University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 7 – Solutions

Due 11:00am Wednesday, May 24, 2023

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1. **Probability current for a charged particle.**

   Check that probability is still conserved $\partial_t \rho + \vec{\nabla} \cdot \vec{j}_A = 0$ for a charged particle, with $\rho(x,t) = |\psi(x,t)|^2$ and

   $$\vec{j}_A \equiv \frac{\hbar}{2m\mathbf{i}} \left(\psi^\star \vec{\nabla} \psi - \psi \vec{\nabla} \psi^\star \right) - \frac{e}{mc} \psi^\star \psi \vec{A}.$$

   Check that $\vec{j}_A$ is gauge invariant if the gauge transformation acts on the wavefunction by

   $$\vec{A} \to \vec{A} + \vec{\nabla}\lambda, \quad \psi \to e^{\pm \mathbf{i} \frac{e\lambda}{\hbar c}} \psi$$

   for one choice of the sign in the exponent.

   (As a check of the sign, you can check that the Schrödinger equation maps to itself under the transformation.)

2. **A charged particle in a uniform magnetic field, quantumly.** Consider an electron constrained to move in the xy plane under the influence of a uniform magnetic field of magnitude B oriented in the $+\hat{z}$ direction. The Hamiltonian for this electron is

   $$\mathbf{H} = \frac{1}{2m} \left(\left(\mathbf{p}_x - \frac{e}{c} A_x \right)^2 + \left(\mathbf{p}_y - \frac{e}{c} A_y \right)^2 \right)$$

   where m and e are the mass and charge of the electron, and c is the speed of light.

   (a) Show that a classical particle in this potential will move in circles at an angular frequency $\omega_0 = \frac{eB}{mc}$ where m is the mass.

   (b) Find a suitable expression for $\vec{A}$ so that $\mathbf{p}_x$ is a constant of motion for the above Hamiltonian.

   We can choose a gauge for $\vec{A}$ with $\vec{\nabla} \times \vec{A} = B\hat{z}$, uniform, so that $\mathbf{x}$ does not appear:

   $$A_x = -B\mathbf{y}, A_y = 0.$$

(c) With this choice for $\vec{A}$, show that the eigenfunctions of $\mathbf{H}$ can be written in the form

$$\Psi(x, y) = e^{\frac{i}{\hbar} p_x x} \Phi(y)$$

where $\Phi(y)$ satisfies the Schrödinger equation for a one-dimensional harmonic oscillator whose equilibrium position is $y = y_0$. Find the effective spring constant k for this oscillator and the shift of the origin y_0 in terms of p_x, B, m, e, c.

The Schrödinger equation is

$$\mathbf{H}\Psi(x, y) = E\psi(x, y) \tag{1}$$

and with the choice of gauge from part (a) we have

$$\mathbf{H} = \frac{1}{2m}\left(\left(\mathbf{p}_x + \frac{e}{c}B\mathbf{y}\right)^2 + \mathbf{p}_y^2\right).$$

Plugging in the given ansatz turns $\mathbf{p}_x$ into a number, and (**??**) becomes:

$$\frac{1}{2m}\left(\left(p_x + \frac{e}{c}By\right)^2 + \mathbf{p}_y^2\right) e^{\frac{i}{\hbar} p_x x}\Phi(y) = E e^{\frac{i}{\hbar} p_x x}\Phi(y)$$

or

$$\left(\frac{\mathbf{p}_y^2}{2m} + \frac{1}{2m}\left(p_x + \frac{e}{c}By\right)^2\right)\Phi(y) = E\Phi(y)$$

or

$$\left(\frac{\mathbf{p}_y^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}\right)^2\left(y + \frac{c}{eB}p_x\right)^2\right)\Phi(y) = E\Phi(y)$$

which is the Schrödinger equation for a simple harmonic oscillator (SHO)

$$\left(\frac{\mathbf{p}_y^2}{2m} + \frac{k}{2}\left(y - y_0\right)^2\right)\Phi(y) = E\Phi(y)$$

with $k = m\left(\frac{eB}{mc}\right)^2$ and $y_0 = -\frac{cp_x}{eB}$.

(d) Find the energy eigenvalues for this system, and indicate degeneracies.

The energy spectrum of the SHO is

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

where

$$\omega = \sqrt{\frac{k}{m}} = \frac{eB}{mc} = \omega_c,$$

the cyclotron frequency. Notice that p_x drops out of the expression for the energy and so there is a big degeneracy, approximately linear in the system size.

(e) For the remainder of the problem, suppose we further restrict the particles to live in a square of side length L. Suppose we demand periodic boundary conditions. What are the possible values of p_x?

Using the boundary condition that the wavefunction should be the same at the end points ($\psi(x = L) = \psi(x = 0)$), we have

$$p_x = \frac{2\pi\ell}{L}, \quad \ell \in \mathbb{Z} \ .$$

(f) In the estimates below, use the wavefunctions found above, despite the fact that they will not satisfy the boundary conditions at $y = 0, L$. Place several electrons in the strip, and assume they only care about each other because of the Pauli exclusion principle.

Estimate the maximum number of electrons that fit in the lowest Landau level. That is: how many electrons can you add before you must put one in the first excited state of the harmonic oscillator encountered above?

For a state with momentum p_x in the x-direction, the center of its orbit in the y direction is at:

$$y = -\frac{p_x}{m\omega_c}.$$

This must be inside the box! The possible values of y in the lowest Landau level are then

$$y_\ell = -\frac{\hbar 2\pi\ell}{Lm\omega_c}.$$

We want to know how many of these will fit on the sheet of width L. So we have

$$N = \frac{L}{\Delta y}, \quad \text{with} \ \ \Delta y = \frac{2\pi\hbar}{Lm\omega_c}.$$

Therefore the number of electrons that fit in the lowest Landau level is approximately

$$N = \frac{L^2 m\omega_c}{2\pi\hbar} = \frac{e\Phi}{hc},$$

where $\Phi \equiv BL^2$ is the amount of magnetic flux through the sample.

3. **Landau Levels in an Electric Field.** [optional since I added it late]

In lecture I gave several arguments that a quantum Hall droplet has a linearly-dispersing edge mode. Here is a fully quantum mechanical argument. We're going to think about the physics in a neighborhood of the boundary of the sample, where the confining potential $V \simeq -Ex$ is slowly varying, and describes an electric field $E = -\partial_x V$.

The Hamiltonian in the Landau gauge (the one used in the previous problem) is

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) - eEx. \tag{2}$$

(a) Using the same ansatz as above, write the Hamiltonian as that of a displaced harmonic oscillator.

(b) Conclude that the eigenstates have the form

$$\psi(x, y) = \psi_{n,k}\left(x - \frac{mE}{eB^2}, y\right) \tag{3}$$

with energies

$$E_{n,k} = \hbar\omega_c\left(n + \frac{1}{2}\right) + eE\left(k\ell_B - \frac{eE}{m\omega_c^2}\right) + \frac{m}{2}\frac{E^2}{B^2}. \tag{4}$$

(c) Plot this spectrum, and interpret $\partial_k E_{n,k}$ as a velocity in the y direction. Compare this drift velocity with the classical behavior of a charged particle in crossed E and B fields.

4. **Topological terms, particle on a ring.** [from Abanov]

The purpose of this problem is to demonstrate that total derivative terms in the action do affect the physics quantum mechanically.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B}\cdot \mathrm{d}\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta \mathrm{d}\tau \left(\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi}\right)} .$$

Here

$$\phi \equiv \phi + 2\pi \tag{5}$$

is a coordinate on the ring. Because of the identification (4), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta}Q\tau + \sum_{\ell\in\mathbb{Z}\backslash 0} \phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{6}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (5), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where $\hat{f}(p) = \int dx e^{-\mathbf{i}px} f(x)$ is the fourier transform of f.]

Applied to this problem, the Poisson formula says

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}tn^2 + \mathbf{i}zn} = \sqrt{\frac{2\pi}{t}} \sum_{\ell\in\mathbb{Z}} e^{-\frac{1}{2t}(z-2\pi\ell)^2}. \;\;]$$

Using the given mode expansion and $\int_0^\beta dt e^{\frac{2\pi\mathbf{i}(l-l')\tau}{\beta}} = \beta\delta_{l,l'}$ the action is

$$S[\phi] = \mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta} + \sum_{\ell\neq 0} \frac{(2\pi\ell)^2 m}{2\beta} \phi_\ell \phi_{-\ell}$$

where $\phi_\ell = \phi^\star_{-\ell}$. Thus

$$Z = \sum_{Q\in\mathbb{Z}} e^{-\mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta}} \prod_{\ell\neq 0} \int d^2\phi_\ell e^{\frac{(2\pi\ell)^2 m}{2\beta}\phi_\ell\phi^\star_\ell} \tag{7}$$

$$= \sum_{Q\in\mathbb{Z}} e^{-\mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta}} \prod_{\ell\neq 0} \left(\frac{\beta}{2\pi\ell^2 m}\right) \tag{8}$$

$$\propto \sum_{n\in\mathbb{Z}} e^{-\beta\frac{1}{2m(2\pi)^2}(\theta-2\pi n)^2} = \sum_{n\in\mathbb{Z}} e^{-\beta\frac{1}{2m}\left(n-\frac{\theta}{2\pi}\right)^2} \tag{9}$$

where in the last step we used the above Poisson summation formula with $z = \theta$ and $t = \frac{m(2\pi)^2}{\beta}$.

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

After the Poisson resummation, this is manifestly the partition function of a system with energies $E_n = \frac{1}{2m}(n - \frac{\theta}{2\pi})^2$.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

Note that the action given above is the *Euclidean* action. The real time action (from which we should derive the hamiltonian) is

$$S = \int dt \left(\frac{1}{2}m\dot{\phi}^2 + \dot{\phi}\frac{\theta}{2\pi}\right).$$

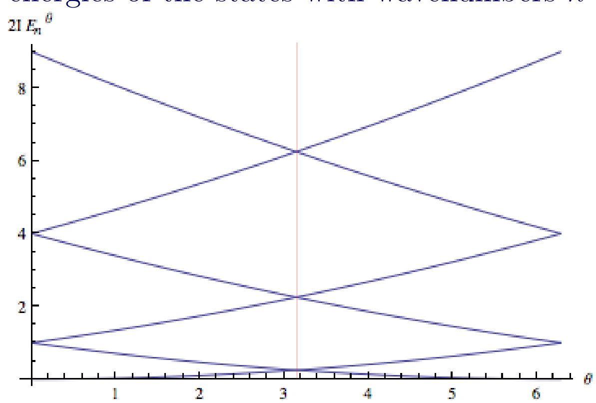
This gives $p = \frac{\partial L}{\partial \dot{\phi}} = m\dot{\phi} + \frac{\theta}{2\pi}$, and hence

$$H = \frac{\left(p - \frac{\theta}{2\pi}\right)^2}{2m}.$$

Now, since $\phi \equiv \phi + 2\pi$, its canonical momentum is quantized, $p \in \mathbb{Z}$, so

$$E_n = \frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2$$

as above. We find the following spectrum for various θ (I am plotting the energies of the states with wavenumbers $n \in [-3, 2]$):



(In the axis label, I is the moment of inertia of the rotor.) Notice that when $\theta = \pi$, the groundstate becomes doubly degenerate.

5. **Aharonov-Casher effect** [Bonus problem]

   [Commins] Consider a neutral particle with spin-$\frac{1}{2}$ (such as a neutron), described by the Lagrangian

$$L = \frac{1}{2}mv^2 + \mu\vec{\sigma} \cdot \left(\vec{v} \times \vec{E}\right)$$

   where $\vec{\sigma}$ is the spin operator and $\vec{v} \equiv \dot{\vec{x}}$.

   (a) Find the canonical momentum and the Hamiltonian.

   (b) Consider an experiment where a cylindrically-symmetric electric field $\vec{E}$ is produced by a line charge with charge-per-unit-length λ extended in the $\check{z}$ direction. Two beams of the particles described by L are sent in paths around the line charge and allowed to interfere. Show that the phase shift between the two waves arising from the line charge is

$$\delta_{\pm} = \pm\frac{4\pi\lambda\mu}{\hbar c}$$

   for spin up/down in the $\check{z}$ basis.