University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 2

Due 11pm Thursday, April 18, 2024

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1. **Anyons in the toric code.**

   (a) Show that when acting on a toric code groundstate the operator

   $$W_C = \prod_{\ell \in C} X_\ell$$

   creates a state which violates only the star operators at the sites in the boundary of C, ∂C, a pair of e-particles.

   (b) Show that when acting on a toric code groundstate the operator

   $$V_{\check{C}} = \prod_{\ell \perp \check{C}} Z_\ell$$

   creates a state which violates only the plaquette operators in the boundary of $\check{C}$, $\partial \check{C}$.

   (c) Show that a boundstate of an e particle and an m particle in the 2d toric code must be a fermion.

2. **Toric code as $\mathbb{Z}_2$ gauge theory with matter.** Consider a model with qubits on the links of a lattice (with Pauli operators X_ℓ, Z_ℓ, $X_\ell Z_\ell = -Z_\ell X_\ell$) and qubits on the sites of the lattice (with Pauli operators σ^x_j, σ^z_j, $\sigma^x_j \sigma^z_j = -\sigma^z_j \sigma^x_j$).

   (a) Show that the operator

   $$G_j \equiv A_j \sigma^z_j$$

   (where A_j is the star operator) generates the gauge transformation[1]

   $$\sigma^x_j \to (-1)^{s_j} \sigma^x_j, \quad Z_{ij} \to (-1)^{s_i} Z_{ij} (-1)^{s_j}, \; \sigma^z_j \to \sigma^z_j, \; X_{ij} \to X_{ij} \qquad (1)$$

   (where i, j are the sites at the ends of the link labelled ij). By *generates* here I mean that an operator $\mathcal{O}$ transforms as

   $$\mathcal{O} \to \mathcal{G}_s^\dagger \mathcal{O} \mathcal{G}_s, \quad \mathcal{G}_s \equiv \prod_j G_j^{s_j}$$

   with $s_j = 0, 1$.

[1] In the first version of this problem set I had reversed my convention about whether A is made of Xs or Zs.

(b) Show that the Hamiltonian

$$\mathbf{H} = -\sum_j G_j - \sum_p B_p - h \sum_{ij} \sigma_i^x Z_{ij} \sigma_j^x - g \sum_{\langle ij \rangle} X_{ij}$$

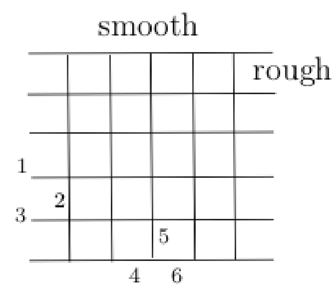
is gauge invariant.

Here we can identify σ_j^x as the operator which creates an e particle at site j. And we can identify $\sigma_j^z = (-1)^{n_j}$ as the parity of the number operator.

(c) Show that if we set $\sigma_j^x = 1$ and $\sigma_j^z = 1$ for all j we get back the (perturbed) toric code.

Bonus problem: interpret this operation as a choice of gauge in the model where $G_j = 1$ is imposed as a constraint on physical states.

3. **A topological qubit.** Consider the toric code on this cell complex:

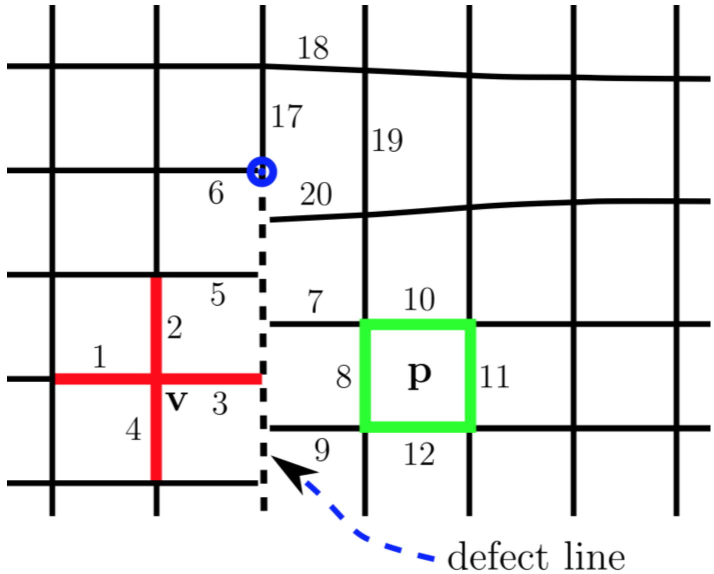


Recall that rough boundary conditions means that plaquette terms get truncated, such as the term $-X_1X_2X_3$, while smooth boundary conditions mean that star terms get truncated, such as the term $-Z_4Z_5Z_6$.

Show that there is a two-dimensional space of groundstates. A good way to do this is using the algebra of string operators which terminate at the various components of the boundary without creating excitations.

4. **Duality wall.** [Bonus problem] Show that the following hamiltonian realizes a

duality wall in the toric code.



What this means is that when crossing the wall, an e particle turns into an m particle and vice versa. (More precisely, there is a string operator which transports an e particle to the wall, and can be completed by a string operator transporting an m particle away from the wall, without creating any excitations.)

To be more precise about the figure: the dotted line carries no degrees of freedom. The terms in the hamiltonian along the wall are of the form

$$H = ... - X_2 X_5 X_3 Z_7 - Z_3 X_7 X_8 X_9$$

and there is a term at the end of the wall (the little blue circle) of the form $-Z_6 Y_{17} X_{18} X_{19} X_{20}$. Show that these terms commute with each other and all the usual star and plaquette terms, such as $A = Z_1 Z_2 Z_3 Z_4$ and $B = X_8 X_{10} X_{11} X_{12}$.

What can you say about the end of the duality wall (the little blue circle)?