University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 2

Due 12:30pm Tuesday, April 18, 2023

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1. **Brain-warmer: oscillator algebra.** Convince yourself that an operator $\mathcal{O}$ made of creation and annihilation operators $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$ for various k commutes with the number operator $\sum_k \mathbf{N}_k$ if and only if it has the same number of $\mathbf{a}$s as $\mathbf{a}^\dagger$s.

2. **Brain-warmer: Heisenberg time evolution of the harmonic chain.** Recall the expression for $\mathbf{q}_n$ in terms of creation and annihilation operators given in the lecture notes. Check that the expression for $\mathbf{p}_n$ in terms of creation and annihilation operators is consistent with the Heisenberg equations of motion

$$\mathbf{p}_n = m\dot{\mathbf{q}}_n = \frac{\mathbf{i}m}{\hbar}[\mathbf{H}, \mathbf{q}_n].$$

(That is, evaluate the right hand side of this expression using the algebra of $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$.

3. **Entropy and thermodynamics.** Consider a quantum system with hamiltonian $\mathbf{H}$ and Hilbert space $\mathcal{H}$. Its behavior in thermal equilibrium at temperature T can be described using the *thermal density matrix*

$$\boldsymbol{\rho}_\beta \equiv \frac{1}{Z} e^{-\beta \mathbf{H}}$$

where $\beta \equiv \frac{1}{T}$ specifies the temperature and Z is a normalization factor. (We can think about this as the density matrix resulting from coupling the system to a heat bath and tracing out the Hilbert space of the heat bath.) Expectation values are computed by $\langle \mathcal{O} \rangle \equiv \mathrm{tr} \boldsymbol{\rho}_\beta \mathcal{O}$.

   (a) Find a formal expression for Z by demanding that $\boldsymbol{\rho}_\beta$ is normalized appropriately. This is called the *partition function*.

   (b) Recall that the von Neumann entropy of a density matrix is defined as

$$S[\rho] = -\mathrm{tr} \rho \log \rho.$$

   Show that the von Neumann entropy of $\boldsymbol{\rho}_\beta$ can be written as

$$S_\beta = E/T + \log Z$$

   where $E \equiv \langle \mathbf{H} \rangle$ is the expectation value for the energy. Convince yourself that this is same as the thermal entropy.

(c) Evaluate Z and E and the heat capacity $C = \partial_T E$ for the case where the system is a simple harmonic oscillator

$$\mathcal{H} = \text{span}\{|n\rangle, n = 0, 1, 2...\}, \quad \mathbf{H} = \hbar\omega\left(\mathbf{n} + \frac{1}{2}\right)$$

with $\mathbf{n}|n\rangle = n|n\rangle$.

(d) Now evaluate the low-temperature equilibrium heat capacity for a harmonic mattress (the d-dimensional version of the harmonic chain). That is, find the heat capacity for a collection of harmonic oscillators labelled by wavenumber $\vec{k}$ in d dimensions,

$$\mathbf{H} = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right)$$

with dispersion relation $\omega_k = v_s|k|$.

4. **Gaussian identity.** Show that for a gaussian quantum system

$$\left\langle e^{\mathbf{i}K\mathbf{q}}\right\rangle = e^{-A(K)\langle\mathbf{q}^2\rangle}$$

and determine $A(K)$. Here $\langle ...\rangle \equiv \langle 0| ... |0\rangle$. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).