d).$$

The underbraced equations are consequences of Lorentz invariance. Then for $\lambda < 1$

$$S(b) - S(a) \geq S(\lambda b) - S(\lambda a). \tag{5.38}$$

Notice that the area law term cancels in the differences. This implies that $\Delta(a, b, c) = (S(ab) - S(a)) + (S(bc) - S(c))$ is monotonic under a rescaling.
From this it follows that the entanglement entropy of an interval of length r

satisfies

$$0 \geq \left(\frac{d}{d\log r}\right)^2 S(r). \tag{5.39}$$

This says that $c(r) \equiv 3r\partial_r S(r)$ satisfies $c'(r)/3 = rS'' + S' \leq 0$. In particular, for a CFT groundstate, we have $S(r) = \frac{c}{3}\log r/\epsilon$, so $c(r) = c$ is the central charge. If we set $\lambda = 1 + \epsilon$ in (5.38), (5.39) follows.

The above theorem (relating the vector fixed point equation to the stationary condition for $c(\psi)$) shows that at a CFT fixed point, c is stationary, not just under variation of couplings, but under arbitrary variations of the state.

For more general choices of regions, monotonicity of $c(\psi)$ is not guaranteed by the above argument, and is in fact a stronger condition on RG flows. Based on preliminary studies of simple examples, it seems that it may nevertheless be true.

Q: can we relate $c''|$ to the spectrum of operator dimensions, and $c'''|$ to the OPE coefficients? Is the gradient flow by this function an RG flow?

5.5 Final words

Principle of Entanglement Bootstrap: all the universal data about a state of matter is encoded in a local region of a single representative wavefunction.

So far, this principle has been applied with some success to liquid bulk topological orders, gapped interfaces between topological orders, 1+1d CFTs, 2+1d chiral states.

Some suggestions for the future might be: Higher-dimensional CFT? Non-relativistic CFT? Non-unitary CFT? Fractons? String theory?

Acknowledgement. Parts of §1, 2, 3 have appeared before in [41]. I am grateful to Tarun Grover for helpful discussions and comments. Thanks to Nabil Iqbal, Bowen Shi, Jin-Long Huang, Isaac Kim, Xiang Li, and Ting-Chun David Lin for collaborations related to these lectures. I've used some figures made by Xiang Li. This work was supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-SC0009919, and by the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (652264, JM).

References

[1] X. G. Wen, "Topological orders in rigid states," *Int. J. Mod. Phys. B* **04** (feb, 1990) 239–271, https://www.worldscientific.com/doi/abs/10.1142/S0217979290000139.

[2] F. J. Wegner, "Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters," *Journal of Mathematical Physics* **12** (1971), no. 10 2259–2272, http://scitation.aip.org/content/aip/journal/jmp/12/10/10.1063/1.1665530.

[3] J. Haah, "A degeneracy bound for homogeneous topological order," *SciPost Phys.* **10** (2021), no. 1 011, `2009.13551`.

[4] T. Grover, Y. Zhang, and A. Vishwanath, "Entanglement entropy as a portal to the physics of quantum spin liquids," *New Journal of Physics* **15** (2013), no. 2 025002, `1302.0899`.

[5] R. M. Nandkishore and M. Hermele, "Fractons," *Annual Review of Condensed Matter Physics* **10** (2019) 295–313, `1803.11196`.

[6] M. Pretko, X. Chen, and Y. You, "Fracton phases of matter," *International Journal of Modern Physics A* **35** (2020), no. 06 2030003, `2001.01722`.

[7] A. Y. Kitaev, "Fault-tolerant quantum computation by anyons," *Annals of Physics* **303** (Jan., 2003) 2–30, `quant-ph/9707021`.

[8] R. Dijkgraaf and E. Witten, "Topological Gauge Theories and Group Cohomology," *Commun. Math. Phys.* **129** (1990) 393.

[9] M. A. Levin and X.-G. Wen, "String net condensation: A Physical mechanism for topological phases," *Phys. Rev.* **B71** (2005) 045110, `cond-mat/0404617`.

[10] C.-H. Lin, M. Levin, and F. J. Burnell, “Generalized string-net models: A thorough exposition,” *Phys. Rev. B* **103** (2021), no. 19 195155, 2012.14424.

[11] X.-G. Wen and A. Zee, “Classification of Abelian quantum Hall states and matrix formulation of topoological fluids,” *Phys. Rev. B* **46** (1992) 2290–2301.

[12] S. Dong, E. Fradkin, R. G. Leigh, and S. Nowling, “Topological Entanglement Entropy in Chern-Simons Theories and Quantum Hall Fluids,” *JHEP* **05** (2008) 016, 0802.3231.

[13] G. Wong, “A note on entanglement edge modes in Chern Simons theory,” *JHEP* **08** (2018) 020, 1706.04666.

[14] M. E. Peskin, “Mandelstam ’t Hooft Duality in Abelian Lattice Models,” *Annals Phys.* **113** (1978) 122.

[15] C. Dasgupta and B. I. Halperin, “Phase Transition in a Lattice Model of Superconductivity,” *Phys. Rev. Lett.* **47** (1981) 1556–1560.

[16] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” *Nucl. Phys. B* **435** (1995) 129–146, hep-th/9411149.

[17] Z. Nussinov and G. Ortiz, “Sufficient symmetry conditions for Topological Quantum Order,” *Proc. Nat. Acad. Sci.* **106** (2009) 16944–16949, cond-mat/0605316.

[18] Z. Nussinov and G. Ortiz, “A symmetry principle for topological quantum order,” *Annals Phys.* **324** (2009) 977–1057, cond-mat/0702377.

[19] A. Kovner and B. Rosenstein, “New look at QED in four-dimensions: The Photon as a Goldstone boson and the topological interpretation of electric charge,” *Phys. Rev. D* **49** (1994) 5571–5581, hep-th/9210154.

[20] T. Pantev and E. Sharpe, “Notes on gauging noneffective group actions,” hep-th/0502027.

[21] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe, and M. Ando, “Cluster decomposition, T-duality, and gerby CFT’s,” *Adv. Theor. Math. Phys.* **11** (2007), no. 5 751–818, hep-th/0606034.

[22] E. Sharpe, “Notes on generalized global symmetries in QFT,” *Fortsch. Phys.* **63** (2015) 659–682, 1508.04770.

[23] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, “Generalized Global Symmetries,” *JHEP* **02** (2015) 172, 1412.5148.

[24] D. M. Hofman and N. Iqbal, "Goldstone modes and photonization for higher form symmetries," 1802.09512.

[25] L. V. Delacretaz, D. M. Hofman, and G. Mathys, "Superfluids as Higher-form Anomalies," 1908.06977.

[26] N. Iqbal and J. McGreevy, "Toward a 3d Ising model with a weakly-coupled string theory dual," *SciPost Phys.* **9** (2020), no. 2 019, 2003.04349.

[27] J. Zhao, Z. Yan, M. Cheng, and Z. Y. Meng, "Higher-form symmetry breaking at Ising transitions," *Physical Review Research* **3** (July, 2021) 033024, 2011.12543.

[28] A. Kapustin and R. Thorngren, "Higher symmetry and gapped phases of gauge theories," 1309.4721.

[29] C. Córdova, D. S. Freed, H. T. Lam, and N. Seiberg, "Anomalies in the Space of Coupling Constants and Their Dynamical Applications I," *SciPost Phys.* **8** (2020), no. 1 001, 1905.09315.

[30] L. P. Kadanoff and H. Ceva, "Determination of an operator algebra for the two-dimensional Ising model," *Physical Review B* **3** (1971), no. 11 3918.

[31] E. Fradkin, "Disorder Operators and their Descendants," *J. Statist. Phys.* **167** (2017) 427, 1610.05780.

[32] X.-G. Wen, "Emergent anomalous higher symmetries from topological order and from dynamical electromagnetic field in condensed matter systems," *Phys. Rev. B* **99** (2019), no. 20 205139, 1812.02517.

[33] E. Lake, "Higher-form symmetries and spontaneous symmetry breaking," 1802.07747.

[34] M. A. Levin and X.-G. Wen, "Colloquium: Photons and electrons as emergent phenomena," *Rev. Mod. Phys.* **77** (2005) 871–879, cond-mat/0407140.

[35] A. Polyakov, "Quark confinement and topology of gauge theories," *Nuclear Physics B* **120** (1977) 429–458.

[36] S. Elitzur, "Impossibility of spontaneously breaking local symmetries," *Phys. Rev. D* **12** (Dec, 1975) 3978–3982, https://link.aps.org/doi/10.1103/PhysRevD.12.3978.

[37] D. Forster, H. B. Nielsen, and M. Ninomiya, “Dynamical Stability of Local Gauge Symmetry: Creation of Light from Chaos,” *Phys. Lett. B* **94** (1980) 135–140.

[38] M. B. Hastings and X.-G. Wen, “Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance,” *Physical review b* **72** (2005), no. 4 045141, cond-mat/0503554.

[39] N. Iqbal and J. McGreevy, “Mean string field theory: Landau-Ginzburg theory for 1-form symmetries,” *SciPost Phys.* **13** (2022) 114, 2106.12610.

[40] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, “Topological quantum memory,” *J. Math. Phys.* **43** (2002) 4452–4505, quant-ph/0110143.

[41] J. McGreevy, “Generalized Symmetries in Condensed Matter,” 2204.03045.

[42] S. D. Pace and X.-G. Wen, “Exact emergent higher-form symmetries in bosonic lattice models,” 2301.05261.

[43] A. Cherman and T. Jacobson, “Emergent 1-form symmetries,” 2304.13751.

[44] A. A. Migdal, “Loop equations and $1/N$ expansion,” *Physics Reports* **102** (1983), no. 4 199–290.

[45] T. Banks, “The Gaussian transformation converts lattice gauge theory into a field theory of strings,” *Phys. Lett. B* **89** (1980) 369–372.

[46] T. Yoneya, “A Path Functional Field Theory of Lattice Gauge Models and the Large N Limit,” *Nucl. Phys. B* **183** (1981) 471–496.

[47] S.-J. Rey, “The Higgs Mechanism for Kalb-ramond Gauge Field,” *Phys. Rev. D* **40** (1989) 3396.

[48] M. Franz, “Vortex-boson duality in four space-time dimensions,” *EPL* **77** (2007), no. 4 47005, cond-mat/0607310.

[49] A. J. Beekman, D. Sadri, and J. Zaanen, “Condensing Nielsen-Olesen strings and the vortex boson duality in 3+1 and higher dimensions,” *New Journal of Physics* **13** (mar, 2011) 033004, https://doi.org/10.1088/1367-2630/13/3/033004.

[50] A. Florio, J. Matos, J. M. V. P. Lopes, and J. Penedones, “Searching for continuous phase transitions in 5D SU(2) lattice gauge theory,” 2103.15242.

[51] A. Kapustin, "Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality," *Phys. Rev.* **D74** (2006) 025005, hep-th/0501015.

[52] J. L. Cardy, "UNIVERSAL PROPERTIES OF U(1) GAUGE THEORIES," *Nucl. Phys. B* **170** (1980) 369–387.

[53] M. Vettorazzo and P. de Forcrand, "Electromagnetic fluxes, monopoles, and the order of the 4-d compact U(1) phase transition," *Nucl. Phys. B* **686** (2004) 85–118, hep-lat/0311006.

[54] S. R. Coleman and E. J. Weinberg, "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking," *Phys. Rev.* **D7** (1973) 1888–1910.

[55] G. 't Hooft, "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking," *NATO Adv.Study Inst.Ser.B Phys.* **59** (1980) 135.

[56] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, "Symmetry protected topological orders and the group cohomology of their symmetry group," *Physical Review B* **87** (2013), no. 15 155114, 1106.4772.

[57] T. Senthil, "Symmetry Protected Topological phases of Quantum Matter," *Ann.Rev.Condensed Matter Phys.* **6** (2015) 299, 1405.4015.

[58] A. Zee, "Quantum Hall fluids," *Lect.Notes Phys.* **456** (1995) 99–153, cond-mat/9501022.

[59] A. Vishwanath and T. Senthil, "Physics of Three-Dimensional Bosonic Topological Insulators: Surface-Deconfined Criticality and Quantized Magnetoelectric Effect," *Phys. Rev. X* **3** (Feb, 2013) 011016, 1209.3058, http://link.aps.org/doi/10.1103/PhysRevX.3.011016.

[60] M. Oshikawa, "Topological Approach to Luttinger's Theorem and the Fermi Surface of a Kondo Lattice," *Phys. Rev. Lett.* **84** (Apr., 2000) 3370–3373, cond-mat/0002392.

[61] M. B. Hastings, "Lieb-Schultz-Mattis in higher dimensions," *Phys. Rev. B* **69** (2004) 104431, cond-mat/0305505.

[62] D. V. Else and R. Thorngren, "Topological theory of Lieb-Schultz-Mattis theorems in quantum spin systems," *Phys. Rev. B* **101** (2020), no. 22 224437, 1907.08204.

[63] A. Kapustin, "Bosonic Topological Insulators and Paramagnets: a view from cobordisms," 1404.6659.

[64] C. Z. Xiong, "Minimalist approach to the classification of symmetry protected topological phases," *Journal of Physics A: Mathematical and Theoretical* **51** (2018), no. 44 445001, 1701.00004.

[65] D. V. Else and T. Senthil, "Critical drag as a mechanism for resistivity," *Phys. Rev. B* **104** (2021), no. 20 205132, 2106.15623.

[66] R. Kobayashi, K. Shiozaki, Y. Kikuchi, and S. Ryu, "Lieb-Schultz-Mattis type theorem with higher-form symmetry and the quantum dimer models," *Phys. Rev. B* **99** (2019), no. 1 014402, 1805.05367.

[67] P.-S. Hsin, H. T. Lam, and N. Seiberg, "Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d," *SciPost Phys.* **6** (2019), no. 3 039, 1812.04716.

[68] F. J. Burnell, "Anyon condensation and its applications," *Ann. Rev. Condensed Matter Phys.* **9** (2018) 307–327, 1706.04940.

[69] M. Qi, L. Radzihovsky, and M. Hermele, "Fracton phases via exotic higher-form symmetry-breaking," *Annals Phys.* **424** (2021) 168360, 2010.02254.

[70] X. Shen, Z. Wu, L. Li, Z. Qin, and H. Yao, "Fracton Topological Order at Finite Temperature," *Phys. Rev. Res.* **4** (2022) L032008, 2109.06887.

[71] B. C. Rayhaun and D. J. Williamson, "Higher-Form Subsystem Symmetry Breaking: Subdimensional Criticality and Fracton Phase Transitions," 2112.12735.

[72] W. Shirley, K. Slagle, Z. Wang, and X. Chen, "Fracton Models on General Three-Dimensional Manifolds," *Physical Review X* **8** (July, 2018) 031051, 1712.05892.

[73] S. Vijay, J. Haah, and L. Fu, "Fracton topological order, generalized lattice gauge theory, and duality," *Phys. Rev. B* **94** (Dec., 2016) 235157, 1603.04442.

[74] X. Ma, W. Shirley, M. Cheng, M. Levin, J. McGreevy, and X. Chen, "Fractonic order in infinite-component Chern-Simons gauge theories," 2010.08917.

[75] X. Qiu, R. Joynt, and A. MacDonald, "Phases of the multiple quantum well in a strong magnetic field: Possibility of irrational charge," *Physical Review B* **40** (1989), no. 17 11943.

[76] J. Haah, "Local stabilizer codes in three dimensions without string logical operators," *Phys. Rev. A* **83** (Apr., 2011) 042330, 1101.1962.

[77] W. Shirley, K. Slagle, and X. Chen, "Fractional excitations in foliated fracton phases," *Annals Phys.* **410** (2019) 167922, 1806.08625.

[78] M. Pretko, "Subdimensional Particle Structure of Higher Rank U(1) Spin Liquids," *Phys. Rev. B* **95** (2017), no. 11 115139, 1604.05329.

[79] A. Gromov, "Towards classification of Fracton phases: the multipole algebra," *Phys. Rev. X* **9** (2019), no. 3 031035, 1812.05104.

[80] N. Seiberg and S.-H. Shao, "Exotic Symmetries, Duality, and Fractons in 2+1-Dimensional Quantum Field Theory," *SciPost Phys.* **10** (2021), no. 2 027, 2003.10466.

[81] N. Seiberg and S.-H. Shao, "Exotic $U(1)$ Symmetries, Duality, and Fractons in 3+1-Dimensional Quantum Field Theory," *SciPost Phys.* **9** (2020), no. 4 046, 2004.00015.

[82] N. Seiberg and S.-H. Shao, "Exotic $\mathbb{Z}_N$ symmetries, duality, and fractons in 3+1-dimensional quantum field theory," *SciPost Phys.* **10** (2021), no. 1 003, 2004.06115.

[83] C. Stahl, E. Lake, and R. Nandkishore, "Spontaneous breaking of multipole symmetries," 2111.08041.

[84] A. Paramekanti, L. Balents, and M. P. Fisher, "Ring exchange, the exciton Bose liquid, and bosonization in two dimensions," *Phys. Rev. B* **66** (Aug., 2002) 054526, cond-mat/0203171.

[85] K. Slagle and Y. B. Kim, "Quantum Field Theory of X-Cube Fracton Topological Order and Robust Degeneracy from Geometry," *Phys. Rev. B* **96** (2017), no. 19 195139, 1708.04619.

[86] D. Bulmash and M. Barkeshli, "The Higgs Mechanism in Higher-Rank Symmetric $U(1)$ Gauge Theories," *Phys. Rev. B* **97** (2018), no. 23 235112, 1802.10099.

[87] D. Bulmash and M. Barkeshli, "Generalized $U(1)$ Gauge Field Theories and Fractal Dynamics," 1806.01855.

[88] N. Seiberg, "Field Theories With a Vector Global Symmetry," *SciPost Phys.* **8** (2020), no. 4 050, 1909.10544.

[89] E. Lake, "Renormalization group and stability in the exciton Bose liquid," *Phys. Rev. B* **105** (2022), no. 7 075115, 2110.02986.

[90] T. Devakul, D. J. Williamson, and Y. You, "Classification of subsystem symmetry-protected topological phases," *Phys. Rev. B* **98** (2018), no. 23 235121, 1808.05300.

[91] F. J. Burnell, T. Devakul, P. Gorantla, H. T. Lam, and S.-H. Shao, "Anomaly Inflow for Subsystem Symmetries," 2110.09529.

[92] D. T. Stephen, J. Garre-Rubio, A. Dua, and D. J. Williamson, "Subsystem symmetry enriched topological order in three dimensions," *Phys. Rev. Res.* **2** (2020), no. 3 033331, 2004.04181.

[93] D. J. Williamson, "Fractal symmetries: Ungauging the cubic code," *Phys. Rev. B* **94** (2016), no. 15 155128, 1603.05182.

[94] B. Yoshida, "Exotic topological order in fractal spin liquids," *Phys. Rev. B* **88** (2013), no. 12 125122, 1302.6248.

[95] T. Devakul, Y. You, F. Burnell, and S. Sondhi, "Fractal Symmetric Phases of Matter," *SciPost Physics* **6** (2018), no. 1 007, 1805.04097.

[96] M. Newman and C. Moore, "Glassy dynamics and aging in an exactly solvable spin model," *Physical Review E* **60** (1999), no. 5 5068.

[97] Z. Zhou, X.-F. Zhang, F. Pollmann, and Y. You, "Fractal Quantum Phase Transitions: Critical Phenomena Beyond Renormalization," 2105.05851.

[98] L. M. Vasiloiu, T. H. Oakes, F. Carollo, and J. P. Garrahan, "Trajectory phase transitions in noninteracting spin systems," *Physical Review E* **101** (2020), no. 4 042115, 1911.11739.

[99] K. Sfairopoulos, L. Causer, J. F. Mair, and J. P. Garrahan, "Boundary conditions dependence of the phase transition in the quantum Newman-Moore model," 2301.02826.

[100] R. Wiedmann, L. Lenke., M. Mühlhauser, and K. P. Schmidt, "Absence of fractal quantum criticality in the quantum Newman-Moore model," 2302.01773.

[101] N. E. Myerson-Jain, S. Liu, W. Ji, C. Xu, and S. Vijay, "Pascal's Triangle Fractal Symmetries," 2110.02237.

[102] I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, "Matrix Product States and Projected Entangled Pair States: Concepts, Symmetries, and Theorems," 2011.12127.

[103] J. Garre-Rubio, L. Lootens, and A. Molnár, "Classifying phases protected by matrix product operator symmetries using matrix product states," 2203.12563.

[104] J. Kaidi, Z. Komargodski, K. Ohmori, S. Seifnashri, and S.-H. Shao, "Higher central charges and topological boundaries in 2+1-dimensional TQFTs," *SciPost Phys.* **13** (2022), no. 3 067, 2107.13091.

[105] R. Thorngren and Y. Wang, "Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases," 1912.02817.

[106] D. V. Else, R. Thorngren, and T. Senthil, "Non-Fermi liquids as ersatz Fermi liquids: general constraints on compressible metals," *Phys. Rev. X* **11** (2021), no. 2 021005, 2007.07896.

[107] C. Holzhey, F. Larsen, and F. Wilczek, "Geometric and renormalized entropy in conformal field theory," *Nucl. Phys. B* **424** (1994) 443–467, hep-th/9403108.

[108] P. Calabrese and J. Cardy, "Entanglement entropy and conformal field theory," *J. Phys. A* **42** (2009) 504005, 0905.4013.

[109] R. C. Myers and A. Sinha, "Holographic c-theorems in arbitrary dimensions," *JHEP* **01** (2011) 125, 1011.5819.

[110] A. Kitaev and J. Preskill, "Topological Entanglement Entropy," *Phys. Rev. Lett.* **96** (Mar., 2006) 110404, hep-th/0510092.

[111] M. Levin and X.-G. Wen, "Detecting Topological Order in a Ground State Wave Function," *Phys. Rev. Lett.* **96** (Mar., 2006) 110405, cond-mat/0510613.

[112] I. H. Kim, M. Levin, T.-C. Lin, D. Ranard, and B. Shi, "Universal lower bound on topological entanglement entropy," 2302.00689.

[113] I. H. Kim, B. Shi, K. Kato, and V. V. Albert, "Chiral Central Charge from a Single Bulk Wave Function," *Phys. Rev. Lett.* **128** (Apr., 2022) 176402, 2110.06932.

[114] I. H. Kim, B. Shi, K. Kato, and V. V. Albert, "Modular commutator in gapped quantum many-body systems," *Phys. Rev. B* **106** (2022), no. 7 075147, 2110.10400.

[115] Y. Zou, B. Shi, J. Sorce, I. T. Lim, and I. H. Kim, "Modular Commutators in Conformal Field Theory," *Phys. Rev. Lett.* **129** (Dec., 2022) 260402, 2206.00027.

[116] R. Fan, R. Sahay, and A. Vishwanath, "Extracting the Quantum Hall Conductance from a Single Bulk Wavefunction," 2208.11710.

[117] B. Shi, K. Kato, and I. H. Kim, "Fusion rules from entanglement," *Annals of Physics* **418** (2020) 168164, 1906.09376.

[118] B. Shi, *Anyon theory in gapped many-body systems from entanglement.* PhD thesis, The Ohio State University, 2020.

[119] J.-L. Huang, J. McGreevy, and B. Shi, "Knots and entanglement," *SciPost Phys.* **14** (2023) 141, https://scipost.org/10.21468/SciPostPhys.14.6.141.

[120] B. Shi and I. H. Kim, "Entanglement bootstrap approach for gapped domain walls," *Phys. Rev. B* **103** (2021), no. 11 115150, 2008.11793.

[121] K. Kawagoe and M. Levin, "Microscopic definitions of anyon data," *Phys. Rev. B* **101** (Mar., 2020) 115113, 1910.11353.

[122] B. Shi, J.-L. Huang, and J. McGreevy, "Remote detectability from entanglement bootstrap II: Generalized symmetry algebras,".

[123] B. Shi, J.-L. Huang, and J. McGreevy, "Remote detectability from entanglement bootstrap I: Kirby's torus trick," 2301.07119.

[124] R. C. Kirby, "Stable Homeomorphisms and the Annulus Conjecture," *Annals of Mathematics* (1969) 575–582.

[125] M. B. Hastings, "Classifying quantum phases with the Kirby torus trick," *Phys. Rev. B* **88** (2013), no. 16 165114, 1305.6625.

[126] M. Freedman and M. B. Hastings, "Classification of Quantum Cellular Automata," *Communications in Mathematical Physics* **376** (Apr., 2020) 1171–1222, 1902.10285.

[127] M. Freedman, D. Kasprowski, M. Kreck, A. W. Reid, and P. Teichner, "Immersions of punctured 4-manifolds," 2208.03064.

[128] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, "Quasi-particle Statistics and Braiding from Ground State Entanglement," *Phys. Rev. B* **85** (2012) 235151, 1111.2342.

[129] Y. Zhang, T. Grover, and A. Vishwanath, "General procedure for determining braiding and statistics of anyons using entanglement interferometry," *Phys. Rev. B* **91** (2015), no. 3 035127, 1412.0677.

[130] C. Shen and L.-Y. Hung, "Defect Verlinde Formula for Edge Excitations in Topological Order," *Phys. Rev. Lett.* **123** (2019), no. 5 051602, 1901.08285.

[131] T.-C. Lin and J. McGreevy, "Conformal Field Theory Ground States as Critical Points of an Entropy Function," *arXiv e-prints* (Mar., 2023) arXiv:2303.05444, 2303.05444.

[132] J. Cardy and E. Tonni, "Entanglement hamiltonians in two-dimensional conformal field theory," *J. Stat. Mech.* **1612** (2016), no. 12 123103, 1608.01283.

[133] M. Gell-Mann, "The Symmetry group of vector and axial vector currents," *Physics Physique Fizika* **1** (1964), no. 1 63.

[134] F. Labourie, "What is... a cross-ratio," *Notices of the AMS* **55** (2008), no. 10 1234–1235.

[135] F. Labourie, "Cross ratios, surface groups, $PSL(n, \mathbf{R})$ and diffeomorphisms of the circle," *Publications Mathématiques de l'IHÉS* **106** (2007) 139–213, http://www.numdam.org/articles/10.1007/s10240-007-0009-5/.