Hierarchical growth of entangled states

or

'*s*-sourcery'

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based on work

(arXiv:1407.8203, 1505.07106, 1602.02805, 1607.05753, in progress)

with

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and

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and

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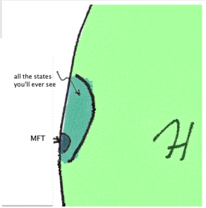
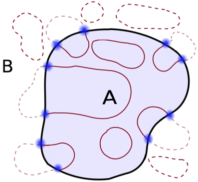
Big goal: Understand the structure of entanglement in physical states of quantum field theories

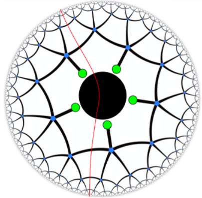
- necessary for numerical simulation
  (What resources are required? where in Hilbert space to look?)



- useful as a diagnostic
  (how to distinguish different phases with the same symmetries?) [Fig: T. Grover]



- point of contact with holographic duality
  (entanglement entropy $\simeq$ bulk area) [Fig: D. Harlow]



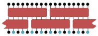
Context

- $\mathcal{H} = \otimes_x \mathcal{H}_x$
- $H = \sum_x H_x$ hamiltonian 'motif'

  (rules out many horrible pathologies). support of H_x is localized.
- families of systems labelled by (linear) system size L: H_L with groundstate(s) $\{|\psi_L\rangle\}$

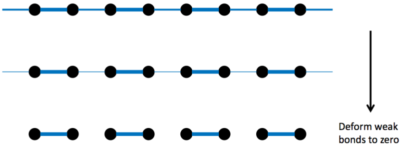
Coarsely-stated, impossible desideratum: low-depth unitary $\mathbf{U}$ which constructs the groundstate *from smaller unentangled subsystems* :

$$|\psi_L\rangle \overset{??}{=} \mathbf{U}|0\rangle^{\otimes L} \qquad \mathbf{U} \overset{??}{=}$$



Warmup example
$(d = 1, s = 0)$:
$H(\eta) = \sum_n (1 + (-1)^n \eta) \mathbf{c}_n^\dagger \mathbf{c}_{n+1} + hc$
adiabatically deform 1d band insulator

to product state



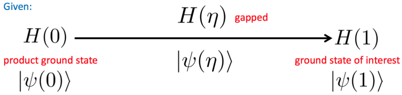
Given:

$H(0) \xrightarrow{H(\eta)\ \text{gapped}} H(1)$

product ground state $|\psi(0)\rangle$ — $|\psi(\eta)\rangle$ — ground state of interest $|\psi(1)\rangle$

Construct: $\mathbf{U} \overset{?}{=} \mathrm{P}e^{\mathbf{i}\int_0^1 d\eta \mathbf{H}(\eta)}$

There are two problems with this plan, in general



Construct: $\mathbf{U} \stackrel{?}{=} \mathrm{P}e^{\mathbf{i}\int_0^1 d\eta \mathbf{H}(\eta)}$

1. (Technical, solvable) Even if $H(\eta)$ all have gap $\geq \Delta > 0$, adiabatic evolution has a nonzero failure probability (per unit time, per unit volume).

Solution [Hastings, Wen]:
Find quasilocal $\mathbf{K}$ such that
$\mathbf{i}\partial_\eta|\psi(\eta)\rangle = \mathbf{K}(\eta)|\psi(\eta)\rangle$
$\rightsquigarrow$ Produce quasi-local $\mathbf{U} = e^{\mathbf{i}\int_0^1 d\eta \mathbf{K}(\eta)}$.

$$K = -\mathbf{i}\int_{-\infty}^{\infty} dt F(t) e^{\mathbf{i}H(\eta)t}\partial_\eta H(\eta) e^{-\mathbf{i}H(\eta)t}$$

$F(t)$ odd, rapidly decaying, $\tilde{F}(0) = 0$,
$\tilde{F}(\omega) = -\frac{1}{\omega}, |\omega| \geq \Delta$.

Quasilocal means:
$U = e^{\mathbf{i}K}, \quad K = \sum_x K_x, \quad K_x = \sum_r K_{x,r}$
$K_{x,r}$ supported on disk of radius r, $\parallel K_{x,r} \parallel \leq e^{-r^{1-d}}$

2. (Crucial, physical) Nontrivial states of matter are *defined* by the inability to find such a gapped path to a product state!

Expanding universe strategy

[Swingle, JM, 1407.8203, PRB]

Instead, we are going to *grow* the system
$|\psi_L\rangle \to |\psi_{2L}\rangle$ with local unitaries.

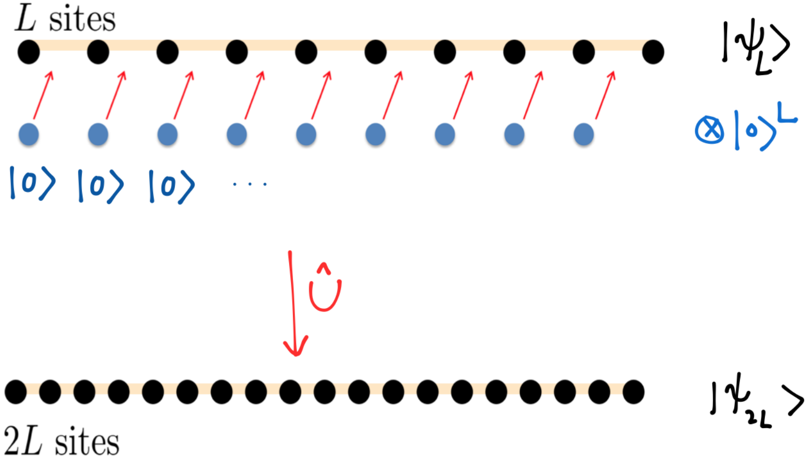
Then, iterate: $\mathbf{U} \sim \cdots \circ U_{4L_0 \leftarrow 2L_0} \circ U_{2L_0 \leftarrow L_0}$.

U will in general not have finite depth.
but **U** will have an RG structure.

Assumptions:

- Raw material: a bath of ‘ancillas’ $\otimes|0\rangle^M$ is freely available.
- For rigorous results, energy gap Δ for all excitations.
- There may be groundstate degeneracy $G(H_L)$
  but the groundstates are *locally indistinguishable*
  (a necessary condition for the state to be stable)

$d = 1, s = 1$ example: (Not like crystal growth!)



$$|\psi_{2L}\rangle = U\left(|\psi_L\rangle \otimes |0\rangle^L\right).$$

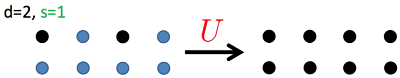
An s-source RG fixed point

(in d dimensions) is a system whose groundstate on $(2L)^d$ sites can be made from s copies of the groundstate on L^d sites (plus unentangled ancillas) using a quasilocal unitary.

[Swingle, JM, 1407.8203, PRB]

$$|\psi_{2L}\rangle = U \left(\underbrace{|\psi_L\rangle \cdots |\psi_L\rangle}_{s} \otimes |0\rangle^M \right)$$

$$M = L^d(2^d - s)$$



How to construct **U**:

By quasiadiabatic evolution from $\sum_1^s H_L + \sum_{\text{ancillas, a}} Z_a$ to H_{2L}:

(*e.g.* for $s = 1$ we must start with $s = 1$ copy at size L.)

Given:

$H(\eta)$ gapped

$H(0) \longrightarrow H(1)$

ground state on size L (and product states) $|\psi(0)\rangle$

$|\psi(\eta)\rangle$

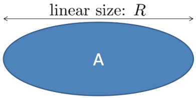
ground state on size 2L $|\psi(1)\rangle$

$\rightsquigarrow$ Produce quasi-local **U** .

[Hastings, Wen]

Reminder: quasilocal means:

$U = e^{\mathbf{i}K}$, $K = \sum_x K_x$, $K_x = \sum_r K_{x,r}$ $K_{x,r}$ supported on disk of radius r, $\| K_{x,r} \| \leq e^{-r^{1-d}}$



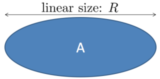
Basic property: Recursive entropy bounds:
(Uses Small Incremental Entangling result of
[Kitaev, Bravyi, van Acoleyn-Marien-Verstraete 2014].)

$$S(2R) \leq sS(R) + kR^{d-1}$$
$$S(2R) \geq sS(R) - k'R^{d-1}$$

Why is s-source RG fixed point a useful notion?

1. Such a circuit controls the growth of entanglement with system size: Area law theorem: any $s \leq 1$ fixed point in $d > 1$ enjoys an area law for EE of subregions.

$$S(A) \equiv -\mathrm{tr}\rho_A \log \rho_A \leq k|\partial A| = kR^{d-1}.$$

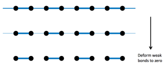


$s \geq 2^{d-1}$ is required to violate the area law.

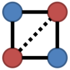
2. The groundstate degeneracy satisfies: $G(2L) = G(L)^s$
3. s (smallest possible) is a property of the phase (since by definition an adiabatic path connects any two representatives) $\implies$ classification axis.
4. The circuit implies a MERA representation of the groundstate.

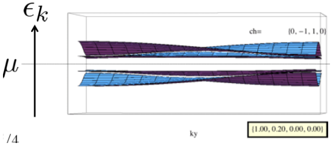
Many interesting states are s-source fixed points

- Mean field symmetry-breaking states $(s = 0)$



- Chern insulators, IQH $(s = 1)$





- Topological states (discrete gauge theory, fractional QH), including chiral ones $(s = 1)$

- Any *topological quantum liquid*

  $\equiv$ insensitive to smooth deformations of space $\simeq$ gapped QFT

  has $s = 1$.

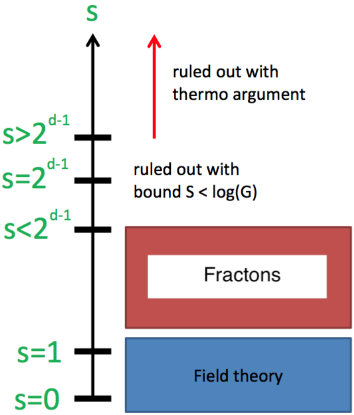
Why: place it in an expanding universe $ds^2 = -d\eta^2 + a(\eta)^2 d\vec{x}^2$

Experimental example: QCD

- Our universe is expanding, $t_{\text{doubling}} \sim 10^{10}$years.
- The QCD gap stays open $(m_\pi, m_p > 0)$.
- This is a gapped path from $|\psi_L\rangle$ to $|\psi_{2L}\rangle$.
- $\implies$ $\exists$ a quasilocal unitary which constructs the QCD groundstate from a small cluster plus ancillas (*i.e.* QCD has $s = 1$).

This suggests a new approach to simulating its groundstate which is in principle very efficient.

Reason to care #3: Classification of gapped states by s



$$S(R) \sim R^{\log_2(s)}$$

$$S(R) \leq R^{d-1} \log(R)$$

$$S(R) \sim R^{d-1}$$

← *e.g.*:
Layers of FQHE,
X-cube model [Chen et al],
Haah's cubic code [Haah]

have $d = 3$, $s = 2$.

$\Longrightarrow$ extensive GSD!

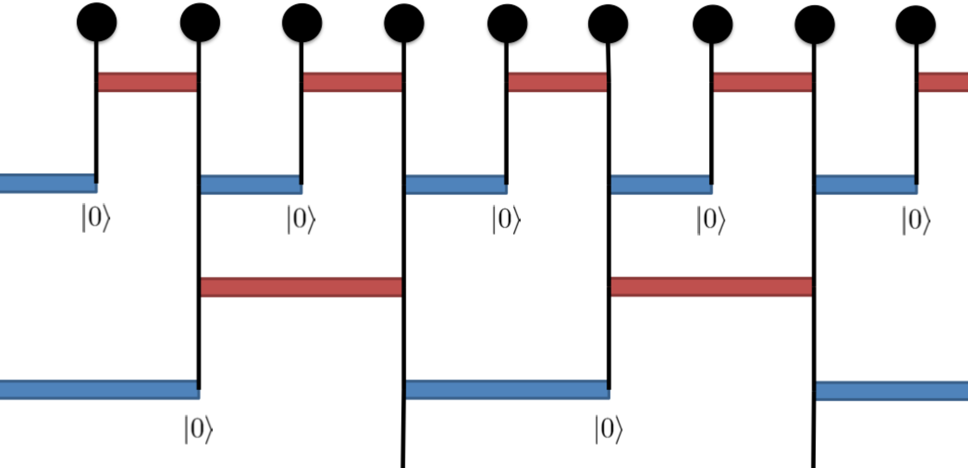
no ordinary TFT!

Reason to care #4: **U** $\rightsquigarrow$ MERA

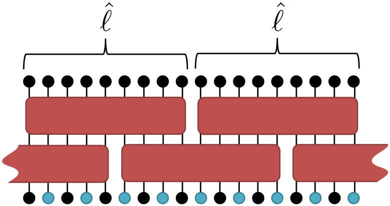
A MERA is a representation of the groundstate which: [Vidal]

- allows efficient computation of observables (few contractions)
- organizes the information by scale (like Wilson and AdS/CFT taught us to do)
- geometrizes the entanglement structure [Swingle]

(Best representation of 1d critical states, very hard to find in $d > 1$.)



MERA representations of $s = 1$ fixed points



Quasilocal $\mathbf{U} \overset{\text{Trotter}}{\rightarrow}$ low-depth circuit:

$$|\psi_L\rangle \simeq \mathbf{U}_{\text{circuit}} |\psi_{L/2}\rangle |0\rangle^{L/2}$$

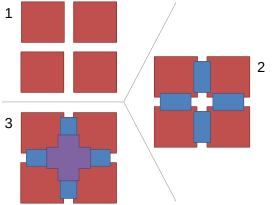
finite overlap requires $\hat{\ell} \sim \log^{1+\delta}(L)$

$$\Longrightarrow$$

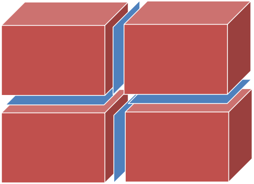
bond dimension $\sim e^{\hat{\ell}^d} \sim e^{c \log^{d(1+\delta)}(L)}$

Crucial point: This construction of $\mathbf{U}_{\text{circuit}}$ requires no variational sweeps on large system.

$d = 2$:



$d = 3$:



Gapless states and s-sourcery

- 'Entanglement Thermodynamics' constrains area law violation by gapless states
- and gives a relation between s and scaling exponents ($s = 2^{\theta}$).
- Examples of RG circuits for nontrivial critical points.

Entanglement bounds for gapless states

The area law is violated in groundstates of metals: $S \sim R^{d-1} \log k_F R$.

This violation is a symptom of many low-energy *extended* modes.

$\implies$ can be seen in thermodynamics.

Result: [Swingle-JM, 1505.07106, PRB]

If: thermal entropy of a scale-invariant state is $s(T) \sim T^{\frac{d-\theta}{z}}$

$z \equiv$ dynamical exponent

$\theta \equiv$ hyperscaling violation exponent

(anomalous dimension of T_{tt})

Then: the groundstate EE obeys the area law when $\theta < d - 1$ and $0 < z < \infty$.

(Recall: a Fermi surface has $\theta = d - 1$.)

Furthermore: $s = 2^{\theta}$.

(Fermi surface has $c_V \sim S \sim T \implies \theta = d - 1$, hence $s = 2^{d-1}$, marginally violates area law. ✓)

Entanglement thermodynamics

Idea: Recast EE as local thermodynamics problem ($T = T_x$)

Find $\boldsymbol{\sigma}_A \simeq Z^{-1} e^{-\sum_x \frac{1}{T_x} \mathbf{H}_x}$ ($\mathbf{H} \equiv \sum_x \mathbf{H}_x$. local Gibbs state)

such that $S(\boldsymbol{\sigma}_A) \geq S(\boldsymbol{\rho}_A)$.

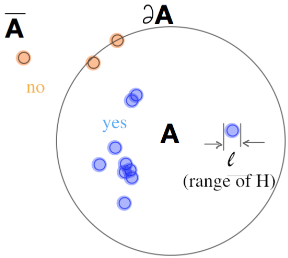
Who is $\boldsymbol{\sigma}_A$?

State of max entropy consistent w $\langle \mathbf{H}_x \rangle$ in A.

(hence $S(\boldsymbol{\sigma}_A) \geq S(\boldsymbol{\rho}_A)$)

[Cramer et al 2010, Swingle-Kim 2014] $\Longrightarrow$ $\boldsymbol{\sigma}_A \propto e^{-\sum_{x \in A} \mathbf{H}_x / T_x}$

$1/T_x$ = Lagrange multipliers



$$\mathrm{tr}\mathbf{H}_A \boldsymbol{\sigma}_A = \mathrm{tr}\mathbf{H}_A \boldsymbol{\rho}_A = \underbrace{E_{g,A}}_{\text{gs energy of } H_A} + \mathcal{O}\left(|\partial A| \ell |\mathbf{H}_x|\right)$$

$\Longrightarrow$ $\boldsymbol{\sigma}_A$ is a state with excitations localized at ∂A, $T_x \to 0$ in interior of A.

Entanglement thermodynamics

Crucial Fact **(local thermodynamics):** For scaling purposes,

$$\begin{aligned} \mathrm{tr}\mathbf{H}_A \boldsymbol{\sigma}_A &\simeq E_{g,A} + \int_A d^d x \; e(T_x) \\ -\mathrm{tr}\boldsymbol{\sigma}_A \log \boldsymbol{\sigma}_A &\simeq \int_A d^d x \; s(T_x) \end{aligned}$$

$e(T_x) = Ts(T_x)$, bulk thermodynamic densities at temp T_x.

Why: True if $1 \gg \frac{\nabla T_x}{T_x} \cdot \xi_x$ (for all x) ($\xi_x \equiv$ local correlation length).

But: let $\boldsymbol{\sigma}_A(\tau) \equiv Z(\tau)^{-1} e^{-\frac{1}{\tau}\sum_x \tilde{\mathbf{H}}_x / T_x} \overset{\tau \to 1}{\to} \boldsymbol{\sigma}_A$.

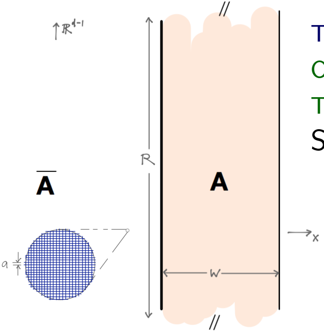
This state has temperature $T_x(\tau) = \tau T_x$, $\implies \xi_x(\tau) \sim T_x(\tau)^{-1/z} \propto \tau^{1/z}$

So (unless $z = \infty$!) the figure of merit for local thermo in state $\boldsymbol{\sigma}_A(\tau)$ is

$$1 \gg \underbrace{\frac{\nabla T_x(\tau)}{T_x(\tau)}}_{\sim \tau^0} \cdot \underbrace{\xi_x(\tau)}_{\sim \tau^{-1/z}} \overset{\tau \to \infty}{\to} 0.$$

$S(\boldsymbol{\sigma}_A(\tau)) = \tau^{\frac{d-\theta}{z}} S(\boldsymbol{\sigma}_A) \implies$ scales the same way with region size.

Scaling in strip geometry



To use local thermo, we need T_x.

Our question is local. Choose convenient geometry.

Translation invariant in $d-1$ dims (PBC). $R \gg w \gg a$.

Scale invariance $\implies$

$$T_x \sim \begin{cases} x^{-z} & \\ \infty & \text{(no)} \\ 0 & \text{(sometimes: frustration free } \mathbf{H}) \end{cases}$$

$$\implies \quad e(T_x) \sim x^{-z+\theta-d}, \quad s(T_x) \sim x^{\theta-d}$$

$$S_A \leq -\mathrm{tr}\boldsymbol{\sigma}_A \ln \boldsymbol{\sigma}_A \quad \sim R^{d-1} \int_a^w dx \; x^{-d+\theta}$$

$$\sim R^{d-1} \quad \left(a^{-d+\theta+1} - w^{-d+\theta+1}\right) \quad \overset{w\to\infty}{\to} \infty \text{ only if } d < 1+\theta$$

Hence: scale invariant states with $\theta < d-1$ obey the area law.

Connection to s-sourcery

[Swingle-JM, 1505.07106]

If our scaling theory is an s-source RG fixed point

$$S(2R) \leq s S(R) + k R^{d-1} \ .$$

Assume saturated (if not, can use smaller s) $\Longrightarrow$

$$\begin{aligned} S_A &= k \left(\frac{R}{a}\right)^{d-1} \sum_{n=0}^{\log_2(w/a)} \left(\frac{s}{2^{d-1}}\right)^n \\ &\overset{R \gg w \gg a}{\simeq} k \left(\frac{R}{a}\right)^{d-1} \left(1 - \left(\frac{a}{w}\right)^{d-1-\log_2 s} + \cdots \right) \end{aligned}$$

Compare subleading terms in EE of strip:

$$\boxed{s = 2^{\theta}}$$

Gapless states with explicit $s = 1$ RG circuits

Expectation: CFTs are $s = 1$ fixed points.

∞ many examples of $d = 2$ quantum critical points
which are exact $s = 1$ fixed points: **'Square-root states'** [Kimball 1979]

• Classical stat mech model in d space dimensions	$\longrightarrow$	• Quantum system in d space dimensions
• configurations s	$\longrightarrow$	• states $\lvert s\rangle$ (orthonormal)
• Boltzmann weight $e^{-\beta h(s)}$ $\mathcal{Z} \equiv \sum_s e^{-\beta h(s)}$	$\longrightarrow$	• g.s. wavefunction $\lvert h, \beta\rangle = \mathcal{Z}^{-1/2} \sum_s e^{-\beta h(s)/2} \lvert s\rangle$
• coolness $\beta = 1/T$	$\longrightarrow$	• coupling
e.g. near-neighbor Ising model: $h(s) = \sum_{\langle ij\rangle} s_i s_j$	$\longrightarrow$	$\mathbf{Z}_i \lvert s\rangle = s_i \lvert s\rangle$. Parent Hamiltonian: $\mathbf{H} = \sum_i \left(-\mathbf{X}_i + e^{-\beta \mathbf{Z}_i \sum_{\langle i\lvert j\rangle} \mathbf{Z}_j} \right)$
• correlations $\langle Z_r Z_{r'}\rangle$	$=$	• correlations of diagonal operators $\langle \text{gs}\lvert \mathbf{Z}_r \mathbf{Z}_{r'} \rvert \text{gs}\rangle$
• classical critical point	$\longrightarrow$	• quantum critical point
• real-space RG scheme	$\longrightarrow$	• quantum RG circuit with $s = 1$

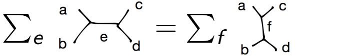
RG circuits for square root states: example

2d classical Ising TRG scheme: $\mathcal{Z} = \sum_{abcd\ldots} T_{abc} T_{ade} \cdots$

Two parts of classical RG step

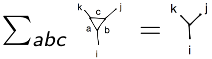
[Levin-Nave]:

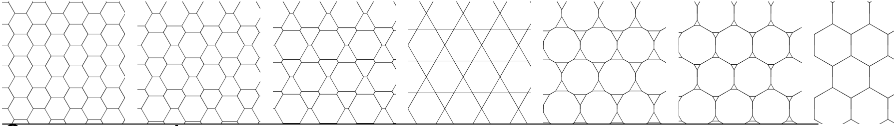
$1: \sum_e T_{abe} T_{cde} = \sum_f S_{acf} S_{bdf}$



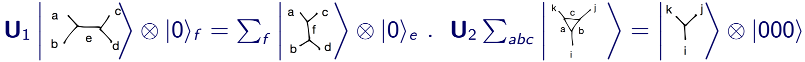
[Different use of related machinery: Evenbly-Vidal, TNR]

$2: \sum_{abc} S_{akc} S_{cjb} S_{bia} = T'_{ijk}$





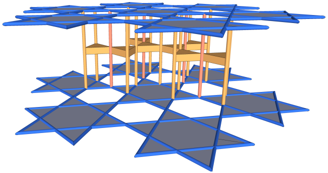
Quantum version:



$\mathbf{U} = \prod \mathbf{U}_2 \prod \mathbf{U}_1$

Fixed point of classical TRG

$\implies \quad s = 1$ fixed point.



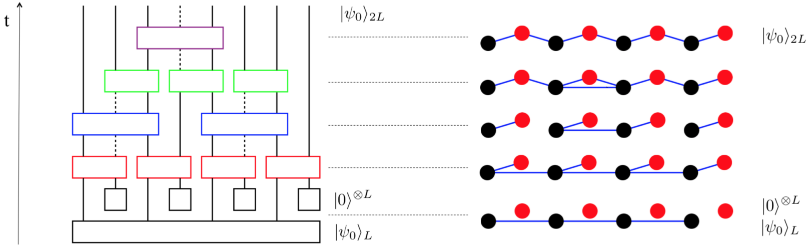
[JM, B Swingle, Shenglong Xu, 1602.02805, PRB]

Numerical *s*-sourcery

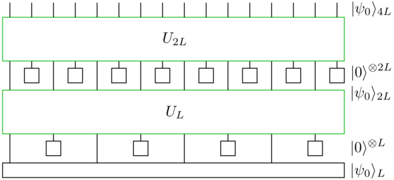
[w/ Chris Olund, Snir Gazit, Norman Yao (Berkeley), in progress]

s-sourcery with teeth

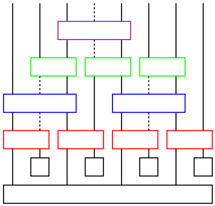
Trotterize expanding-universe time evolution $U_L : L \to 2L$.



Iterate:



'Control theory' implementation



First step: ↑ find $L = 4$ groundstate $|\psi_0\rangle_L$ (DMRG or ED).

Second step: find tensors.

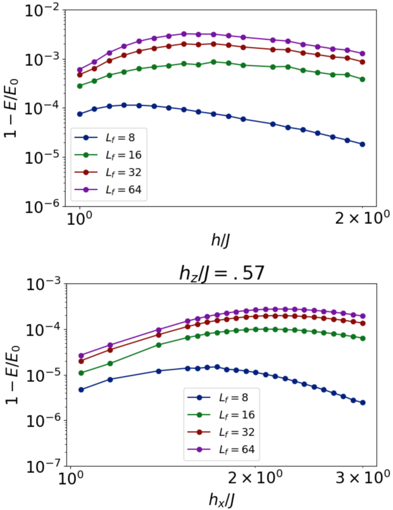
idea #1: find the tensors which optimize the fidelity in going from $L = 4$ to $L = 8$ and repeat.

bad: (1) fidelity is too stringent a metric. (2) away from fixed points, circuit should not be scale invariant.

idea #2: greedy algorithm. first optimize $4 \to 8$ layer. then optimize $8 \to 16$ layer, etc.

Results

For Ising chain, $H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - h\sum_i \sigma_i^x - h_z\sum_i \sigma_i^z$ (integrability-breaking does not qualitatively change the outcome).



Some noise and basin-hopping required to avoid local minima (not used for MERA optimization).

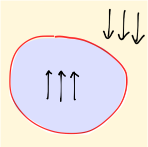
The maximum of the error as a function of h is not at the max correlation length.

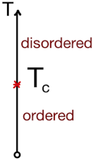
Mixed s-sourcery

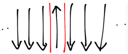
Motivation: finite-T quantum memory/topological order

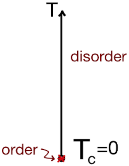
Energy cost to flip bit $\propto L^{d-1}$ vs $d = 1$.

Stable finite-T **classical** memory:
Ising model in $d > 2$

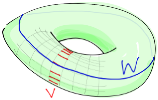




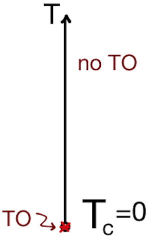


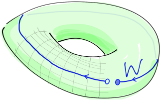


$T = 0$ quantum memory:
e.g. toric code in $d = 2, 3$



$WV = -VW$

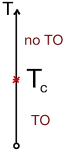




At $T > 0$, density of anyons $\propto e^{-\Delta/T}$.

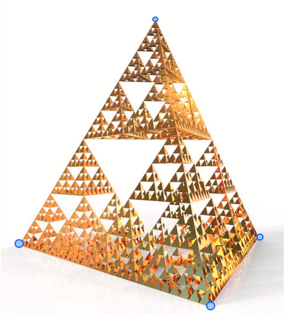
Stable finite-T quantum memory:
2-form toric code in $d = 4$

Anyonic excitations are *strings*: density $\propto e^{-\sigma L/T} \overset{L\to\infty}{\to} 0$

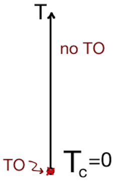


$\exists$ finite-T stable quantum memory in $d \leq 3$?

Best attempts so far: gapped fracton models [Haah, Vijay, Fu, Williamson, **Chen**, **Hermele**...].



Anyonic excitations can be *immobile*.
But [Bravyi-Haah]:
in known models, there's a lot of them.
Entropy wins in $F = E - TS$.
It seems $T_c = 0$ still.



[fig: A. Pasieka]

Further motivation for mixed *s*-sourcery:

The extension of tensor network ideas to open quantum systems will be useful.
Even for thermal equilibrium, given $\boldsymbol{\rho} = Z^{-1} e^{-\beta \mathbf{H}}$,
expectations are not, in general, computable.

Mixed s-sourcery

[Swingle-JM, 1607.05753]

What should replace the unitaries in the s-source RG circuit?

A sequence of states $\{\rho_L\}$ form a **purified** s source fixed point if there exists a sequence of purifications $\{|\sqrt{\rho_L}\rangle_{12}\}$ with $\mathrm{tr}_2(|\sqrt{\rho_L}\rangle\langle\sqrt{\rho_L}|_{12}) = \rho_L$ and

$$|\sqrt{\rho_{2L}}\rangle = \tilde{V}\left(\underbrace{|\sqrt{\rho_L}\rangle \otimes \ldots \otimes |\sqrt{\rho_L}\rangle}_{s \text{ times}} \otimes |0\ldots0\rangle\right)$$

where $|0\ldots0\rangle$ is a product state of the appropriate size and $\tilde{V}$ is a quasi-local unitary on A^sE. *i.e.*: $\exists$ a quasilocal channel $\boldsymbol{\rho}_{2L} = \mathcal{N}\left(\boldsymbol{\rho}_L^{\otimes s} \otimes |0\ldots0\rangle\langle 0\ldots0|\right)$.

- The entropy can be volume law, but the mutual information ($I(A:B) \equiv S(A) + S(B) - S(AB)$ bounds correlations) is still area law:

$$I(A_{2R} : A_{2R}^c) \leq sI(A_R : A_R^c) + kR^{d-1}.$$

- Local channel preserves locality of operators $\implies$ efficiently contractible.

Local free fermions are mixed $s = 0$

[Swingle-JM, 1607.05753]

$$H = \sum_{xy} c_x^\dagger h_{xy} c_y + h.c., \qquad \text{with } h_{xy} \to 0 \text{ for } |x-y| \gg a$$

thermal eqbm: $\boldsymbol{\rho}_T = e^{-\mathbf{H}/T}/Z = \mathrm{tr}_2 \underbrace{\sum_E \sqrt{\frac{e^{-\beta E}}{Z}} |E\rangle_1 |\tilde{E}\rangle_2}_{\equiv |T\rangle}$ is $s = 0$.

$|T\rangle$ is the groundstate of $\left(f_k = \frac{1}{e^{\epsilon_k}+1}\right)$

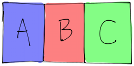
$$H_T \equiv \sum_k \left(-d_k^\dagger d_k + \tilde{d}_k^\dagger \tilde{d}_k\right), \qquad \left(\begin{smallmatrix} d_k \equiv \sqrt{f_k} c_k + \sqrt{1-f_k} \tilde{c}_k, \\ \tilde{d}_k \equiv -\sqrt{f_k} c_k + \sqrt{f_k} \tilde{c}_k \end{smallmatrix}\right)$$

which is gapped, local and adiabatically connected to

$$H_\infty = -\sum_x \left(c_x^\dagger c_x + \tilde{c}_x^\dagger \tilde{c}_x\right), \quad |\mathrm{gs}_\infty\rangle = \prod_x \frac{c_x^\dagger + \tilde{c}_x^\dagger}{\sqrt{2}} |0\rangle \qquad \text{(ultralocal).}$$

So the resulting a quasiadiabatic $\mathbf{U}$ gives a quasilocal channel:
$\boldsymbol{\rho}_T \to \mathrm{tr}_2 \mathbf{U} |T\rangle\langle T| \mathbf{U}^\dagger =$ product state.

Approximate quantum Markov chains



Basic idea: at finite T, correlations are short-ranged.

$\Longrightarrow$ The full state can be reconstructed by local operations on its parts.

- classical Markov chain $A \to B \to C$ has no memory: $p(c|ba) = p(c|b)$.

$0 = I(A:C|B) \equiv H(AB) + H(BC) - H(ABC) - H(B)$

$$\text{reconstruction (Bayes, 1750s)}: \quad p(abc) = \frac{p(ab)p(bc)}{p(b)}$$

- exact quantum Markov chain (saturates SSA):

$0 = I(A:C|B) \equiv S(AB) + S(BC) - S(ABC) - S(B)$

$$\text{reconstruction (Petz, 1980s)}: \quad \log \rho_{ABC} = \log \rho_{AB} - \log \rho_B + \log \rho_{BC}.$$

- **approximate** quantum Markov chain: $0 \simeq I(A:C|B)$

$$\text{reconstruction (Fawzi-Renner, 2015)}: \quad \rho_{ABC} \simeq (\mathbb{1}_A \otimes \mathcal{N}_{BC})(\rho_{AB}).$$

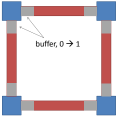
This is a refinement of SSA: $I(A:C|B) \geq D(\rho_{ABC}, \text{reconstruction})$. ($D$ = distance)

[Fawzi-Renner]: approximate quantum Markov chains can be reconstructed from marginals via a channel on the buffer.

A sufficient condition for mixed $s = 0$

$$S(A) = c_1 \mathrm{vol}(A) + \int_{\partial A} \left(c_2 + \sum_{i>2} c_i f_i(K, R) \right) + \mathcal{O}(\ell^d e^{-\ell/\xi}) \quad (\star)$$

$\ell \equiv$ linear size of A, $K, R \equiv$ curvatures of ∂A.

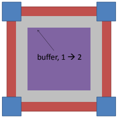


$\implies I(A : C|B) \approx 0$ if $\begin{matrix} AB + BC - B - ABC = 0 \\ \text{and } \partial B + \partial(AC) = \emptyset \end{matrix}$.

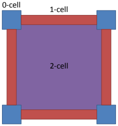
Make a cellular decomposition of space

$d = 1$: 

(all regions $> \xi$)



$\overset{(d=2)}{\longrightarrow}$

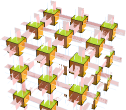


$$I(p\text{-cells} : (p-1)\text{-cells}|\text{buffer}) \approx \mathcal{O}(N_{\text{cells}} e^{-\ell/\xi}).$$

If so, then here is the state:

$$\rho = \rho_{\text{2-cells}\cup\text{1-cells}\cup\text{0-cells}} \approx \mathcal{N}_{1\to2}(\mathcal{N}_{0\to1}(\mathcal{N}_{\emptyset\to0}(\cdot)))$$

$d = 3$:

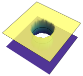


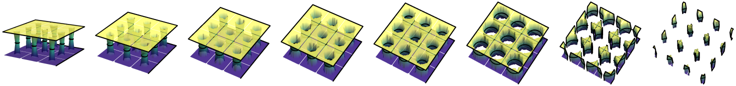
When is cellular reconstruction possible?

($\star$) is true for:

- invertible states.
- CFT at finite temperature.
- states with classical gravity duals.
- states which are not finite-T quantum memories [Hastings def of TO] :
  adiabatically connected to $T = \infty \implies$ quasilocal channel to product.

Run the construction backwards: an array of bubbles-of-Nothing.

bubble of Nothing:  [Witten 1985]



Two possible obstructions: edge modes and TEE [Preskill-Kitaev].

For p-form gauge theory at $T = 0$, $I_{p-1 \to p}, I_{d-p-1 \to d-p} \neq 0$

This construction was used in [Mahajan et al, 1608.05074] to make efficient representations of non-eqbm steady states associated with dissipative transport.

The idea: despite extensive von Neumann entropy, such states have low entanglement, hence tensor network representations.

Questions

Q: Is the thermal double $\sum_n \sqrt{\frac{e^{-\beta \mathbf{H}}}{Z}} |n\rangle |n\rangle$ always the groundstate of a local, gapped $\mathbf{H}$?

We showed 'yes' for free fermions and for sqrt states.

'Yes' lets us use groundstate s-sourcery.

Q: Can we improve the structure of the channel? The range of the resulting circuits is the thermal correlation length ($\to \infty$ as $T \to 0$).

Fawzi-Renner result doesn't take advantage of locality within the buffer B.

U will be more local if we incorporate the $s = 1$ groundstate circuit near the IR.

Q: Can we show explicitly that known fracton models are mixed $s = 0$?

Q: How to extend mixed s-sourcery to ultra-quantum non-equilibrium states?

The end.

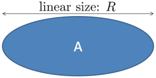
Thank you for listening.

State of matter	z	s	θ	EE
Insulators, etc.	Gap	0	n/a	Area
SSB, discrete	Gap	0	n/a	Area
IQHE (invertible)	Gap	1	n/a	Area
FQHE	Gap	1	n/a	Area
Topological states	Gap	1	n/a	Area
Fracton states (w gap, $d = 3$)	Gap	2	n/a	Area
SSB, continuous ($d > 1$)	1	1	0	Area
QCP (conformal), $d = 1$	1	1	0	Area*Log
QCP (conformal), $d > 1$	1	1	0	Area
Quadratic band touching	2	≤ 1	0	Area
Fermi liquids	1	2^{d-1}	$d - 1$	Area*Log
Spinon Fermi surface	3/2?	2^{d-1}	$d - 1$	Area*Log
Diffusive metal, $d = 3$	2	2^{d-2}	$d - 2$	Area
QED	1	1	0	Area
QCD	Gap	1^*	n/a	Area

Digression about dragons vs motives

There are nefarious counterexamples ('dragons') to many of these statements, but they all involve some *localized cleverness*.

An example of a pathology that goes away upon '**H**-motivization':

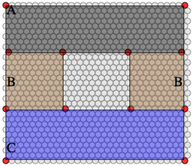


Expectation: Any 2d gapped state: $S_A = R/\epsilon - \gamma$

$\gamma \equiv$ 'topological entanglement entropy' $\epsilon \propto$ UV cutoff.

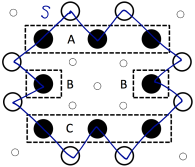
signature of topological order (determines $\mathcal{D}^2 \overset{\text{abelian}}{=} \#$ of T^2-groundstates: $\gamma = \log \mathcal{D}$) [Preskil-Kitaev, Levin-Wen 04].

γ can be extracted by inclusion-exclusion (cancel boundaries):



$$2\gamma = S_{AB} + S_{BC} - S_B - S_{ABC}$$

Counterexample [S. Bravyi]:



a qbit at each site ($\mathbf{X} \equiv \sigma^x, \mathbf{Z} \equiv \sigma^z$ to save eyesight) **pick** S

$$\mathbf{H}_{\text{snake}} = -\sum_{n \in S} \mathbf{Z}_{n-1}\mathbf{X}_n\mathbf{Z}_{n+1} - \sum_{n \notin S} \mathbf{X}_n$$

This state does *not* have topological order (unique gs)

$$|\text{gs}\rangle = |\text{snake state}\rangle_S \otimes_{n \notin S} |+\rangle_n$$

but has $\gamma = 1 \neq \log(\mathcal{D} = 1) = 0$. Not motivic!

Digression about dragons vs motives

Make the Bravyi snakes hop [D. Ben-Zion, D. Das, JM, 1511.01539, PRB]:

$$H = -\sum_i A_i - \sum_p B_p S_p - \sum_i X_i \prod_{l \in i} (Z_{i+l})^{\frac{1-\sigma_l^z}{2}} .$$

$\equiv \boldsymbol{\sigma}_\ell^z = -1$

a solvable model with a **H**-motif made from the Bravyi snake.

$$|\text{gs}\rangle = \sum_{\{S\}} |S\rangle \otimes \left(|\text{snake state}\rangle_S \otimes_{n \notin S} |+\rangle_n \right)$$

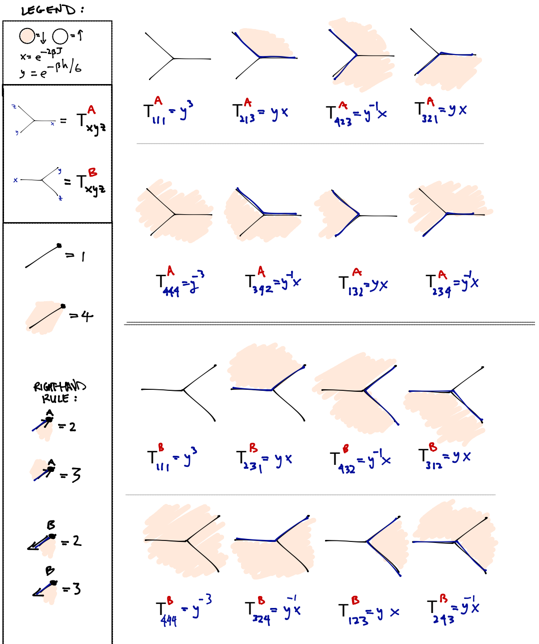
But this is in the same phase as $\mathbb{Z}_2$ gauge theory: $\gamma = \ln 2$. ($\checkmark$)

(**Side remark:** However, in the presence of symmetry ($\mathbb{Z}_2 \times \mathbb{Z}_2$), it is distinct. This is an example of a *symmetry-enriched topological phase*: the anyons come in doublets of $\mathbb{Z}_2 \times \mathbb{Z}_2$!)

The point of the Wagnerian digression: it is the motivic structure itself (not Siegfried) which slays the dragon.

Tensors for 2d ising model

$$Z = \sum_{abcd\ldots} T^A_{abc} T^B_{ade} \cdots$$



Further payoff: Invertible states

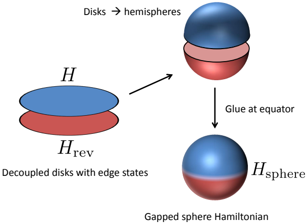
- A robust notion of 'short-range-entangled' Related ideas: [Kitaev, Freed]

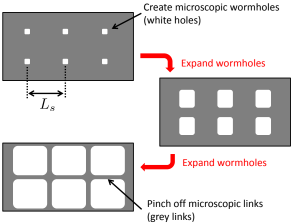
  'Invertible states,' $|\psi\rangle$ means $\exists |\psi^{-1}\rangle, \mathbf{U}$ s.t.

$$|\psi\rangle \otimes |\psi^{-1}\rangle = \mathbf{U}|0\rangle^{\otimes 2L^d} \text{ has } s = 0.$$

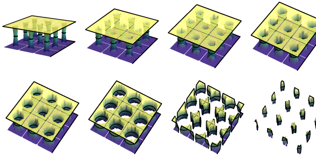
- Weak area law: a unique groundstate on any closed manifold (no topological order, but can still be interesting as SPTs) implies the existence of an inverse state and the area law.

Graphical proof of weak area law:





side view of $H + H_{\text{rev}}$ + wormholes :



step 1: 'edge inverse' kills edge states

step 2: make adiabatic path to $|0\rangle^{\otimes}$ on T^d