University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 2 – Solutions

Due 11pm Thursday, April 18, 2024

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1. **Anyons in the toric code.**

   (a) Show that when acting on a toric code groundstate the operator

   $$W_C = \prod_{\ell \in C} X_\ell$$

   creates a state which violates only the star operators at the sites in the boundary of C, ∂C, a pair of e-particles.

   Clearly W_C commutes with B_p since they are both made of just Xs. If a site j touches C in the middle somewhere, A_j shares two edges with W_C and therefore $[A_j, W_C] = 0$. At the end of the curve is a site which shares only one edge with A_j and therefore they anticommute. Therefore acting with W_C changes the A_j-eigenvalue of the state from 1 to -1.

   (b) Show that when acting on a toric code groundstate the operator

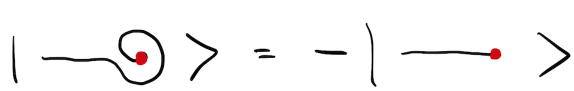
   $$V_{\check{C}} = \prod_{\ell \perp \check{C}} Z_\ell$$

   creates a state which violates only the plaquette operators in the boundary of $\check{C}$, $\partial\check{C}$.

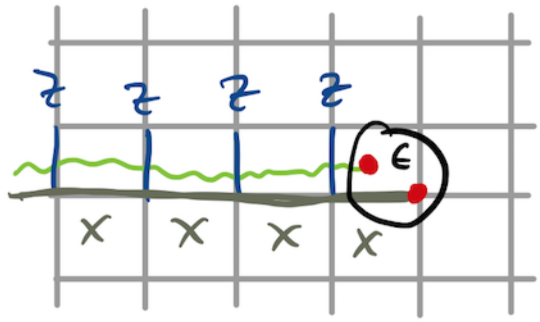
   This question is related to the previous by the duality map which takes the lattice to the dual lattice and interchanges $X \leftrightarrow Z$.

   (c) Show that a boundstate of an e particle and an m particle in the 2d toric code must be a fermion.

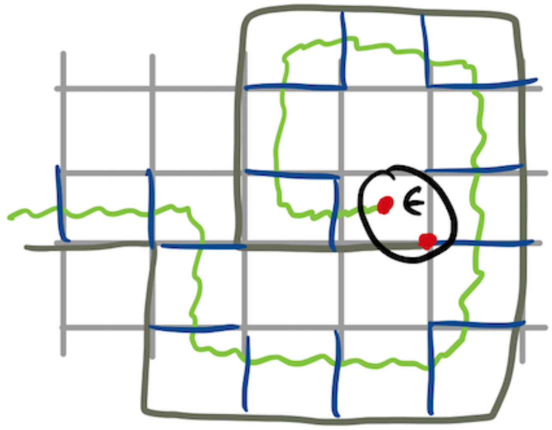
   Recall that a particle is a fermion if after rotating it by 2π its state picks up a minus sign:



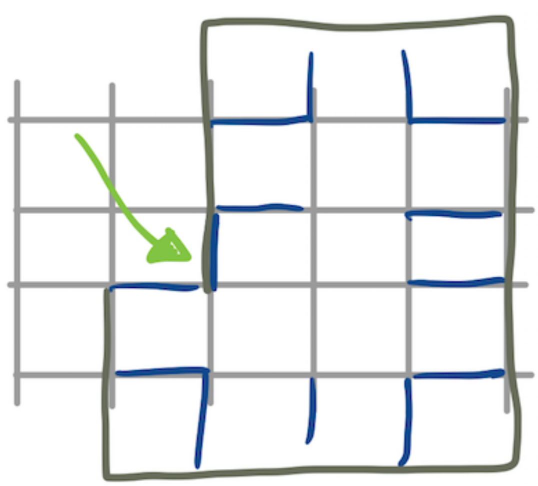
The operator which creates the epsilon particle looks like this, call it $\mathcal{O}_1$:



The operator which creates an epsilon particle and rotates it by 2π looks like this, call it $\mathcal{O}_2$:

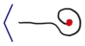


The product of the two is this:

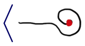


(1)

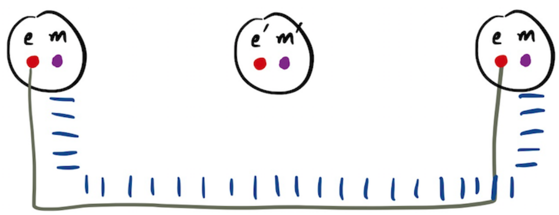
except I need to tell you the order in which the X and Z act on the indicated link. Therefore

$\langle$  $|$ $\rangle = \langle \text{gs} | \mathcal{O}_2 \mathcal{O}_1 | \text{gs} \rangle .$

If we can separate the two loops of Xs and Zs in (**??**) they will give $W_C V_{\check{C}}$ for contractible curves, which gives 1 when acting on the groundstate. But: the X and Z on the indicated link need to be moved past each other first. This costs a minus sign and therefore

$\langle$  $|$ $\rangle = \langle \text{gs} | \mathcal{O}_2 \mathcal{O}_1 | \text{gs} \rangle = -1.$

Alternatively, we can think about *exchange*. Exchanging two particles can be accomplished by first rotating one around the other by a π rotation, and then translating both of them by their separation. As you can see in this figure:



the first step requires the string creating the e particle to cross that creating the m particle on an odd number of links. (The second step is innocuous.)

2. **Toric code as $\mathbb{Z}_2$ gauge theory with matter.** Consider a model with qubits on the links of a lattice (with Pauli operators X_ℓ, Z_ℓ, $X_\ell Z_\ell = -Z_\ell X_\ell$) and qubits on the sites of the lattice (with Pauli operators σ^x_j, σ^z_j, $\sigma^x_j \sigma^z_j = -\sigma^z_j \sigma^x_j$).

   (a) Show that the operator

$$G_j \equiv A_j \sigma^z_j$$

   (where A_j is the star operator) generates the gauge transformation[1]

$$\sigma^x_j \to (-1)^{s_j} \sigma^x_j, \quad Z_{ij} \to (-1)^{s_i} Z_{ij} (-1)^{s_j}, \; \sigma^z_j \to \sigma^z_j, \; X_{ij} \to X_{ij} \qquad (2)$$

   (where i, j are the sites at the ends of the link labelled ij). By *generates* here I mean that an operator $\mathcal{O}$ transforms as

$$\mathcal{O} \to \mathcal{G}_s^\dagger \mathcal{O} \mathcal{G}_s, \quad \mathcal{G}_s \equiv \prod_j G_j^{s_j}$$

[1] In the first version of this problem set I had reversed my convention about whether A is made of Xs or Zs.

with $s_j = 0, 1$.

The relations involving Z and σ^z follow because they commute with G_j. X_ℓ transforms under G_j if the site $j \in \partial\ell$.

(b) Show that the Hamiltonian

$$\mathbf{H} = -\sum_j G_j - \sum_p B_p - h \sum_{ij} \sigma_i^x Z_{ij} \sigma_j^x - g \sum_{\langle ij \rangle} X_{ij}$$

is gauge invariant.

G_j commutes with $G_{j'}$. B_p commutes since it is a closed loop of Zs. The third term is a kinetic term for the e particles, which is invariant because the transformation of the σ^xs cancels that of the Z. The last term is invariant because it is made of Xs.

Here we can identify σ_j^x as the operator which creates an e particle at site j. And we can identify $\sigma_j^z = (-1)^{n_j}$ as the parity of the number operator.

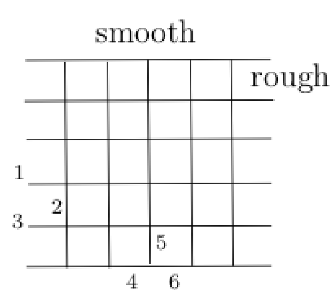
(c) Show that if we set $\sigma_j^x = 1$ and $\sigma_j^z = 1$ for all j we get back the (perturbed) toric code.

Bonus problem: interpret this operation as a choice of gauge in the model where $G_j = 1$ is imposed as a constraint on physical states.

Clearly if we just erase all the σ_j^xs and σ_j^zs we get back the toric code Hamiltonian.

And clearly we can choose the gauge parameter s_j in (1) to set $\sigma^x = 1$. The tricky part of this is that we can also erase all the appearances of σ^z. This wouldn't make sense on the full Hilbert space $\otimes \mathcal{H}_2$, since σ^z and σ^x do not commute. The claim is that on the space of physical states of the gauge theory, which satisfy $G_j |\text{phys}\rangle = |\text{phys}\rangle$ for all j, we can do this. It's because on such states, the action of σ_j^z can be replaced by A_j. If everything is gauge-invariant, any appearance of G_j can be moved onto the states, and replaced with 1.

3. **A topological qubit.** Consider the toric code on this cell complex:

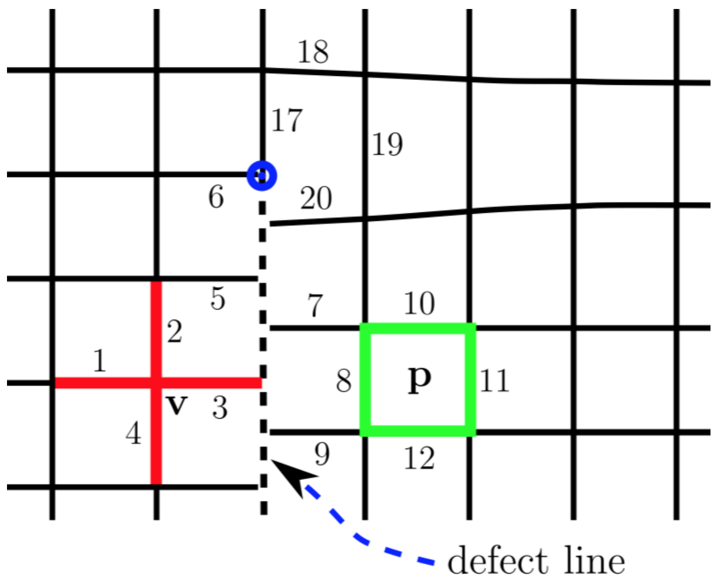


Recall that rough boundary conditions means that plaquette terms get truncated, such as the term $-X_1X_2X_3$, while smooth boundary conditions mean that star terms get truncated, such as the term $-Z_4Z_5Z_6$.

Show that there is a two-dimensional space of groundstates. A good way to do this is using the algebra of string operators which terminate at the various components of the boundary without creating excitations.

An electric string $W_C = \prod_{\ell \in C} X_\ell$ can end without creating excitations on the rough boundaries. A magnetic string $V_{\hat{C}} = \prod_{\ell \perp \hat{C}} Z_\ell$ can end without creating excitations on the smooth boundaries. $VW = -WV$, and they commute with the toric code Hamiltonian, so there is a pair of degenerate groundstates.

4. **Duality wall.** [Bonus problem] Show that the following hamiltonian realizes a *duality wall* in the toric code.



What this means is that when crossing the wall, an e particle turns into an m particle and vice versa. (More precisely, there is a string operator which transports an e particle to the wall, and can be completed by a string operator transporting an m particle away from the wall, without creating any excitations.)

To be more precise about the figure: the dotted line carries no degrees of freedom. The terms in the hamiltonian along the wall are of the form

$$H = \dots - X_2X_5X_3Z_7 - Z_3X_7X_8X_9$$

and there is a term at the end of the wall (the little blue circle) of the form $-Z_6Y_{17}X_{18}X_{19}X_{20}$. Show that these terms commute with each other and all the usual star and plaquette terms, such as $A = Z_1Z_2Z_3Z_4$ and $B = X_8X_{10}X_{11}X_{12}$.

What can you say about the end of the duality wall (the little blue circle)?

The figure (and the result) comes from this paper by Kitaev and Kong.

A string operator which is the usual W operator made of Xs along the curve on the left (and hence transports an e particle) can be connected to a V operator made of a string of Zs crossing the curve on the right (and hence transports an m particle) without violating any of the terms in the hamiltonian. And vice-versa.

The endpoint of the duality wall can absorb a fermion (the boundstate of e and m particles).