University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 8 – Solutions

Due 11:00am Wednesday, May 31, 2023

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1. **Landau Levels in an Electric Field.** [If you did this problem last week, please hand in your solution again.]

   In lecture I gave several arguments that a quantum Hall droplet has a linearly-dispersing edge mode. Here is a fully quantum mechanical argument. We're going to think about the physics in a neighborhood of the boundary of the sample, where the confining potential $V \simeq -Ex$ is slowly varying, and describes an electric field $E = -\partial_x V$.

   The Hamiltonian in the Landau gauge (the one used on the last homework) is

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) - eEx. \quad (1)$$

   (a) Using the same ansatz as in the last homework, write the Hamiltonian as that of a displaced harmonic oscillator.

   (b) Conclude that the eigenstates have the form

$$\psi(x,y) = \psi_{n,k}\left(x - \frac{mE}{eB^2}, y\right) \quad (2)$$

   with energies

$$E_{n,k} = \hbar\omega_c\left(n + \frac{1}{2}\right) + eE\left(k\ell_B^2 - \frac{eE}{m\omega_c^2}\right) + \frac{m}{2}\frac{E^2}{B^2}. \quad (3)$$

   (c) Plot this spectrum, and interpret $\partial_k E_{n,k}$ as a velocity in the y direction.

   (d) Compare this drift velocity with the classical behavior of a charged particle in crossed E and B fields.

2. **Interacting particles on a very small lattice.**

   Consider the Hamiltonian

$$\mathbf{H} = -t\sum_{i=1}^{N}\left(\mathbf{a}_i^\dagger \mathbf{a}_{i+1} + \mathbf{a}_{i+1}^\dagger \mathbf{a}_i\right) + V\sum_i \mathbf{n}_i\mathbf{n}_{i+1}$$

   describing particles on a circular chain ($\mathbf{a}_{i+N} = \mathbf{a}_i$). Here $\mathbf{n}_i \equiv \mathbf{a}_i^\dagger \mathbf{a}_i$. Assume $t, V > 0$.

(a) Suppose that the operators $\mathbf{a}$ are fermionic ($\{\mathbf{a}_i, \mathbf{a}_j\} = \delta_{ij}$). Suppose there are only three (N=3) sites. Write the matrix form of the Hamiltonian acting on the sector with exactly two fermions. Beware of signs. Find its eigenvalues and eigenvectors. Feel free to use some software (*e.g.* Mathematica or Sympy). Compare to the case with exactly one fermion.

There are three such states, which I will label

$$|1\rangle \equiv \mathbf{a}_2^\dagger \mathbf{a}_3^\dagger |0\rangle, |2\rangle \equiv \mathbf{a}_1^\dagger \mathbf{a}_3^\dagger |0\rangle, |3\rangle \equiv \mathbf{a}_1^\dagger \mathbf{a}_1^\dagger |0\rangle .$$

Notice that I have specified the signs of these basis states. Note also that they are labelled by the location of the *hole*.

The diagonal matrix elements are

$$\langle a| \mathbf{H} |a\rangle = V, \ a = 1 \ldots 3$$

since each state has one pair of occupied neighboring sites. The off-diagonal entries are

$$\langle 2| \mathbf{H} |1\rangle = \langle 0| \mathbf{a}_3 \mathbf{a}_1 \left(-t \mathbf{a}_1^\dagger \mathbf{a}_2\right) \mathbf{a}_2^\dagger \mathbf{a}_3^\dagger |0\rangle = -t \langle 0| \mathbf{a}_1 \mathbf{a}_1^\dagger \mathbf{a}_2 \mathbf{a}_2^\dagger \mathbf{a}_3 \mathbf{a}_3^\dagger |0\rangle = -t.$$

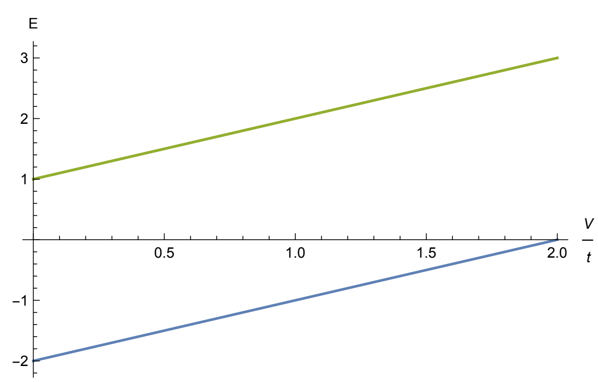
$$\langle 3| \mathbf{H} |2\rangle = \langle 0| \mathbf{a}_2 \mathbf{a}_1 \left(-t \mathbf{a}_2^\dagger \mathbf{a}_3\right) \mathbf{a}_1^\dagger \mathbf{a}_3^\dagger |0\rangle = -t \langle 0| \mathbf{a}_2 \mathbf{a}_2^\dagger \mathbf{a}_3 \mathbf{a}_3^\dagger \mathbf{a}_1 \mathbf{a}_1^\dagger |0\rangle = -t.$$

$$\langle 3| \mathbf{H} |1\rangle = \langle 0| \mathbf{a}_2 \mathbf{a}_1 \left(-t \mathbf{a}_1^\dagger \mathbf{a}_3\right) \mathbf{a}_2^\dagger \mathbf{a}_3^\dagger |0\rangle = -t \langle 0| \mathbf{a}_1 \mathbf{a}_1^\dagger \mathbf{a}_2 \mathbf{a}_3 \mathbf{a}_1 \mathbf{a}_2^\dagger \mathbf{a}_3^\dagger |0\rangle = +t.$$

Therefore the matrix to diagonalize is

$$H = \begin{pmatrix} V & -t & +t \\ -t & V & -t \\ t & -t & V \end{pmatrix}.$$

Its eigenvalues are $-t + V, -t + V, 2t + V$.



This is not quite just $\delta_{ab}V$ plus the matrix we would find for a single fermion. For a single fermion, all the hopping matrix elements are of the same sign, since no fermions need to move through each other. In that case, the spectrum is $-2t\cos ka$ with $ka = \frac{2\pi j}{N}$, $j = 1 \cdots N$ and $N = 3$. This has the effect of flipping the signs of the terms linear in t.

(b) Consider general N sites and exactly $N-1$ particles. Again compare to the case of a single particle.

The same is true for general *odd* N with $N-1$ fermions. In that case, the amplitude for moving the hole from the Nth site to the first site comes with an extra sign. This is the same as saying that the hole has *antiperiodic* boundary conditions. This changes the spectrum to $-2t\cos(ka+\pi) = +2t\cos ka$.

To see this, again define

$$|n\rangle = \mathbf{a}_1^\dagger \cdots \widehat{\mathbf{a}_n^\dagger} \cdots \mathbf{a}_N^\dagger |0\rangle$$

(where the hat means the operator is omitted) to be the state with a hole at site n. Then for $n < N$

$$\langle n| \mathbf{H} |n+1\rangle = \langle 0| \mathbf{a}_N \cdots \widehat{\mathbf{a}_n} \cdots \mathbf{a}_1 \left(-t\mathbf{a}_{n+1}^\dagger \mathbf{a}_n\right) \mathbf{a}_1^\dagger \cdots \widehat{\mathbf{a}_{n+1}^\dagger} \cdots \mathbf{a}_N^\dagger |0\rangle \quad (4)$$

$$= -t(-1)^{n+1} \langle 0| \mathbf{a}_N \cdots \mathbf{a}_n \cdots \mathbf{a}_1 \left(\mathbf{a}_{n+1}^\dagger\right) \mathbf{a}_1^\dagger \cdots \widehat{\mathbf{a}_{n+1}^\dagger} \cdots \mathbf{a}_N^\dagger |0\rangle \quad (5)$$

$$= -t(-1)^{n+1}(-1)^n \langle 0| \mathbf{a}_N \cdots \mathbf{a}_n \cdots \mathbf{a}_1 \mathbf{a}_1^\dagger \cdots \mathbf{a}_{n+1}^\dagger \cdots \mathbf{a}_N^\dagger |0\rangle = -t. \quad (6)$$

In contrast, when hopping around the back of the circle, we get

$$\langle N| \mathbf{H} |1\rangle = \langle 0| \mathbf{a}_{N-1} \cdots \mathbf{a}_1 \left(-t\mathbf{a}_1^\dagger \mathbf{a}_N\right) \mathbf{a}_2^\dagger \cdots \mathbf{a}_N^\dagger |0\rangle \quad (7)$$

$$= -t(-1)^{N-2} \langle 0| \mathbf{a}_N \cdots \mathbf{a}_1 \left(\mathbf{a}_1^\dagger\right) \mathbf{a}_2^\dagger \cdots \mathbf{a}_N^\dagger |0\rangle \quad (8)$$

$$= -t(-1)^{N-2} \langle 0| \mathbf{a}_N \cdots \mathbf{a}_1 \mathbf{a}_1^\dagger \cdots \mathbf{a}_N^\dagger |0\rangle = (-1)^{N-1} t. \quad (9)$$

For even N the spectrum is the same as for a single particle, shifted by the diagonal term $(N-2)V$.

(c) Consider again $N = 3$ and exactly two particles, but now suppose that the particles are bosons. Write down the matrix representation of the Hamiltonian in this case. Plot the spectrum as a function of V/t.

Now there are six possible states: three states with two particles at one site, and three single-hole states $|n\rangle$ where site n is missing a particle.

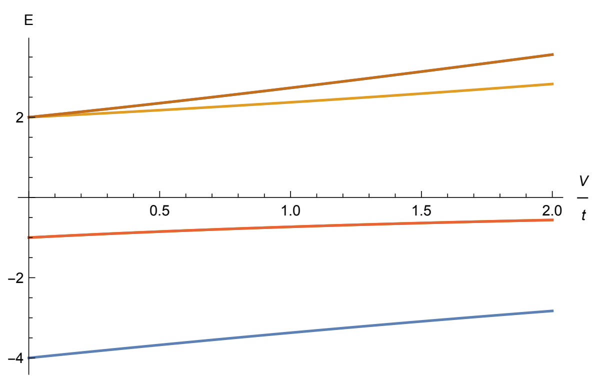
Now there are no signs in the hopping matrix elements:

$$\langle n|\,\mathbf{H}\,|n\rangle = V, \quad \langle n|\,\mathbf{H}\,|n+1\rangle = -t.$$

The full matrix, in the basis $011, 101, 110, 200, 020, 002$ is

$$H = \begin{pmatrix} V & -t & -t & 0 & -t\sqrt{2} & -t\sqrt{2} \\ -t & V & -t & -t\sqrt{2} & 0 & -t\sqrt{2} \\ t & -t & V & -t\sqrt{2} & -t\sqrt{2} & 0 \\ 0 & -t\sqrt{2} & -t\sqrt{2} & 0 & 0 & 0 \\ -t\sqrt{2} & 0 & -t\sqrt{2} & 0 & 0 & 0 \\ -t\sqrt{2} & -t\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

whose spectrum looks like this:



3. **Brain-warmer: Spin rotations.** The goal of this problem is to give a different perspective on mean field theory for the Transverse Field Ising Model. In lecture, we described it as a variational calculation. Here we'll give a self-consistency argument.

   (a) Show that

$$\mathbf{H}(\theta) \equiv -K\sum_i (\sin\theta \mathbf{X}_i + \cos\theta \mathbf{Z}_i) = -K\mathbf{U}\sum_i \mathbf{Z}_i \mathbf{U}^\dagger$$

   where

$$\mathbf{U} = e^{-\mathbf{i}\frac{1}{2}\theta\sum_i \mathbf{Y}_i}.$$

   This is a global rotation about the y-axis.

   Using $\text{ad}_Y(Z) \equiv [Y, Z] = 2\mathbf{i}X$, we have

$$e^{-\mathbf{i}\alpha Y} Z e^{\mathbf{i}\alpha Y} = Z - \mathbf{i}\alpha\,\text{ad}_Y(Z) + \frac{(\mathbf{i}\alpha)^2}{2!}\text{ad}_Y^2 Z + \ldots = \cos 2\alpha Z + \sin 2\alpha X.$$

(b) Conclude that the groundstate of $\mathbf{H}(\theta)$ is

$$|\theta\rangle \equiv \mathbf{U} \otimes_i |\uparrow\rangle_i .$$

If the groundstate of H is $|\text{gs}\rangle$, the groundstate of $UHU^\dagger$ is $U|\text{gs}\rangle$.

(c) Compute $m = \langle\theta| \mathbf{Z}_i |\theta\rangle$.

$$m = \langle\theta| \mathbf{Z}_i |\theta\rangle = \langle\uparrow| U^\dagger Z U |\uparrow\rangle = \langle\uparrow| (\cos\theta Z - \sin\theta X) |\uparrow\rangle = \cos\theta$$

.

(d) Impose the self-consistency condition that m is the expectation value used to determine the mean field in

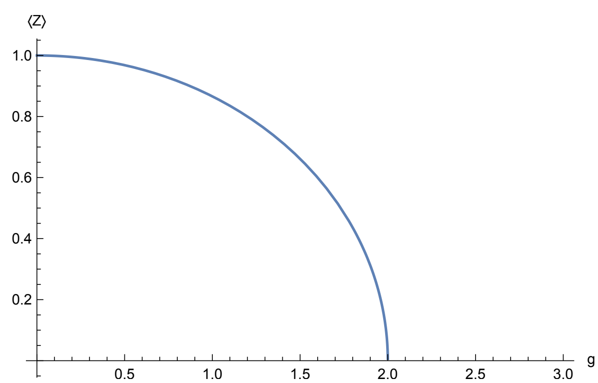
$$\mathbf{H}_{\text{TFIM}} \simeq \mathbf{H}_{\text{MFT}} = -J \sum_i \left(g\mathbf{X}_i + \mathbf{Z}_i \left(\sum_{\text{neighbors } j \text{ of } i} \langle \mathbf{Z}_j \rangle \right) \right) = -J \sum_i \left(g\mathbf{X}_i + zm\mathbf{Z}_i \right).$$

[Note that I had an extra factor of $\frac{1}{2}$ in my statement of the problem. I forgot to account for the fact that in each pair $Z_i Z_j$, we should include both the term where we replace Z_i with $\langle Z_i \rangle$, and the term where we replace Z_j with $\langle Z_j \rangle$.]

Plot θ as a function of g.

Comparing to the mean-field hamiltonian, $\tan\theta = \frac{g}{zm}$. Using $m = \cos\theta$, this says $\sin\theta = \frac{g}{z}$ when the RHS has absolute value less than one, else $m = 0$.

It is more instructive to plot $\langle Z \rangle = \cos\theta$ (for $z = 2$):



This is the same answer we found in lecture (though with a different parameterization of the angles.

4. **Two coupled spins.** [based on Le Bellac problem 6.5.4]

This is a very useful warmup for the next problem. Consider a four-state system consisting of two qbits,

$$\mathcal{H} = \text{span}\{|\epsilon_1\rangle \otimes |\epsilon_2\rangle \equiv |\epsilon_1 \epsilon_2\rangle, \epsilon = \uparrow_z, \downarrow_z\}.$$

(a) For each qbit, define $\boldsymbol{\sigma}^\pm \equiv \frac{1}{2}(\boldsymbol{\sigma}^x \pm \mathbf{i}\boldsymbol{\sigma}^y)$. (These are raising and lowering operators for $\boldsymbol{\sigma}^z$: $[\boldsymbol{\sigma}^z, \boldsymbol{\sigma}^\pm] = \pm 2\boldsymbol{\sigma}^\pm$. Check this.)

Show that

$$\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2 = 2\left(\boldsymbol{\sigma}_1^+ \boldsymbol{\sigma}_2^- + \boldsymbol{\sigma}_1^- \boldsymbol{\sigma}_2^+\right) + \boldsymbol{\sigma}_1^z \boldsymbol{\sigma}_2^z.$$

Here, by for example $\boldsymbol{\sigma}_1^x$ I mean the operator $\boldsymbol{\sigma}^x \otimes \mathbb{1}$ which acts as

$$\boldsymbol{\sigma}^x \otimes \mathbb{1} |\uparrow \epsilon_2\rangle = |\downarrow \epsilon_2\rangle, \quad \boldsymbol{\sigma}^x \otimes \mathbb{1} |\downarrow \epsilon_2\rangle = |\uparrow \epsilon_2\rangle.$$

(b) Determine the action of the operator $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ on the basis states

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle.$$

Respectively, we find, using the previous part,

$$|\uparrow\uparrow\rangle, 2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle, 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle.$$

(c) Show that the four vectors

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad |1,1\rangle \equiv |\uparrow\uparrow\rangle, \quad |1,0\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |1,-1\rangle \equiv |\downarrow\downarrow\rangle$$

are orthonormal and are eigenvectors of $\vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2$ with eigenvalues 1 or -3.

Actually, the easiest way to do this is not using the previous part but using group theory. The spin-half generator on a single site is $\vec{S}_1 = \vec{\boldsymbol{\sigma}}_1/2$, which satisfies $\vec{S}_1^2 = j(j+1) = 3/4$, since $j = 1/2$. The total spin of two sites is

$$\vec{J} = \vec{S}_1 + \vec{S}_2.$$

The tensor product of two spin-half representations decomposes into a singlet ($j = 0$) and a triplet ($j = 1$), on which $\vec{J}^2 = j(j+1) = 0, 2$ respectively. But

$$\vec{J}^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2 = \frac{3}{4} + \frac{3}{4} + \frac{1}{2}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2.$$

Therefore on the singlet and triplet respectively,

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = 2\left(J^2 - \frac{3}{2}\right) = -3, 1.$$

Of course, this is also the answer you find if you use brute force.

(d) Show that they are also eigenvectors of $\mathbf{J}^2 \equiv (\vec{\boldsymbol{\sigma}}_1 + \vec{\boldsymbol{\sigma}}_2)^2$ and $\mathbf{J}^z \equiv \boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z$ and find their eigenvalues.

They are also eigenstates of $\sum_i \boldsymbol{\sigma}_i^z$, with the eigenvalues indicated in the second entry of the ket.

(e) Consider the operator

$$\mathcal{P}_{1,2} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2\right)$$

acting on the two spins. Show that $\mathcal{P}_{1,2}$ acts by exchanging the states of the two spins:

$$\mathcal{P}_{1,2}\left|\epsilon_1\epsilon_2\right\rangle = \left|\epsilon_2\epsilon_1\right\rangle \ .$$

Well, from above we see that it does nothing to $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ and interchanges $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$.

(f) Show that the operator

$$Q_{1,2} \equiv \frac{1}{4}\left(\mathbb{1} - \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{\sigma}}_2\right)$$

acts as a projector onto the (singlet) state $|0, 0\rangle$.

For any m, $Q_{1,2}\,|1, m\rangle = 0$, while $Q_{1,2}\,|0, 0\rangle = \frac{1+3}{4}\,|0, 0\rangle = |0, 0\rangle$.

5. **Spin chains and spin waves.** [Related to Le Bellac problem 6.5.5 on page 200]

A one-dimensional ($\mathsf{SU}(2)$-symmetric) *ferromagnet* can be represented as a chain of N qbits (spin-1/2 particles) numbered $n = 0, ...N-1, N \gg 1$, fixed along a line with a spacing ℓ between each successive pair. It is convenient to use periodic boundary conditions, where the Nth spin is identified with the 0th spin: $n + N \equiv n$. Suppose that each spin interacts only with its two nearest neighbors, so the Hamiltonian can be written as

$$\mathbf{H} = \frac{1}{2}NJ\mathbb{1} - \frac{1}{2}J\sum_{n=0}^{N-1}\vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1} \ .$$

where J is a *coupling constant* determining the strength of the interactions. Note that $\vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1} \equiv X_n X_{n+1} + Y_n Y_{n+1} + Z_n Z_{n+1}$.

(a) Show that all eigenvalues E of $\mathbf{H}$ are non-negative, and that the minimum energy E_0 (the *ground state*) is obtained in the state where all the spins point in the same direction. A possible choice for the ground state $|\Phi_0\rangle$ is then

$$|\Phi_0\rangle = |\uparrow_z\rangle_{n=0} \otimes |\uparrow_z\rangle_{n=1} \otimes ... \otimes |\uparrow_z\rangle_{N-1} \equiv |\uparrow\uparrow ... \uparrow\rangle \, .$$

The eigenvalues are all non-negative since $\mathbf{H}$ can be written in the form

$$\sum_n J\frac{1}{2}\left(\mathbb{1} - \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1}\right) \equiv +\sum_n 2JQ_{n,n+1}$$

where the eigenvalues of $Q_{ij} \equiv \frac{1}{4}\left(\mathbb{1} - \vec{\boldsymbol{\sigma}}_i \cdot \vec{\boldsymbol{\sigma}}_j\right)$ are 0 and 1, since it projects onto the singlet of spins i and j. This means that any state where every two neighbors are in a relative triplet will be a groundstate with energy zero. In particular in the state $|\uparrow\uparrow\cdots\uparrow\rangle$ every pair of neighbors is in a relative triplet.

(b) Show that any state obtained from $|\Phi_0\rangle$ by rotating each of the spins by the same angle is also a possible ground state.

[Hint: the generator of spin rotations $\vec{\mathbf{J}} \equiv \sum_n \vec{\boldsymbol{\sigma}}_n$ commutes with the Hamiltonian.]

Any other state where every pair of neighbors is in a relative triplet is obtained by such a rotation

$$e^{\mathbf{i}\theta\hat{n}\cdot\vec{\mathbf{J}}}\,|\uparrow\uparrow\cdots\uparrow\rangle\,.$$

This is also a groundstate because $[\mathbf{H}, \vec{\mathbf{J}}] = 0$.

[Cultural remark: the phenomenon of a ground state which does not preserve a symmetry of the Hamiltonian is called *spontaneous symmetry breaking.*]

(c) Now we wish to find the low-energy excitations above the ground state $|\Phi_0\rangle$. Show that $\mathbf{H}$ can be written

$$\mathbf{H} = NJ\mathbb{1} - J\sum_{n=0}^{N-1}\mathcal{P}_{n,n+1} = J\sum_{n=0}^{N-1}\left(\mathbb{1} - \mathcal{P}_{n,n+1}\right).$$

where

$$\mathcal{P}_{n,n+1} \equiv \frac{1}{2}\left(\mathbb{1} + \vec{\boldsymbol{\sigma}}_n \cdot \vec{\boldsymbol{\sigma}}_{n+1}\right)\ .$$

$\mathbb{1} - \mathcal{P} = \frac{1}{2}\left(\mathbb{1} - \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\right)$ so

$$\mathbf{H} = \frac{1}{2}J\sum_{n=0}^{N-1}\left(\mathbb{1} - \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}\right) = J\sum_{n=0}^{N-1}\left(\mathbb{1} - \mathcal{P}_{n,n+1}\right).$$

Using the result of the problem 4, show that the eigenvectors of $\mathbf{H}$ are linear combinations of vectors in which the number of up spins minus the number of down spins is fixed. Let $|\Psi_n\rangle$ be the state in which the spin n is down with all the other spins up. What is the action of $\mathbf{H}$ on $|\Psi_n\rangle$?

Consider a state which is an eigenstate of σ_n^z for all n. The previous problem means that $\mathbf{H}$ acts to permute the locations of the down spins. It does not change the number of down spins. The operator $\mathbb{1} - \mathcal{P}_{j,j+1}$ annihilates $|\Psi_n\rangle$ unless $j = n$ or $n-1$. Those two terms give

$$\mathbf{H} |\Psi_n\rangle = J (2 |\Psi_n\rangle - |\Psi_{n-1}\rangle - |\Psi_{n+1}\rangle) .$$

(d) We are going to construct eigenvectors $|k_s\rangle$ of $\mathbf{H}$ out of linear combinations of the $|\Psi_n\rangle$. Let

$$|k_s\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\mathbf{i} k_s n \ell} |\Psi_n\rangle$$

with

$$k_s = \frac{2\pi s}{N\ell}, \quad s = 0, 1, ...N-1 .$$

Show that $|k_s\rangle$ is an eigenvector of $\mathbf{H}$ and determine the energy eigenvalue E_k. Show that the energy is proportional to k_s^2 as $k_s \to 0$. This state describes an elementary excitation called a *spin wave* or *magnon* with wave-vector k_s.

This is the same problem we've seen several times now.