University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 3 – Solutions

Due 11:00am Wednesday, April 26, 2023

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1. **Brain-warmer: fluctuations of the EM field.** Let us focus on a single mode, and a single polarization, of the EM field, at a point in space:

$$\mathbf{E}(t) \equiv \mathbf{E}(\vec{r}=0,t) = \mathbf{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\mathbf{a}e^{-\mathbf{i}\omega t} - \mathbf{a}^\dagger e^{\mathbf{i}\omega t}\right)$$

where V is the volume of the cavity in which this mode lives. Define the *variance* of an operator in a state ψ as

$$\Delta_\psi E \equiv \sqrt{\langle\psi|\mathbf{E}^2(t)|\psi\rangle - \left(\langle\psi|\mathbf{E}(t)|\psi\rangle\right)^2}.$$

(a) Find the variance of $\mathbf{E}$ in the vacuum, $\Delta_0 E$.

Let's write $\mathcal{N} \equiv \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$. Then $\langle 0|\mathbf{E}(t)|0\rangle = 0, \langle 0|\mathbf{E}(t)^2|0\rangle = \mathcal{N}^2$, so $(\Delta E)_0 = \mathcal{N}$.

(b) Find the variance of $\mathbf{E}$ in the state of exactly n photons (all in this mode).

$\langle n|\mathbf{E}(t)|n\rangle = 0, \langle n|\mathbf{E}(t)^2|n\rangle = \mathcal{N}^2(2n+1)$, so $(\Delta E)_n = \mathcal{N}(\sqrt{2n+1})$.

(c) A more realistic state, for both a single-mode laser, and for a classical source of light, is a coherent state. (In a classical source of light, different modes have random phases relative to each other.) In this state, what is the expected number of photons? What is its variance?

$$\langle z|\hat{N}|z\rangle = \langle z|a^\dagger a|z\rangle = |z|^2.$$

$$\langle z|\hat{N}^2|z\rangle = \langle z|a^\dagger a a^\dagger a|z\rangle = \langle z|a^\dagger\left(a^\dagger a + 1\right)a|z\rangle = |z|^4 + |z|^2.$$

$$(\Delta N)_z = |z|.$$

Find the variance of $\mathbf{E}$ in a coherent state $|z\rangle$, where $\mathbf{a}|z\rangle = z|z\rangle$.

$$\langle z|\mathbf{E}(t)|z\rangle = \mathbf{i}\mathcal{N}e^{-\mathbf{i}\omega t}\langle z|a|z\rangle + h.c. = \mathbf{i}\mathcal{N}e^{-\mathbf{i}\omega t}z + h.c.,$$

$$\langle z|\mathbf{E}(t)^2|z\rangle = \left(\langle z|\mathbf{E}(t)|z\rangle\right)^2 + \mathcal{N}^2,$$

so $(\Delta E)_n = \mathcal{N}$, same as in vacuum.

2. **Maxwell's equations, quantumly.**

   (a) Check that the oscillator algebra for the photon creation and annihilation operators

$$[\mathbf{a}_{ks}, \mathbf{a}^\dagger_{k's}] = \delta_{kk'}\delta_{ss'}. \tag{1}$$

   implies (using the mode expansion for $\mathbf{A}$) that

$$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -\mathbf{i}\hbar \int đ^3k \; e^{\mathbf{i}\vec{k}\cdot(\vec{r}-\vec{r}')} \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)/\varepsilon_0$$

   (and also $[\mathbf{A}_i(\vec{r}), \mathbf{A}_j(\vec{r}')] = 0$ and $[\mathbf{E}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = 0$).

   Conclude that it's not possible to simultaneously measure $E_x(\vec{r})$ and $B_y(\vec{r})$.

   (b) Using the result of the previous part, check that the wave equation for $\mathbf{A}_i(x)$ follows from the Heisenberg equations of motion

$$-\partial_t\vec{\mathbf{E}} = \frac{\mathbf{i}}{\hbar}[\mathbf{H}, \vec{\mathbf{E}}].$$

3. **A charged particle, classically.** [I am postponing this problem until we discuss path integrals.] This problem is an exercise in calculus of variations, as well as an important ingredient in our discussion of particles in electromagnetic fields.

   Consider the following action functional for a particle in three dimensions:

$$S[x] = \int dt \left(\frac{m}{2}\dot{\vec{x}}^2 - e\Phi(\vec{x}) + \frac{e}{c}\dot{\vec{x}}\cdot\vec{A}(x)\right) .$$

   (a) Show that the extremization of this functional gives the equation of motion:

$$\frac{\delta S[x]}{\delta x^i(t)} = -m\ddot{x}^i(t) - e\partial_{x^i}\Phi(x(t)) + \frac{e}{c}\dot{x}^j F_{ij}(x(t))$$

   where $F_{ij} \equiv \partial_{x^i}A_j - \partial_{x^j}A_i$. Show that this is the same as the usual Coulomb-Lorentz force law

$$\vec{F} = e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

   with $B_i \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}$.

   (b) Show that the canonical momenta are

$$\Pi_i \equiv \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + \frac{e}{c}A_i(x).$$

   Here $S = \int dtL$. (I call them Π rather than p to emphasize the difference from the 'mechanical momentum' $m\dot{x}$.) Show that the resulting Hamiltonian is

$$H \equiv \sum_i \dot{x}^i\Pi^i - L = \frac{1}{2m}\left(\Pi_i - \frac{e}{c}A_i(x(t))\right)^2 + e\Phi.$$

4. **Phonons in salt.** Consider a model of a more complex solid, where there are two kinds of atoms, of masses m and M, connected by springs of strength κ_1 and κ_2, as in the figure. (This is a cartoon of an ionic solid, like NaCl.)

The *unit cell*, *i.e.* the pattern that is repeated, contains two atoms. Let q_n be the deviation from equilibrium position of the nth light atom and Q_n be the deviation from equilibrium position of the nth heavy atom. Assume periodic boundary conditions. You may wish to rescale the variables q, Q to simplify the dependence on the parameters. (If you prefer, study the special case where $m = M$, but $\kappa_1 \neq \kappa_2$.)

By making the Fourier ansatz

$$\begin{pmatrix} q_n \\ Q_n \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_k e^{-\mathbf{i}kna} \begin{pmatrix} q_k \\ Q_k \end{pmatrix}$$

find the spectrum of normal modes. Introduce creation and annihilation operators and reduce the problem to a collection of decoupled oscillators.

In addition to the *acoustic phonons*, which have dispersion $\omega_k \sim v_s|k|$ near $k = 0$, and represent the quantum avatar of the sound mode, you should find a branch of *optical phonons* which have $\omega_k \overset{k\to 0}{\to}$ a finite number. Interpret the optical phonon mode in terms of the motion of the atoms.

The Lagrangian is

$$L = \frac{1}{2}\sum_n \left(m\dot{q}_n^2 + M\dot{Q}_n^2 - \kappa_1 (q_n - Q_n)^2 - \kappa_2 (Q_n - q_{n+1})^2 \right).$$

Rescaling $q \to \sqrt{m}q, Q \to \sqrt{M}Q$ it becomes

$$L = \frac{1}{2}\sum_n \left(\dot{q}_n^2 + \dot{Q}_n^2 - \kappa_1 \left(q_n/\sqrt{m} - Q_n/\sqrt{M}\right)^2 - \kappa_2 \left(Q_n/\sqrt{M} - q_{n+1}/\sqrt{m}\right)^2 \right) \tag{2}$$

$$= \frac{1}{2}\sum_n \left(\dot{q}_n^2 + \dot{Q}_n^2 - \frac{\kappa_1+\kappa_2}{m} q_n^2 - \frac{\kappa_1+\kappa_2}{M} Q_n^2 - \frac{2}{\sqrt{mM}} \left(\kappa_1 q_n Q_n + \kappa_2 Q_n q_{n+1}\right) \right). \tag{3}$$

We don't miss any physics by setting $m = M$, and we avoid some annoying algebra. When $m = M$, the Hamiltonian has the form

$$\mathbf{H} = \sum_n \left(\frac{\mathbf{p}_n^2}{2} + \frac{\mathbf{P}_n^2}{2} + \frac{1}{2} g_1 (q_n - Q_n)^2 + \frac{1}{2} g_2 (Q_n - q_{n-1})^2 \right).$$

Here $g_{1/2} = \frac{\kappa_{1/2}}{m}$. When $g_1 = g_2$, we return to the periodic chain with a single atom per unit cell.

Using the stated Fourier ansatz, we have

$$\mathbf{H} = \frac{1}{2} \sum_k \left(q_k q_{-k} + Q_k Q_{-k} + (q_k, Q_k) \begin{pmatrix} g_1 + g_2 & h_1(k) \\ h_1^\star(k) & g_1 + g_2 \end{pmatrix} \begin{pmatrix} q_{-k} \\ Q_{-k} \end{pmatrix} \right)$$

where $h_1(k) \equiv -2g_1 - 2g_2 e^{\mathbf{i}ka}$. To see this, consider for example the cross-term

$$\sum_n Q_n q_{n+1} = \sum_{kk'} \frac{1}{N} \sum_k e^{\mathbf{i}kna} e^{\mathbf{i}k'(n+1)a} Q_k q_{k'} = \sum_k e^{-\mathbf{i}k} Q_k q_{-k}.$$

To diagonalize the matrix here, we let

$$\begin{pmatrix} q_k \\ Q_k \end{pmatrix} = U(k) \begin{pmatrix} q_k^+ \\ q_k^- \end{pmatrix}$$

and hence

$$\begin{pmatrix} q_{-k} \\ Q_{-k} \end{pmatrix} = U(-k) \begin{pmatrix} q_{-k}^+ \\ q_{-k}^- \end{pmatrix}$$

where $U(k)$ is the unitary matrix that diagonalizes the matrix

$$M(k) = \begin{pmatrix} g_1 + g_2 & h_1 \\ h_1^\star & g_1 + g_2 \end{pmatrix}$$

for given k, with eigenvalues

$$\varepsilon^\pm(k) = g_1 + g_2 \pm \sqrt{g_1^2 + g_2^2 + 2g_1 g_2 \cos ka}.$$

We make a similar redefinition of

$$\begin{pmatrix} p_k \\ P_k \end{pmatrix} = U(k) \begin{pmatrix} p_k^+ \\ p_k^- \end{pmatrix}.$$

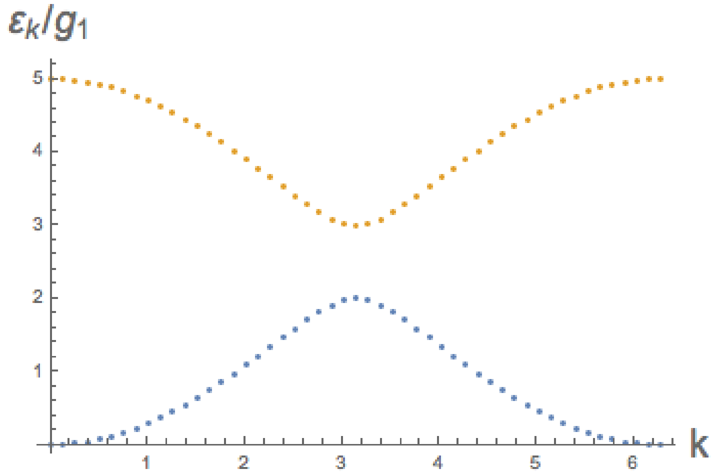
A further definition, as usual,

$$q_k^\pm = \sqrt{\frac{\hbar}{2\omega_k}} \left(\mathbf{a}_{\pm,k} + \mathbf{a}_{\pm,-k}^\dagger \right) p_k^\pm = -\mathbf{i} \sqrt{\frac{\hbar\omega_k}{2}} \left(\mathbf{a}_{\pm,k} - \mathbf{a}_{\pm,-k}^\dagger \right)$$

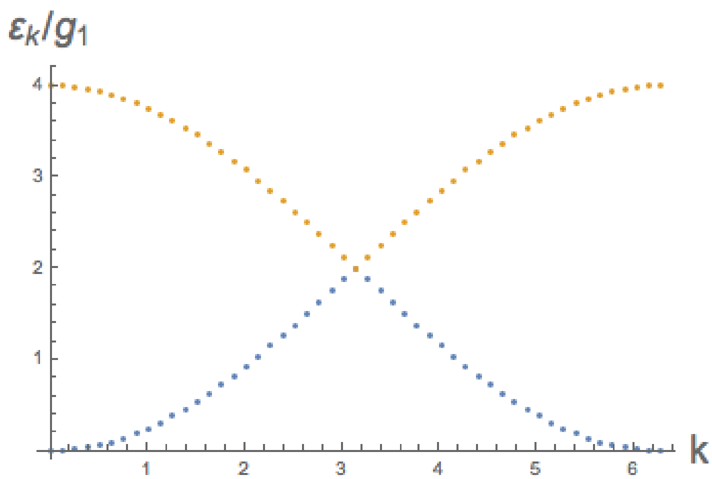
decouples modes with $\pm k$, so that

$$\mathbf{H} = \sum_{k,\alpha=\pm} \varepsilon^{\alpha}(k)\mathbf{a}^{\dagger}_{\alpha,k}\mathbf{a}_{\alpha,k}.$$

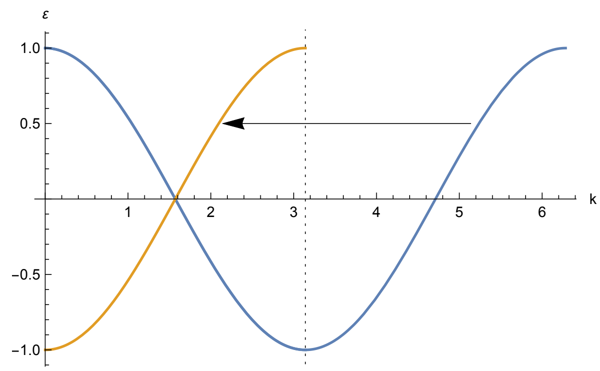
The spectrum looks like this, for $g_1/g_2 = 1.5$:



The optical band describes the mode where the heavy and light atoms move relative to each other within the unit cell. No symmetry guarantees that this is ever massless, and indeed it is not. When $g_1 = g_2$, the unit cell is halved, and the Brillouin zone is twice as big. The optical mode at $k = 0$ here is then just the mode at $k = \frac{\pi}{a_{\frac{1}{2}}}$ where $a_{\frac{1}{2}}$ is the new lattice spacing, as you can see from this plot:



That is, the single band on the larger BZ gets folded on itself when described in terms of the enlarged unit cell, like this:



5. **How big is the Hilbert space?** [Bonus problem] Show that the Hilbert space of N bosons in D orbitals (*i.e.* D possible single-particle states) has dimension

$$\mathcal{D}_B(N, D) = \frac{(N + D - 1)!}{N!(D - 1)!}.$$

Place the N particles in a line. Think of the choice of which of the D modes to put them in as placing $D - 1$ dividers between the D groups of bosons. Since there can be zero bosons in a given mode, there can be more than one divider between two bosons. So we can think about this as putting $N + D - 1$ spots in a line, and choosing $D - 1$ of them to be dividers. There are $\frac{N+D-1}{D-1}$ ways to do this.

6. **Casimir force in real E&M.** [Bonus problem]

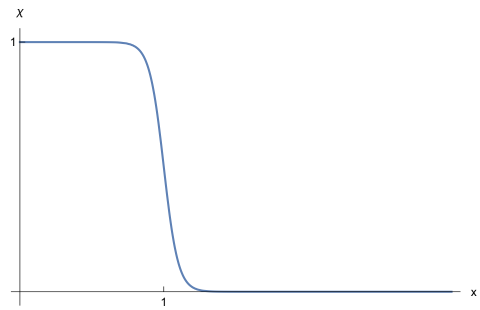
We've shown that the vacuum energy of the electromagnetic field between two perfectly-conducting parallel plates separated by distance d is

$$f(d) = L_x L_y \sum_n{}' \int đ^2 k \omega_n(k) \tag{4}$$

with $\sum_n' \equiv \frac{1}{2}\sum_{n=0} + \sum_{n\geq 1}$ and $\omega_n(k) = c\sqrt{\left(\frac{\pi n}{d}\right)^2 + k_x^2 + k_y^2}$. To account for the failure of the plates to impose boundary conditions on the high-frequency modes, we replace this with

$$f(d) = L_x L_y \sum_n{}' \int đ^2 k \omega_n(k) \chi\left(\frac{\omega_n(k)}{\omega_c}\right) \tag{5}$$

where ω_c is a cutoff energy, and $\chi(x)$ is a function like this:



with $\chi(0) = 1$.

I got this problem from Le Bellac.

(a) Convince yourself that

$$\frac{f(d)}{L_x L_y} = \sum_n{}' \int_{\omega_n}^{\infty} \omega^2 d\omega \chi\left(\frac{\omega}{\omega_c}\right) \tag{6}$$

with $\omega_n \equiv \frac{\pi c n}{d}$.

(b) Show that the pressure on the plate, $P = \frac{-\partial_d E_0}{L_x L_y} = -\frac{(f'(d) - f'(L-d))}{L_x L_y}$, is

$$P \stackrel{L \gg d}{=} -\frac{\hbar c \pi^2}{2d^4} \left(\sum_n{}' n^3 \chi(\omega_n/\omega_c) - \int_0^{\infty} dn n^3 \chi(\omega_n/\omega_c) \right). \tag{7}$$

(c) Use the Euler-Maclaurin theorem

$$\sum_n{}' g(n) - \int_0^{\infty} dn g(n) = -\frac{1}{12} g'(0) + \frac{1}{6!} g'''(0) + \cdots \tag{8}$$

to find the Casimir force.

(d) Note that we did not have to choose a particular function $\chi(x)$.