University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 4 – Solutions

Due 11:00am Wednesday, May 3, 2023

---

1. **Commutation relations of creation operators for general one-particle states.** Show that

$$\mathbf{a}(\varphi_1)\mathbf{a}^\dagger(\varphi_2) - \zeta\mathbf{a}^\dagger(\varphi_2)\mathbf{a}(\varphi_1) = \langle\varphi_2|\varphi_1\rangle \,,$$

where these objects are as defined in the lecture notes.

$$[\mathbf{a}(\varphi_1), \mathbf{a}^\dagger(\varphi_2)]_\zeta = \sum_{k_1,k_2}[\mathbf{a}_{k_1}, \mathbf{a}_{k_2}]_\zeta\varphi_1(k_1)\varphi_2(k_2)^\star = \sum_k \varphi_1(k)\varphi_2(k)^\star = \langle\varphi_2|\varphi_1\rangle \,.$$

2. **Fermion creation and annihilation algebra.**

Consider a single fermion mode $\mathbf{c}$. We showed in lecture that the associated Hilbert space is two-dimensional, and is spanned by

$$|0\rangle\,, \quad \text{with } \mathbf{c}\,|0\rangle = 0 \;\text{ and }\; |1\rangle = \mathbf{c}^\dagger\,|0\rangle\,.$$

(a) Check that the two states are orthogonal.

$$\langle 1|0\rangle = \langle 0|\,\mathbf{c}\,|0\rangle = 0.$$

(b) Show that acting on this Hilbert space it is indeed true that

$$\mathbf{c}^\dagger\mathbf{c} + \mathbf{c}\mathbf{c}^\dagger = \mathbb{1},$$

as long as $\langle 1|1\rangle = \langle 0|0\rangle$.

A resolution of the identity is the sum of projectors $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$. $\mathbf{c}^\dagger\mathbf{c}$ gives zero when acting on $|0\rangle$, and gives back $|1\rangle$ when acting on $|1\rangle$. Therefore it acts as the projector

$$\mathbf{c}^\dagger\mathbf{c} = |1\rangle\langle 1|.$$

Similarly, $\mathbf{c}\mathbf{c}^\dagger$ gives zero when acting on $|1\rangle$, and gives back $|0\rangle$ when acting on $|0\rangle$. Therefore it acts as

$$\mathbf{c}\mathbf{c}^\dagger = |0\rangle\langle 0|.$$

Therefore

$$\mathbf{c}^\dagger\mathbf{c} + \mathbf{c}\mathbf{c}^\dagger = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}.$$

Actually, we haven't specified the overall normalization of $\mathbf{c}$ so far, that is, $\mathbf{c}' = z\mathbf{c}$ for $z \in \mathbb{C}$ would also satisfy these demands. This would give

$$\mathbf{c}^\dagger\mathbf{c} + \mathbf{c}\mathbf{c}^\dagger = |z|^2|0\rangle\langle 0| + |z|^2|1\rangle\langle 1| = |z|^2\mathbb{1}.$$

But now consider

$$\langle 1|1\rangle = \langle 0|\,\mathbf{c}\mathbf{c}^\dagger\,|0\rangle = \langle 0|\left(|z|^2\mathbb{1} - \mathbf{c}^\dagger\mathbf{c}\right)|0\rangle = |z|^2\,\langle 0|0\rangle\,.$$

So we must have $|z| = 1$. The overall phase of $\mathbf{c}$ is ambiguous.

(c) Check that

$$[\mathbf{N}, \mathbf{c}] = -\mathbf{c}, \quad [\mathbf{N}, \mathbf{c}^\dagger] = \mathbf{c}^\dagger$$

where $\mathbf{N} = \mathbf{c}^\dagger\mathbf{c}$ is the number operator. Notice that this is the same algebra satisfied by bosonic modes.

There is a useful fermionic version of the Liebniz rule for commutators ($[AB, C] = A[B, C] + [A, C]B$), namely

$$\{AB, C\} = A\{B, C\} - \{A, C\}B.$$

Check: $ABC + CAB = ABC + ACB - ACB - CAB$.

Applying this here, we get

$$[\mathbf{N}, \mathbf{c}] = [\mathbf{c}^\dagger\mathbf{c}, \mathbf{c}] = \mathbf{c}^\dagger\{\mathbf{c}, \mathbf{c}\} - \{\mathbf{c}^\dagger, \mathbf{c}\}\mathbf{c} = -\mathbf{c}$$

while

$$[\mathbf{N}, \mathbf{c}^\dagger] = \mathbf{c}^\dagger\{\mathbf{c}, \mathbf{c}^\dagger\} - \{\mathbf{c}^\dagger, \mathbf{c}^\dagger\}\mathbf{c} = +\mathbf{c}^\dagger.$$

3. **Majorana modes.** Given a collection of fermionic operators $\mathbf{c}_A$, satisfying the fermionic creation-annihilation algebra

$$\{\mathbf{c}_A, \mathbf{c}_B^\dagger\} = \delta_{AB}\mathbb{1} \qquad \text{and} \qquad \{\mathbf{c}_A, \mathbf{c}_B\} = 0,$$

we can decompose them into their real and imaginary parts

$$\gamma_{A1} \equiv \frac{1}{2}\left(\mathbf{c}_A + \mathbf{c}_A^\dagger\right), \quad \gamma_{A2} \equiv \frac{1}{2\mathbf{i}}\left(\mathbf{c}_A - \mathbf{c}_A^\dagger\right).$$

These are called *Majorana modes*.

(a) Show that the Majorana modes satisfy the algebra

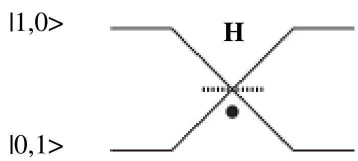
$$\{\gamma_a, \gamma_b\} = 2\Upsilon \delta_{ab} \mathbb{1},$$

where here a is a multi-index running over both A and $\alpha = 1, 2$. In particular, notice that $\gamma_a^2 = \Upsilon \mathbb{1}$. Find the constant Υ.

(b) Write the number operator $\mathbf{c}_A^\dagger \mathbf{c}_A$ in terms of the Majorana modes. Show that it is hermitian.

For each complex mode, $c^\dagger c = \mathbf{i}\gamma_1\gamma_2$. This is hermitian because $(\mathbf{i}\gamma_1\gamma_2)^\dagger = -\mathbf{i}\gamma_2\gamma_1 = +\mathbf{i}\gamma_1\gamma_2$.

4. **Multiple photons on paths of an interferometer.**



One way to make a qubit is out of the two states of a photon moving on the upper and lower paths of an interferometer. On such a qbit, a half-silvered mirror **H** acts as a unitary gate, as indicated at left. (The dot below the mirror specifies a sign convention, to be explained below.)

On the other hand, photons are bosons. This means that if

$$\mathbf{a}^\dagger |0,0\rangle \equiv |1,0\rangle \text{ is a state with one photon on the upper path}$$

of the interferometer, then

$$\frac{(\mathbf{a}^\dagger)^n}{\sqrt{n!}} |0,0\rangle \equiv |n,0\rangle \text{ is a state with } n \text{ photons on the upper path.}$$

Similarly, define

$$\frac{(\mathbf{b}^\dagger)^n}{\sqrt{n!}} |0,0\rangle \equiv |0,n\rangle \text{ to be a state with } n \text{ photons on the lower path}$$

of the interferometer. (Note that $[\mathbf{a}, \mathbf{b}] = 0 = [\mathbf{a}, \mathbf{b}^\dagger]$ – they are independent modes.)

Now suppose we direct these two paths through a half-silvered mirror, as in the figure. A half-silvered mirror acts as a Hadamard gate

$$\mathbf{H} \equiv \frac{1}{\sqrt{2}} (\boldsymbol{\sigma}^x + \boldsymbol{\sigma}^z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

on the qubit made from the one-photon states. (The dot tells us where to put the negative entry.)

Some warm-up questions:

(a) What is the state $|0,0\rangle$? How does $\mathbf{H}$ act on $|0,0\rangle$?

All answers below part d.

(b) How does $\mathbf{H}$ act on $|2,0\rangle$ and $|0,2\rangle$?

(c) How does $\mathbf{H}$ act on the operators $\mathbf{a}^\dagger$ and $\mathbf{b}^\dagger$ (in order that the above relations are realized)?

Here's a more interesting question:

(d) A coherent state is a good cartoon of the state of light in a laser beam. What is the state which results upon sending a coherent state of photons

$$|\alpha, \beta) \equiv \mathcal{N}_\alpha \mathcal{N}_\beta \; e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger} |0,0\rangle$$

through a half-silvered mirror? ($\mathcal{N}_\alpha \equiv e^{-|\alpha|^2/2}$ is a normalization constant.) [Hint: it may be useful to insert $\mathbb{1} = \mathbf{H}^2$ in between the $e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger}$ and the $|0,0\rangle$.]

The hilbert space under discussion here is that of two harmonic oscillators, and above we have defined $|n,m\rangle$ to be the state where the respective number operators $\mathbf{a}^\dagger\mathbf{a}$ and $\mathbf{b}^\dagger\mathbf{b}$ have eigenvalues n, m respectively. From the definition of the photon-path-as-qbit, we have:

$$\mathbf{H}|1,0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle) = \frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger + \mathbf{b}^\dagger\right)|0,0\rangle,$$

$$\mathbf{H}|0,1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle) = \frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger - \mathbf{b}^\dagger\right)|0,0\rangle.$$

Now $|0,0\rangle$ is a state with $n=0$ photons on the upper path and $n=0$ photons on the lower path. No photons at all. So we have $\mathbf{H}|0,0\rangle = |0,0\rangle$ since a mirror does nothing to no photons! (It just sits there.) This is a Zen koan: what does a mirror do to no photons. Actually there could be a phase; it would not affect any of the answers below.

This means further that $\mathbf{H}$ acts on the creation operators by

$$\mathbf{H}\mathbf{a}^\dagger\mathbf{H} = \frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger + \mathbf{b}^\dagger\right), \mathbf{H}\mathbf{b}^\dagger\mathbf{H} = \frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger - \mathbf{b}^\dagger\right),$$

in order to be consistent with the action on the one-photon states. So we can conclude that

$$\mathbf{H}|2,0\rangle = \mathbf{H}\frac{\left(\mathbf{a}^\dagger\right)^2}{\sqrt{2!}}|0,0\rangle = \left(\mathbf{H}\mathbf{a}^\dagger\mathbf{H}\right)^2\frac{1}{\sqrt{2!}}|0,0\rangle = \frac{1}{2}\frac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger + \mathbf{b}^\dagger\right)^2|0,0\rangle = \frac{1}{2}\left(|2,0\rangle + \sqrt{2}|1,1\rangle + |0,2\rangle$$

$$\mathbf{H}|0,2\rangle = \mathbf{H}\frac{(\mathbf{b}^\dagger)^2}{\sqrt{2!}}|0,0\rangle = (\mathbf{H}\mathbf{b}^\dagger\mathbf{H})^2\frac{1}{\sqrt{2!}}|0,0\rangle = \frac{1}{2}\frac{1}{\sqrt{2}}(\mathbf{a}^\dagger - \mathbf{b}^\dagger)^2|0,0\rangle = \frac{1}{2}\left(|2,0\rangle - \sqrt{2}|1,1\rangle + |0,2\rangle\right)$$

And finally,

$$\mathbf{H}e^{\alpha\mathbf{a}^\dagger + \beta\mathbf{b}^\dagger}|0,0\rangle = e^{\alpha\mathbf{H}\mathbf{a}^\dagger\mathbf{H} + \beta\mathbf{H}\mathbf{b}^\dagger\mathbf{H}}|0,0\rangle = e^{\frac{1}{\sqrt{2}}\left(\alpha(\mathbf{a}^\dagger + \mathbf{b}^\dagger) + \beta(\mathbf{a}^\dagger - \mathbf{b}^\dagger)\right)}|0,0\rangle = e^{\frac{\alpha+\beta}{\sqrt{2}}\mathbf{a}^\dagger + \frac{\alpha-\beta}{\sqrt{2}}\mathbf{b}^\dagger}|0,0\rangle$$

It acts on the coherent state labels just like it does on the quantum amplitudes. These coherent state labels are the data that label the lightwave in *e.g.* a laser.

The half-silvered mirror is a special case of the more general notion called a beam-splitter. Suppose instead that the action on the mode operators were[1]

$$\begin{aligned}\mathbf{U}^\dagger(\theta)\mathbf{a}\mathbf{U}(\theta) &= \mathbf{a}\cos\theta + \mathbf{i}\mathbf{b}\sin\theta \\ \mathbf{U}^\dagger(\theta)\mathbf{b}\mathbf{U}(\theta) &= \mathbf{b}\cos\theta + \mathbf{i}\mathbf{a}\sin\theta\ .\end{aligned} \tag{1}$$

(e) Show that $\mathbf{U}(\theta)$ can be written as an evolution operator, in the form:

$$\mathbf{U}(\theta) = e^{\mathbf{i}\theta G}, \quad G = \mathbf{a}^\dagger\mathbf{b} + \mathbf{b}^\dagger\mathbf{a}. \tag{2}$$

Write

$$U^\dagger\mathbf{a}U = e^{-\mathbf{i}\theta G}\mathbf{a}e^{\mathbf{i}\theta G} = e^{-\mathbf{i}\theta\,\mathrm{ad}_G}\mathbf{a} \tag{3}$$

where ad_G is defined to be the *adjoint action* of G, that is,

$$\mathrm{ad}_G\mathcal{O} \equiv [G, \mathcal{O}].$$

The expression (3) follows by Taylor expansion. So we just need to figure out $\mathrm{ad}_G(\mathbf{a}), \mathrm{ad}_G^2(\mathbf{a})$ etc... But this is very simple:

$$\mathrm{ad}_G(\mathbf{a}) = [G, \mathbf{a}] = [\mathbf{a}^\dagger\mathbf{b}, \mathbf{a}] = -\mathbf{b}.$$

This means

$$\mathrm{ad}_G^2(\mathbf{a}) = [G, [G, \mathbf{a}]] = [G, -\mathbf{b}] = [\mathbf{b}^\dagger\mathbf{a}, -\mathbf{b}] = +\mathbf{a}.$$

And therefore the exponential series $e^{-\mathbf{i}\theta\,\mathrm{ad}_G}\mathbf{a}$ is just

$$e^{-\mathbf{i}\theta\,\mathrm{ad}_G}\mathbf{a} = \mathbf{a} + \frac{(-\mathbf{i}\theta)^2}{2!}\mathbf{a} + \frac{(-\mathbf{i}\theta)^4}{4!}\mathbf{a} + \cdots - \mathbf{i}\theta(-\mathbf{b}) + \frac{(-\mathbf{i}\theta)^3}{3!}(-\mathbf{b}) + \cdots = \cos\theta\mathbf{a} + \mathbf{i}\sin\theta\mathbf{b}.$$

Similarly,

$$e^{-\mathbf{i}\theta\,\mathrm{ad}_G}\mathbf{b} = \cos\theta\mathbf{b} + \mathbf{i}\sin\theta\mathbf{a}.$$

[1] The operation H in the previous parts is not $\mathbf{U}(\theta)$ for some θ; it is similar. I apologize for any confusion this caused. To get H we would have to write $\mathbf{U}'(\theta) = e^{\mathbf{i}\theta G'}$, with $G' \equiv \mathbf{i}\mathbf{a}^\dagger\mathbf{b} - \mathbf{i}\mathbf{b}^\dagger\mathbf{a}$, and set $\theta = \pi/2$.

(f) Show that when $\theta = \pi/4$ this beam-splitter takes the state $|1,1\rangle$ with one boson in each mode to the state

$$\frac{1}{\sqrt{2}}\left(|2,0\rangle + |0,2\rangle\right).$$

The beam-splitter takes the state to

$$\mathbf{U}^\dagger |1,1\rangle = \mathbf{U}^\dagger \mathbf{a}^\dagger \mathbf{U}\mathbf{U}^\dagger \mathbf{b}^\dagger \mathbf{U}\mathbf{U}^\dagger |0\rangle \tag{4}$$

$$= \left(\cos\theta \mathbf{a}^\dagger - \mathbf{i}\sin\theta \mathbf{b}^\dagger\right)\left(\cos\theta \mathbf{b}^\dagger - \mathbf{i}\sin\theta \mathbf{a}^\dagger\right)|0\rangle \tag{5}$$

$$\overset{\theta=\frac{\pi}{4}}{=} \left(\frac{1}{\sqrt{2}}\right)^2 \left(\mathbf{a}^\dagger\mathbf{b}^\dagger(1-1) - \mathbf{i}\left(\mathbf{a}^\dagger\right)^2 - \mathbf{i}\left(\mathbf{b}^\dagger\right)^2\right)|0\rangle = -\mathbf{i}\frac{1}{\sqrt{2}}\left(|2,0\rangle + |0,2\rangle\right). \tag{6}$$

This is sometimes called the *Hong-Ou-Mandel effect*.

(g) What if the operators $\mathbf{a}$ and $\mathbf{b}$ were instead fermionic operators? That is, suppose we send fermionic particles through the same beam-splitter, defined by (1). What is

$$\mathbf{U}_F(\theta = \pi/4)^\dagger |1,1\rangle$$

in this case? Hint: the form of the generator is different

$$\mathbf{U}_F(\theta) = e^{\mathbf{i}\theta G_F}, \quad G_F = \mathbf{a}^\dagger\mathbf{b} - \mathbf{a}\mathbf{b}^\dagger.$$

(Notice that G_F is still hermitian.)

[I got this last part of the problem from Le Bellac.]

The hermitian conjugate is

$$G_F^\dagger = (\mathbf{a}^\dagger\mathbf{b} - \mathbf{a}\mathbf{b}^\dagger)^\dagger = \mathbf{b}^\dagger\mathbf{a} - \mathbf{b}\mathbf{a}^\dagger = -\mathbf{a}\mathbf{b}^\dagger + \mathbf{a}^\dagger b = G_F.$$

In this case, using the identity $[AB, C] = A\{B, C\} - \{A, C\}B$

$$[G_F, \mathbf{a}] = -\{\mathbf{a}^\dagger, \mathbf{a}\}\mathbf{b} = -\mathbf{b}, \quad [G_F, \mathbf{b}] = -\mathbf{a}\{\mathbf{b}^\dagger, \mathbf{b}\} = -\mathbf{a}.$$

So the series is again

$$e^{-\mathbf{i}\theta \mathrm{ad}_{G_F}}\mathbf{a} = \mathbf{a} + \mathbf{i}\theta\mathbf{b} + \frac{(-\mathbf{i}\theta)^2}{2!}\mathbf{a} - \frac{(-\mathbf{i}\theta)^3}{3!}\mathbf{b} + \cdots = \cos\theta\mathbf{a} + \mathbf{i}\sin\theta\mathbf{b}.$$

Notice that even though $\mathbf{a}$ and $\mathbf{b}$ are fermionic operators (*e.g.*, $\mathbf{a}^2 = 0$), the exponential $e^{-\mathbf{i}\theta G_F}$ is still an infinite series, because it contains terms which alternate between $\mathbf{a}$ and $\mathbf{a}^\dagger$.

So actually the whole calculation is the same up to the last step:

$$\mathbf{U}_F^\dagger |1,1\rangle = \mathbf{U}^\dagger \mathbf{a}^\dagger \mathbf{U} \mathbf{U}^\dagger \mathbf{b}^\dagger \mathbf{U} \mathbf{U}^\dagger |0\rangle \tag{7}$$

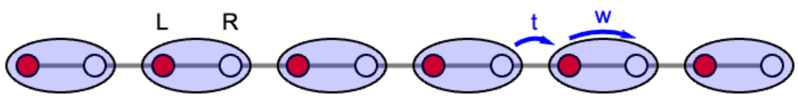
$$= \left(\cos\theta \mathbf{a}^\dagger - \mathbf{i}\sin\theta \mathbf{b}^\dagger\right)\left(\cos\theta \mathbf{b}^\dagger - \mathbf{i}\sin\theta \mathbf{a}^\dagger\right)|0\rangle \tag{8}$$

$$\stackrel{\theta=\frac{\pi}{4}}{=} \left(\frac{1}{\sqrt{2}}\right)^2 \left(\mathbf{a}^\dagger \mathbf{b}^\dagger - \mathbf{b}^\dagger \mathbf{a}^\dagger - \mathbf{i}\left(\mathbf{a}^\dagger\right)^2 - \mathbf{i}\left(\mathbf{b}^\dagger\right)^2\right)|0\rangle = |1,1\rangle \tag{9}$$

since $\{\mathbf{a}^\dagger, \mathbf{b}^\dagger\} = 0$. In the case of Fermions, the state is taken to itself by this beamsplitter.

5. **Slightly more interesting bandstructure.**

Consider a particle hopping on a chain of sites where each site involves two orbitals, one on the left and one on the right.



So the single-particle hamiltonian is

$$\mathbf{H} = \sum_n t\left(|n,R\rangle\langle n+1,L| + |n+1,L\rangle\langle n,R|\right) + (w|n,L\rangle\langle n,R| + h.c.), \tag{10}$$

where w, t are two quantities with dimensions of energy.

(a) Write down the many-body hamiltonian in terms of annihilation $\mathbf{c}_{n,\alpha}$ and creation $\mathbf{c}_{n,\alpha}^\dagger$ operators.

$$\mathbf{H} = \sum_n \left(t\mathbf{c}_{R,n}^\dagger \mathbf{c}_{L,n+1} + h.c. + w\mathbf{c}_{L,n}^\dagger \mathbf{c}_{R,n} + h.c.\right). \tag{11}$$

(b) Suppose there are N sites and N fermions and suppose $w > t$ (for simplicity take w real). Is it a metal or an insulator? Find the energy difference between the groundstate and the first excited state in the thermodynamic ($N \to \infty$) limit.

Since it is translation-invariant and an eigenvalue problem is linear, the single-particle hamiltonian can be diagonalized by going to momentum space. Acting on $|k,\alpha\rangle$, it acts as the matrix

$$h(k) = \begin{pmatrix} 0 & te^{-\mathbf{i}k} + w \\ w^\star + te^{\mathbf{i}k} & 0 \end{pmatrix} = (t\cos k + w_1)X + (t\sin k + w_2)Y \tag{12}$$

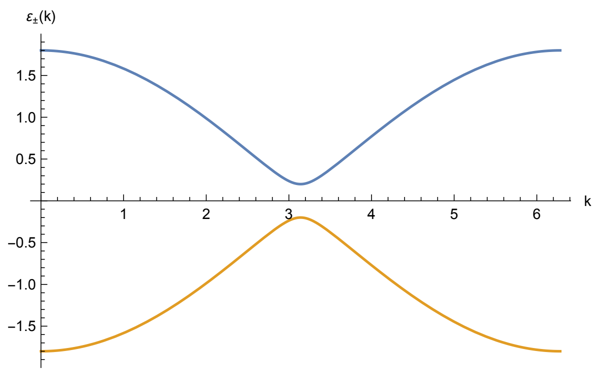
where $w = w_1 + \mathbf{i}w_2$ and $X \equiv \sigma^x, Y \equiv \sigma^y$. That is, using $\mathbb{1} = \int_0^{2\pi/a} \sum_{\alpha=L,R} đk |k,\alpha\rangle\langle k,\alpha|$,

$$\mathbf{H} = \mathbf{H} \int_0^{2\pi/a} đk \sum_\alpha |k,\alpha\rangle\langle k,\alpha| = \int_0^{2\pi/a} đk |k,\alpha\rangle\langle k,\beta| h(k)_{\alpha\beta}. \tag{13}$$

Taking w real, the spectrum of the matrix $h(k)$ is

$$\epsilon_\pm(k) = \pm\sqrt{t^2 + |w|^2 + 2w\cos k}, \tag{14}$$

which looks like this for $|w| \neq t$.

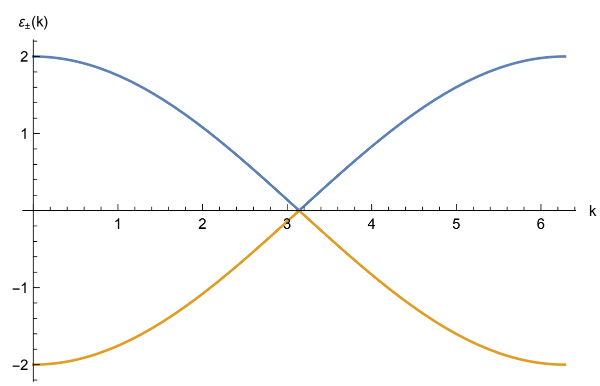


The system is an insulator, since we fill the bottom band, and then the energy cost to excite an electron in the upper band is

$$\min_k (\epsilon_+(k) - \epsilon_-(k)) = 2|t - w|. \tag{15}$$

(c) What happens when $w = t$?

In this case the problem is really just a chain of sites with translation symmetry under a single step. The bandstructure looks like this



which is the single band $\epsilon(k) = 2\cos 2k$ folded on itself into the smaller Brillouin zone. In this case, the system is a metal, since we must fill up half the band.

6. **Normalization.** Check that if $\Psi(r_1 \cdots r_n)$ is a normalized and (anti)symmetric wavefunction on n particles, then

$$|\Psi\rangle \equiv \sum_{r_1 \cdots r_n} \Psi(r_1 \cdots r_n) \, |r_1 \cdots r_n\rangle \tag{16}$$

is normalized, $\langle \Psi | \Psi \rangle = 1$.

(Interpret the sum over r as an integral if you like.)

The overlap is

$$\langle \Psi | \Psi \rangle = \sum_{r_1 \cdots r_n} \sum_{r'_1 \cdots r'_n} \Psi(r_1 \cdots r_n)^\star \underbrace{\langle r_1 \cdots r_n | r_1 \cdots r'_n \rangle}_{= \frac{1}{n!} \sum_\pi s^\pi \delta_{r_1 r_{\pi(1)'}} \cdots \delta_{r_n r'_{\pi(n)}}} \Psi(r'_1 \cdots r'_n) \tag{17}$$

$$= \sum_r |\Psi(r_1 \cdots r_n)|^2 = 1. \tag{18}$$

where the last equation is the normalization condition for Ψ. In the first step, we used only the defintion (16) (twice).