University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 8

Due 11:00am Wednesday, May 31, 2023

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1. **Landau Levels in an Electric Field.** [If you did this problem last week, please hand in your solution again.]

   In lecture I gave several arguments that a quantum Hall droplet has a linearly-dispersing edge mode. Here is a fully quantum mechanical argument. We're going to think about the physics in a neighborhood of the boundary of the sample, where the confining potential $V \simeq -Ex$ is slowly varying, and describes an electric field $E = -\partial_x V$.

   The Hamiltonian in the Landau gauge (the one used on the last homework) is

   $$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) - eEx. \tag{1}$$

   (a) Using the same ansatz as in the last homework, write the Hamiltonian as that of a displaced harmonic oscillator.

   (b) Conclude that the eigenstates have the form

   $$\psi(x, y) = \psi_{n,k}\left(x - \frac{mE}{eB^2}, y\right) \tag{2}$$

   with energies

   $$E_{n,k} = \hbar\omega_c\left(n + \frac{1}{2}\right) + eE\left(k\ell_B - \frac{eE}{m\omega_c^2}\right) + \frac{m}{2}\frac{E^2}{B^2}. \tag{3}$$

   (c) Plot this spectrum, and interpret $\partial_k E_{n,k}$ as a velocity in the y direction.

   (d) Compare this drift velocity with the classical behavior of a charged particle in crossed E and B fields.

2. **Interacting particles on a very small lattice.**

   Consider the Hamiltonian

   $$\mathbf{H} = -t\sum_{i=1}^{N}\left(\mathbf{a}_i^\dagger \mathbf{a}_{i+1} + \mathbf{a}_{i+1}^\dagger \mathbf{a}_i\right) + V\sum_i \mathbf{n}_i \mathbf{n}_{i+1}$$

   describing particles on a circular chain ($\mathbf{a}_{i+N} = \mathbf{a}_i$). Here $\mathbf{n}_i \equiv \mathbf{a}_i^\dagger \mathbf{a}_i$. Assume $t, V > 0$.

(a) Suppose that the operators $\mathbf{a}$ are fermionic ($\{\mathbf{a}_i, \mathbf{a}_j\} = \delta_{ij}$). Suppose there are only three (N=3) sites. Write the matrix form of the Hamiltonian acting on the sector with exactly two fermions. Beware of signs. Find its eigenvalues and eigenvectors. Feel free to use some software (*e.g.* Mathematica or Sympy). Compare to the case with exactly one fermion.

(b) Consider general N sites and exactly $N-1$ particles. Again compare to the case of a single particle.

(c) Consider again $N = 3$ and exactly two particles, but now suppose that the particles are bosons. Write down the matrix representation of the Hamiltonian in this case. Plot the spectrum as a function of V/t.

3. **Brain-warmer: Spin rotations.** The goal of this problem is to solve the Transverse Field Ising Model in the mean field approximation.

(a) Show that

$$\mathbf{H}(\theta) \equiv -K \sum_i (\sin\theta \mathbf{X}_i + \cos\theta \mathbf{Z}_i) = -K\mathbf{U} \sum_i \mathbf{Z}_i \mathbf{U}^\dagger$$

where

$$\mathbf{U} = e^{-\mathbf{i}\theta \sum_i \mathbf{Y}_i}.$$

This is a global rotation about the y-axis.

(b) Conclude that the groundstate of $\mathbf{H}(\theta)$ is

$$|\theta\rangle \equiv \mathbf{U} \otimes_i |\uparrow\rangle_i .$$

(c) Compute $m = \langle\theta| \mathbf{Z}_i |\theta\rangle$.

(d) Impose the self-consistency condition that m is the expectation value used to determine the mean field in

$$\mathbf{H}_{\text{TFIM}} \simeq \mathbf{H}_{\text{MFT}} = -J \sum_i g\mathbf{X}_i - \sum_i \mathbf{Z}_i \left(\frac{1}{2} \sum_{\text{neighbors } j \text{ of } i} \langle \mathbf{Z}_j \rangle \right) = -J \sum_i \left(g\mathbf{X}_i - \frac{1}{2} zm\mathbf{Z}_i \right).$$

Plot θ as a function of g.

4. **Two coupled spins.**

This is a very useful warmup for the next problem. Consider a four-state system consisting of two qbits,

$$\mathcal{H} = \text{span}\{|\epsilon_1\rangle \otimes |\epsilon_2\rangle \equiv |\epsilon_1\epsilon_2\rangle, \epsilon = \uparrow_z, \downarrow_z\}.$$

(a) For each qbit, define $\boldsymbol{\sigma}^{\pm} \equiv \frac{1}{2}(\boldsymbol{\sigma}^x \pm \mathbf{i}\boldsymbol{\sigma}^y)$. (These are raising and lowering operators for $\boldsymbol{\sigma}^z$: $[\boldsymbol{\sigma}^z, \boldsymbol{\sigma}^{\pm}] = \pm 2\boldsymbol{\sigma}^{\pm}$. Check this.)

Show that

$$\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2 = 2\left(\boldsymbol{\sigma}_1^+ \boldsymbol{\sigma}_2^- + \boldsymbol{\sigma}_1^- \boldsymbol{\sigma}_2^+\right) + \boldsymbol{\sigma}_1^z \boldsymbol{\sigma}_2^z.$$

Here, by for example $\boldsymbol{\sigma}_1^x$ I mean the operator $\boldsymbol{\sigma}^x \otimes \mathbb{1}$ which acts as

$$\boldsymbol{\sigma}^x \otimes \mathbb{1} \left|\uparrow \epsilon_2\right\rangle = \left|\downarrow \epsilon_2\right\rangle, \quad \boldsymbol{\sigma}^x \otimes \mathbb{1} \left|\downarrow \epsilon_2\right\rangle = \left|\uparrow \epsilon_2\right\rangle.$$

(b) Determine the action of the operator $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ on the basis states

$$\left|\uparrow\uparrow\right\rangle, \left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle, \left|\downarrow\downarrow\right\rangle.$$

(c) Show that the four vectors

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right), \quad |1,1\rangle \equiv \left|\uparrow\uparrow\right\rangle, \quad |1,0\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right), \quad |1,-1\rangle \equiv \left|\downarrow\downarrow\right\rangle$$

are orthonormal and are eigenvectors of $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ with eigenvalues 1 or -3.

(d) Show that they are also eigenvectors of $\mathbf{J}^2 \equiv (\vec{\boldsymbol{\sigma}}_1 + \vec{\boldsymbol{\sigma}}_2)^2$ and $\mathbf{J}^z \equiv \boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z$ and find their eigenvalues.

(e) Consider the operator

$$\mathcal{P}_{1,2} \equiv \frac{1}{2}(\mathbb{1} + \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2)$$

acting on the two spins. Show that $\mathcal{P}_{1,2}$ acts by exchanging the states of the two spins:

$$\mathcal{P}_{1,2}\left|\epsilon_1\epsilon_2\right\rangle = \left|\epsilon_2\epsilon_1\right\rangle \; .$$

(f) Show that the operator

$$Q_{1,2} \equiv \frac{1}{4}(\mathbb{1} - \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2)$$

acts as a projector onto the (singlet) state $|0,0\rangle$.

5. **Spin chains and spin waves.**

A one-dimensional (SU(2)-symmetric) *ferromagnet* can be represented as a chain of N qbits (spin-1/2 particles) numbered $n = 0, ...N-1, N \gg 1$, fixed along a line with a spacing ℓ between each successive pair. It is convenient to use periodic boundary conditions, where the Nth spin is identified with the 0th spin: $n + N \equiv n$. Suppose that each spin interacts only with its two nearest neighbors, so the Hamiltonian can be written as

$$\mathbf{H} = \frac{1}{2}NJ\mathbb{1} - \frac{1}{2}J\sum_{n=0}^{N-1} \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1} \; .$$

where J is a *coupling constant* determining the strength of the interactions.

(a) Show that all eigenvalues E of $\mathbf{H}$ are non-negative, and that the minimum energy E_0 (the *ground state*) is obtained in the state where all the spins point in the same direction. A possible choice for the ground state $|\Phi_0\rangle$ is then

$$|\Phi_0\rangle = |\uparrow_z\rangle_{n=0} \otimes |\uparrow_z\rangle_{n=1} \otimes ... \otimes |\uparrow_z\rangle_{N-1} \equiv |\uparrow\uparrow ... \uparrow\rangle .$$

(b) Show that any state obtained from $|\Phi_0\rangle$ by rotating each of the spins by the same angle is also a possible ground state.

[Hint: the generator of spin rotations $\vec{\mathbf{J}} \equiv \sum_n \vec{\boldsymbol{\sigma}}_n$ commutes with the Hamiltonian.]

[Cultural remark: the phenomenon of a ground state which does not preserve a symmetry of the Hamiltonian is called *spontaneous symmetry breaking.*]

(c) Now we wish to find the low-energy excitations above the ground state $|\Phi_0\rangle$. Show that $\mathbf{H}$ can be written

$$\mathbf{H} = NJ\mathbb{1} - J\sum_{n=0}^{N-1} \mathcal{P}_{n,n+1} = J\sum_{n=0}^{N-1} \left(\mathbb{1} - \mathcal{P}_{n,n+1}\right) .$$

where

$$\mathcal{P}_{n,n+1} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1}\right) \; .$$

Using the result of the problem 4, show that the eigenvectors of $\mathbf{H}$ are linear combinations of vectors in which the number of up spins minus the number of down spins is fixed. Let $|\Psi_n\rangle$ be the state in which the spin n is down with all the other spins up. What is the action of $\mathbf{H}$ on $|\Psi_n\rangle$?

(d) We are going to construct eigenvectors $|k_s\rangle$ of $\mathbf{H}$ out of linear combinations of the $|\Psi_n\rangle$. Let

$$|k_s\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\mathbf{i}k_s n\ell} \, |\Psi_n\rangle$$

with

$$k_s = \frac{2\pi s}{N\ell}, \quad s = 0, 1, ...N-1 \; .$$

Show that $|k_s\rangle$ is an eigenvector of $\mathbf{H}$ and determine the energy eigenvalue E_k. Show that the energy is proportional to k_s^2 as $k_s \to 0$. This state describes an elementary excitation called a *spin wave* or *magnon* with wave-vector k_s.