Fractons and Chern-Simons Theory

John McGreevy (UCSD)

based on 2010.08917 with

Xiuqi Ma, Wilbur Shirley, Meng Cheng, Michael Levin, Xie Chen

$$\begin{pmatrix}
15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 \\
-5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 \\
2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 \\
-843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 \\
322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 \\
-123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 \\
47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 \\
-18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 \\
7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 \\
-3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 \\
2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 \\
-3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 \\
7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 & -18 \\
-18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 & 47 \\
47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 & -123 \\
-123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 & 322 \\
322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 & -843 \\
-843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 & 2207 \\
2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127 & -5778 \\
-5778 & 2207 & -843 & 322 & -123 & 47 & -18 & 7 & -3 & 2 & -3 & 7 & -18 & 47 & -123 & 322 & -843 & 2207 & -5778 & 15127
\end{pmatrix}$$

2021-09

What is a fracton phase?

[Chamon 05, Haah 11, Vijay-Haah-Fu, Pretko, Shirley-Slagle-Chen ... ,

reviews: Nandkishore-Hermele 1803.11196, Pretko-Chen-You 2001.01722]

Def: A phase with excitations that cannot be moved by any local operator.
(Compare: Topological order $\equiv$ a phase with excitations that cannot be *created* by any local operator.)

Symmetry-based def: A phase that spontaneously breaks a $(p \geq 1)$-form *subsystem* symmetry.
Subsystem symmetry: acts only on a subregion (such as a plane).
SSB of $(p \geq 1)$-form symmetry: deconfined gauge theory, topological order.

Symptoms:

- particles with restricted mobility.

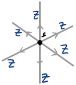
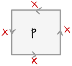
  For gapped fracton phases in 3+1d:

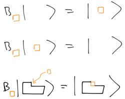
- log(GSD on a T^3 of linear size L) $\sim L$
- S(ball of radius R) $\sim R^2 - \gamma R$

X-cube model.

[Haah-Vijay-Fu 16]

Compare: $\mathbb{Z}_p$ gauge theory aka toric code. Any cell complex, $\mathcal{H} = \otimes_{\text{links},\,\ell}\mathcal{H}_p$. Focus on 3d cubic lattice.

$$H_{\mathrm{TC}} = -\sum_{\text{sites},\, s}$$

$$-\sum_{\text{plaquettes},\, p}$$


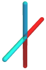


Star term $\overset{!}{=} 1 \implies$ closed strings.

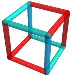
Plaquette term $\overset{!}{=} 1 \implies$ uniform superposition of all of them.

8 (locally-indistinguishable) gs on T^3 of any size. Topological order.

X-cube model: Cubic lattice, $\mathcal{H} = \otimes_{\text{links},\,\ell}\mathcal{H}_p$.

$$H_{\text{X-cube}} = -\sum_{\text{sites}} \Bigg($$

$+$

$+$

$$\Bigg) - \sum_{\text{cubes}}$$


Has $2^{2(L_x+L_y+L_z)-3}$ (locally-indistinguishable) gs on a T^3.

vs: L_x layers of 2d TC has 2^{2L_x} groundstates.

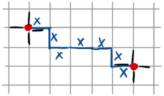
Important point: this GSD is robust, since no local operator relates the gs.

Compare topological excitations.

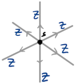
Act on gs with the op.
to create top. excitations.

3d Toric code:

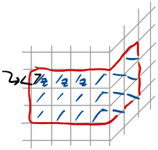
$H_{\rm TC} = -\sum_{\text{sites},\, s}$  $-\sum_{\text{plaquettes},\, p}$ 



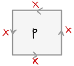
violates



at endpoints.



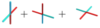
violates



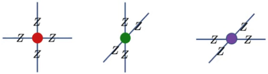
at boundary.

Flexible string operators $\Longrightarrow$ ordinary particles and strings.

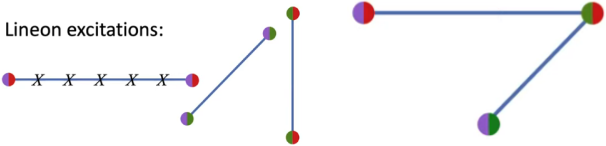
X-cube model:

$H_{\text{X-cube}} = -\sum_{\text{sites}} \Big($  $\Big) - \sum_{\text{cubes}}$ 

color code the star terms:

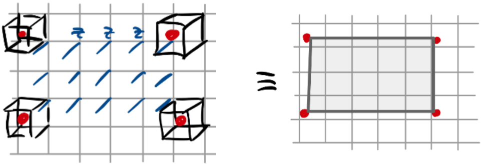


Lineon excitations:

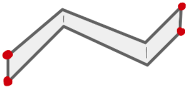


Rigid string operators $\Longrightarrow$ restricted mobility.

Other topological excitations of X-cube model.



Fractons



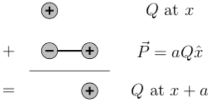
(fracton dipole)

Planeons

(lineon dipole)

Tensor gauge theory perspective on fractons.

If dipole moment P_x is conserved,
charge can't move in $\hat{x}$.



Ordinary (rank 1 vector) gauge theory: $E_i, i = x, y, z$

Gauss law: $\partial_i E_i = \rho \implies Q = \int_B \rho d^3x = \oint_{\partial B} E_i dn_i$

(Charge is locally conserved, can only change by flux through ∂B.)

Rank 2 tensor gauge theory: $E_{ij}, i, j = x, y, z$

Gauss law: $\partial_i \partial_j E_{ij} = \rho \implies Q = \int_B \rho d^3x = \oint_{\partial B} \partial_i E_{ij} dn_j$ and

$$P^i = \int_B x^i \rho d^3x = \oint_{\partial B} \left(x^i \partial_j E_{jk} - E_{ik} \right) dn_j.$$

(Charge and dipole moment are locally conserved, can only change by flux through boundary of region.)

Note that dipoles are perfectly mobile.

Gapped fracton phases are Higgs phases of (gapless) tensor gauge theories.

Why are fracton models interesting? (1 of 3)

• They are counterexamples to the prejudice that gapped states of matter are governed by TFT at low energy

(these are ordinary-looking lattice models!)

...which suggests there are useful generalizations of field theory to be found.

[Pretko, Slagle-Kim, Seiberg-Shao et al]

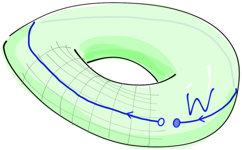
Why are fracton models interesting? (2 of 3)

- A route to finite-temp quantum memory? [Haah 2011]

All known topological orders in $d \leq 3$ are zero-temperature phenomena.

Why: they have top. quasiparticles with energy $\sim L^0$.

$\implies$ a finite density at any $T > 0$. Their wanderings generate logical operators W.



So the idea is that topological order without (flexible) string operators might do better.

Disclaimers:

- today I am discussing type I fracton models, which do have string operators.
- even type II fracton models in $d = 3$ seem to have only logarithmic barriers [Bravyi-Haah]

Why are fracton models interesting? (3 of 3)

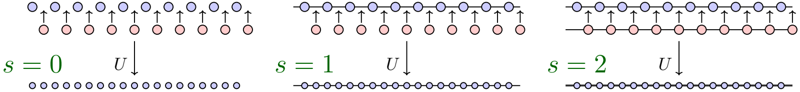
- They require a new definition of phase of matter and of RG fixed-point. [Swingle-JM 2014, Shirley-Slagle-Chen 2018]

Usual notion of phase of matter: equivalence class of families of systems at various linear system sizes L:

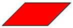
$H_A \sim H_B$ if they are related by adiabatic evolution and addition of decoupled dofs.

A gapped phase contains a unique RG fixed point representative.

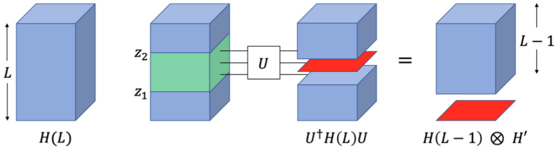
s-source RG fixed point: Fractons require $s > 1$



Foliated fixed point:



= 2d TO layer! = free resource



Fractons and Chern-Simons Theory

Abelian Chern-Simons theory.

$$\mathcal{L} = \frac{K_{IJ}}{4\pi}\epsilon_{\mu\nu\rho}\mathcal{A}^I_\mu \partial_\nu \mathcal{A}^J_\rho - \frac{1}{4e^2}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{\mu\nu J}$$

K is a symmetric matrix of integers. Gauge group is $\mathsf{U}(1)^M$.

Maxwell is an irrelevant op, e^2 is an energy scale.

Groundstate degeneracy (GSD) on a torus is $|\det K|$.

Anyon statistics: $\theta^{IJ} = 2\pi \left(K^{-1}\right)^{IJ}$

Examples:

- $K = (1)$ is IQH.
- $K = (3)$ is Laughlin FQH.
- $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is $\mathbb{Z}_2$ gauge theory/ toric code.

Representative wavefunction:

$$\Psi\left(\{z_i^I\}\right) = \prod_{i<j,I,J} (z_i^I - z_j^J)^{K_{IJ}} e^{-\sum |z|^2/4}$$

Deconstruct a third spatial dimension.

a bit like [Arkani-Hamed, Cohen, Georgi hep-th/0104005]

$$\mathcal{L} = \frac{K_{IJ}}{4\pi}\epsilon_{\mu\nu\rho}\mathcal{A}^I_\mu \partial_\nu \mathcal{A}^J_\rho - \frac{1}{4e^2}\mathcal{F}^I_{\mu\nu}\mathcal{F}^{\mu\nu J}$$

Take $I, J = 1...L$. Think of it as $z = aI$ for some lattice spacing a.

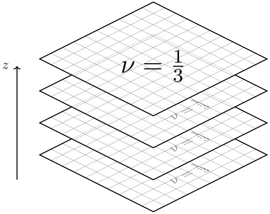
Assume:

- translation symmetry: $K_{IJ} = K_{(I+x)(J+x)}$.
- a version of locality: $K_{IJ} = 0$ for $|I - J| >$ some finite number.
  (‘quasidiagonal’)

Note: we can add the components of the Maxwell term with indices along z without affecting conclusions.

Trivial example.

$$K = \begin{pmatrix} \ddots & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$



Defining fracton model features:

- GSD on T^2_{xy} is 3^L.
- Quasiparticles move only within layers
- $S(\text{ball of radius } R) \sim R^2 - \gamma R$

The first and third features follow from $s = 2$.

Interesting example.

$$K(131) \equiv \begin{pmatrix} 3 & 1 & & & 1 \\ 1 & 3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3 & 1 \\ 1 & & & 1 & 3 \end{pmatrix}$$

Striking phenomena:

- GSD $= \left(\frac{1}{2}\left(3+\sqrt{5}\right)\right)^L + \left(\frac{1}{2}\left(3-\sqrt{5}\right)\right)^L - 2(-1)^L$ (integers!)

$\overset{L\to\infty}{\sim} \left(\frac{1}{2}\left(3+\sqrt{5}\right)\right)^L$

- Planon statistics: as $L \to \infty$,

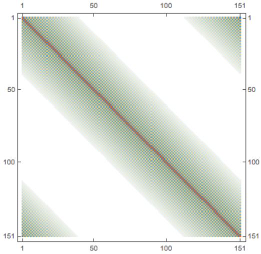
$$\frac{\theta_{IJ}}{2\pi} = K^{-1}_{IJ} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-3}{2}\right)^{|I-J|}$$

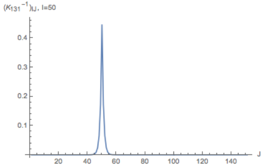
- Planon fusion group is $\mathbb{Z}_{F_L} \times \mathbb{Z}_{5F_L}$ ($F_L \equiv L$th Fibonacci number).

θ_{IJ} is

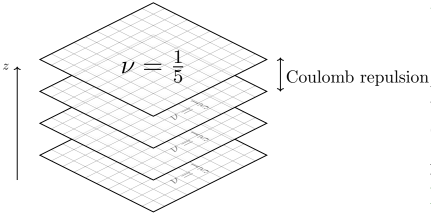
(a) Irrational!

(b) Not ultra-local in $|I-J|$!





Experimental realization. (!)



Actually the $K(131)$ model was studied 30 years ago! [Qiu-Joynt-Macdonald 89, 90. Also Naud-Pryadko-Sondhi 00]

Layers of $\nu = \frac{1}{5}$ FQH with no tunneling but Coulomb repulsion $\implies$ mutual zeros in neighboring layers.

Recall:

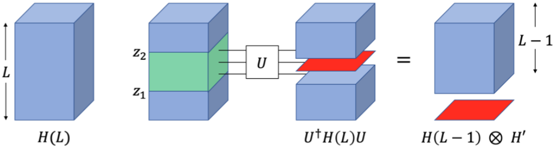
$$\Psi\left(\{z_i^I\}\right) = \prod_{i<j,I,J} (z_i^I - z_j^J)^{K_{IJ}} e^{-\sum |z|^2/4}$$

Contrast with foliated fracton TO.

If $\log(\text{GSD}) \sim L$, what do we mean by 'phase'? In answer to the question "what is a fracton phase?" [Shirley-Slagle-Chen 18]

Foliated fixed point:

= 2d TO layer! = free resource



New notion of "stable equivalence" allows the addition of layers of 2d TO.

Consequences of existence of this foliation structure:

- Exponential scaling of GSD is exact
- Statistics are short-ranged
- Planon fusion group has finite order (independent of L)

Conclusion: $K(131)$ is not foliated!

Some other choices of K are foliated.

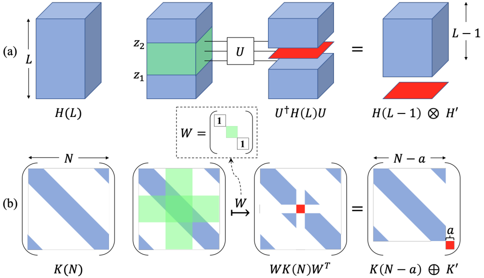
[Shirley-Slagle-Chen, 1907.09048]

$$
K_{\mathrm{F}} = \begin{array}{c} \begin{array}{ccccccccc} & e_1 & m_1 & e_2 & m_2 & e_3 & m_3 & e_4 & \end{array} \\ \begin{pmatrix} \ddots & & & & & & & & \\ & 0 & 2 & -1 & & & & & \\ & 2 & 0 & & & & & & \\ & -1 & & 0 & 2 & -1 & & & \\ & & & 2 & 0 & & & & \\ & & & -1 & & 0 & 2 & -1 & \\ & & & & & 2 & 0 & & \\ & & & & & -1 & & 0 & \\ & & & & & & & & \ddots \end{pmatrix} \end{array}
\rightarrow \quad K_{\mathrm{F}}^{-1} = \frac{1}{4} \begin{array}{c} \begin{array}{ccccccccc} & m_0 & e_1 & m_1 & e_2 & m_2 & e_3 & m_3 & \end{array} \\ \begin{pmatrix} \ddots & & & & & & & & \\ & 0 & & 1 & & & & & \\ & & 0 & 2 & & & & & \\ & 1 & 2 & 0 & & 1 & & & \\ & & & & 0 & 2 & & & \\ & & & 1 & 2 & 0 & & 1 & \\ & & & & & & 0 & 2 & \\ & & & & & 1 & 2 & 0 & \\ & & & & & & & & \ddots \end{pmatrix} \end{array} .
$$

versus: $$K_{131}^{-1} = \frac{1}{65} \begin{pmatrix} 29 & -11 & 4 & -1 & -1 & 4 & -11 \\ -11 & 29 & -11 & 4 & -1 & -1 & 4 \\ 4 & -11 & 29 & -11 & 4 & -1 & -1 \\ -1 & 4 & -11 & 29 & -11 & 4 & -1 \\ -1 & -1 & 4 & -11 & 29 & -11 & 4 \\ 4 & -1 & -1 & 4 & -11 & 29 & -11 \\ -11 & 4 & -1 & -1 & 54 & -11 & 29 \end{pmatrix}$$

K-matrix formulation exhibits foliation structure.

$K' \cong W^T K W, W \in \mathsf{SL}(M, \mathbb{Z})$.



$$K_{\mathrm{F}} = \begin{array}{c} \begin{array}{ccccccc} e_1 & m_1 & e_2 & m_2 & e_3 & m_3 & e_4 \end{array} \\ \begin{pmatrix} \ddots & & & & & & & \\ & 0 & 2 & -1 & & & & \\ & 2 & 0 & & & & & \\ & -1 & & 0 & 2 & -1 & & \\ & & & 2 & 0 & & & \\ & & & -1 & & 0 & 2 & -1 \\ & & & & & 2 & 0 & \\ & & & & & -1 & & 0 \\ & & & & & & & & \ddots \end{pmatrix} \end{array}$$

If W is only nontrivial in a finite block, this is local in z.

$$W = \begin{array}{c|c} & \begin{array}{ccccccccc} \tilde{e}_1 & \tilde{m}_1 & \tilde{e}^A & \tilde{m}^A & \tilde{e}^B & \tilde{m}^B & \tilde{e}_2 & \tilde{m}_2 & \tilde{e}_3 \end{array} \\ \begin{array}{c} \\ e_1 \\ m_1 \\ e_2 \\ m_2 \\ e_3 \\ m_3 \\ e_4 \\ m_4 \\ e_5 \\ \\ \end{array} & \begin{pmatrix} \ddots & & & & & & & & & \\ & 1 & & & & -1 & & & -1 & & \\ & & 1 & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & 1 & & & & 1 & & \\ & & & & & 1 & & & & & \\ & & 1 & & & & 1 & & & & \\ & & & -1 & & & & 1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & 1 & \\ & & & & & & & & & & \ddots \end{pmatrix} \end{array} \to \quad W K_{\mathrm{F}} W^T = \begin{array}{c} \begin{array}{ccccccccc} \tilde{e}_1 & \tilde{m}_1 & \tilde{e}^A & \tilde{m}^A & \tilde{e}^B & \tilde{m}^B & \tilde{e}_2 & \tilde{m}_2 & \tilde{e}_3 \end{array} \\ \begin{pmatrix} \ddots & & & & & & & & & \\ & 0 & 2 & & & & & -1 & & \\ & 2 & 0 & & & & & & & \\ & & & 0 & 2 & -1 & 0 & & & \\ & & & 2 & 0 & 0 & 0 & & & \\ & & & -1 & 0 & 0 & 2 & & & \\ & & & 0 & 0 & 2 & 0 & & & \\ & -1 & & & & & & 0 & 2 & -1 \\ & & & & & & & 2 & 0 & \\ & & & & & & & -1 & & 0 \\ & & & & & & & & & & \ddots \end{pmatrix} \end{array}$$

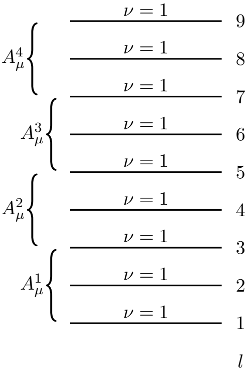
$= \mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory

(Open Q: what are the equivalence classes?)

Lattice construction.

Q: Can these QFTs really arise from local 3d lattice models?
$[\mathcal{A}_x(I), \mathcal{A}_y(J)] \sim \left(K^{-1}\right)^{IJ} \neq 0$ for $|I-J| \gg 1$.

A: Yes. Take layers of IQH, $l \in \mathbb{Z}$, coupled to $\mathsf{U}(1)$ gauge fields A^I like this: $\longrightarrow \quad \longrightarrow \quad \longrightarrow$



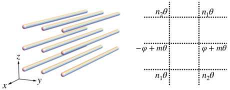
$$H = \sum_l \sum_{\langle \mathbf{rr}' \rangle} u_{l,\mathbf{rr}'} e^{\mathbf{i} \sum_I q^{Il} A^I_{\mathbf{rr}'}} c^\dagger_{l,\mathbf{r}'} c_{l,\mathbf{r}} + \sum_I \left[\sum_{\langle \mathbf{rr}' \rangle} g_E \left(E^I_{\mathbf{rr}'}\right)^2 - g_B \sum_p \cos B^I_p + g_Q \sum_{\mathbf{r}} \left(Q^i_{\mathbf{r}}\right)^2 \right]$$

$$Q^i_{\mathbf{r}} \equiv (\nabla \cdot \mathbf{E})^i_{\mathbf{r}} - \sum_l q^{il} c^\dagger_{l,\mathbf{r}} c_{l,\mathbf{r}}$$

Integrate out the fermions:
$\mathcal{L} = -\frac{1}{4\pi} \sum_l \epsilon^{\mu\nu\lambda} a^l_\mu \partial_\nu a^l_\lambda + \frac{1}{2\pi} \sum_{Il} q^{Il} \epsilon^{\mu\nu\lambda} A^I_\mu \partial_\nu a^l_\lambda$.
A local unitary transf W gives $K(131)$ plus decoupled IQH layers.

(Analogous construction exists for any **quasidiagonal** K.)

[also $\exists$ a coupled-wire construction: Sullivan-Dua-Cheng, 2010.15148]



Spectrum.

Include Maxwell terms. Gap = smallest |eigenvalue| of K:

$\omega^2 = k_x^2 + k_y^2 + \left(\frac{e^2}{2\pi} K_q\right)^2$ where K_q are the eigenvalues of K.

For $K(1n1) \equiv \begin{pmatrix} n & 1 & & & 1 \\ 1 & n & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & n & 1 \\ 1 & & & 1 & n \end{pmatrix}$, the eigenvalues are $K_q = n + 2\cos q$

$K_q > 0$ for $n > 2$.

For $n = 2$:

K^{-1}_{IJ} decays linearly in $|I - J|$.

$$K(121)^{-1} = \frac{1}{4} \begin{pmatrix} 11 & -9 & 7 & -5 & 3 & -1 & -1 & 3 & -5 & 7 & -9 \\ -9 & 11 & -9 & 7 & -5 & 3 & -1 & -1 & 3 & -5 & 7 \\ 7 & -9 & 11 & -9 & 7 & -5 & 3 & -1 & -1 & 3 & -5 \\ -5 & 7 & -9 & 11 & -9 & 7 & -5 & 3 & -1 & -1 & 3 \\ 3 & -5 & 7 & -9 & 11 & -9 & 7 & -5 & 3 & -1 & -1 \\ -1 & 3 & -5 & 7 & -9 & 11 & -9 & 7 & -5 & 3 & -1 \\ -1 & -1 & 3 & -5 & 7 & -9 & 11 & -9 & 7 & -5 & 3 \\ 3 & -1 & -1 & 3 & -5 & 7 & -9 & 11 & -9 & 7 & -5 \\ -5 & 3 & -1 & -1 & 3 & -5 & 7 & -9 & 11 & -9 & 7 \\ 7 & -5 & 3 & -1 & -1 & 3 & -5 & 7 & -9 & 11 & -9 \\ -9 & 7 & -5 & 3 & -1 & -1 & 3 & -5 & 7 & -9 & 11 \end{pmatrix}$$

Previous related gapless models: [Levin-Fisher 2009]

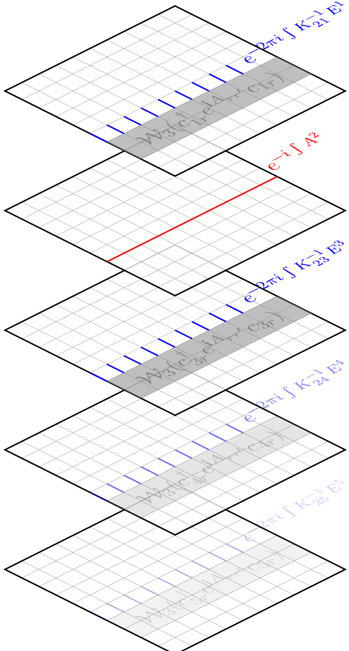
String operators

Nice bonus of the lattice construction: explicit operators that create and transport the planons.

To create planons in layer I:

$$\mathcal{W}^I = \underbrace{e^{-\mathbf{i}\int A^I}}_{\text{creates charge at endpoint}} \cdot \underbrace{e^{-2\pi\mathbf{i}K^{-1}_{IJ}\int E^J}}_{\text{creates flux at endpoint}} \cdot \underbrace{\mathcal{W}_3^I}_{\text{quasi-adiabatic response of fermions}}$$

For $K(131)$, they have tails.



Summary.

Depending on K, infinite-component CS theories can be

- gapped and foliated.
  Interesting math problem: classify
  $K' \sim W(K \oplus \sigma^x)W^T, W \in \mathsf{SL}(M, \mathbb{Z})$, local.

- gapped and not foliated.
  Examples of fracton TO beyond exactly solvable models (and beyond 'topological defect network' constructions).

- gapless.
  ???
  Weak SSB – no local order parameter [Dua-Sullivan-Cheng, Ma-Lam-Chen, to appear]

More open questions.

- Is there an isotropic version of this construction?

- Relation to quiver gauge theories [Razamat, 2107.06465] and D-branes [Geng-Kachru-Karch-Nally-Rayhaun 2108.08322]?

The end.

Thanks for listening.