University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 212C QM Spring 2023
Assignment 1 – Solutions

Due 11:00am Tuesday, April 11, 2023

- Homework will be handed in electronically. Please do not hand in photographs of hand-written work. The preferred option is to typeset your homework. It is easy to do and you need to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples is provided here. If you need help getting set up or have any other questions please email me.

- To hand in your homework, please submit a pdf file through the course's canvas website, under the assignment labelled hw01.

Thanks in advance for following these guidelines. Please ask me if you have any trouble.

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1. **Brain-warmer: oscillation of excited oscillator states.**

   Consider a 1d harmonic oscillator of frequency ω. Consider the initial state

   $$|\psi_{n,s}(0)\rangle \equiv \mathbf{T}(s)\,|n\rangle$$

   where $|n\rangle \equiv \frac{1}{\sqrt{n!}}\left(\mathbf{a}^\dagger\right)^n|0\rangle$ is the nth excited state and $\mathbf{T}(s) \equiv e^{-\mathbf{i}\mathbf{P}s}$ is the displacement operator ($\mathbf{P}$ is the momentum operator).

   Describe (plot it as a function of q for some $n, t, s > 0$) the time evolution of the probability distribution: $\rho(q,t) = |\psi_{n,s}(q,t)|^2$ where $\psi_{n,s}(q,t) \equiv \left\langle q|e^{-\mathbf{i}\mathbf{H}t}|\psi_{n,s}(0)\right\rangle$, and $\langle q|$ is a position eigenstate. Does it keep its shape like it does for $n = 0$?

   There are many ways to do this problem. In retrospect, the easiest way I've found to do this problem is using coherent states, so I should have put it after the next problem.

   We want to know

   $$\psi_{n,s}(q,t) = \left\langle q|e^{-\mathbf{i}\mathbf{H}t}|\psi_{n,s}(0)\right\rangle = \left\langle q|e^{-\mathbf{i}\mathbf{H}t}e^{-\mathbf{i}\mathbf{P}s}|n\right\rangle.$$

First let's move the time evolution operator through the translation operator so it can get at the eigenstate on the right:

$$e^{-\mathbf{iH}t}e^{-\mathbf{iP}s}e^{\mathbf{iH}t} = \exp\left(-\mathbf{i}se^{-\mathbf{iH}t}\mathbf{P}e^{\mathbf{iH}t}\right) \tag{1}$$

$$= \exp\left(-\mathbf{i}se^{-\mathbf{iH}t}\frac{1}{\mathbf{i}}\sqrt{\frac{1}{2}}\left(\mathbf{a}-\mathbf{a}^\dagger\right)e^{\mathbf{iH}t}\right) \tag{2}$$

$$= \exp\left(-\mathbf{i}s\frac{1}{\mathbf{i}}\sqrt{\frac{1}{2}}\left(e^{\mathbf{i}\hbar\omega t}\mathbf{a}-e^{-\mathbf{i}\hbar\omega t}\mathbf{a}^\dagger\right)\right) \tag{3}$$

$$\equiv e^{z\mathbf{a}^\dagger - z^\star\mathbf{a}} \equiv D(z) \tag{4}$$

for appropriate $z = e^{-\mathbf{i}\hbar\omega t}s/\sqrt{2}$. Therefore

$$\psi_{n,s}(q,t) = \langle q|D(z)e^{-\mathbf{iH}t}|n\rangle = \left\langle q|D(z)e^{-\mathbf{i}\hbar\omega(n+\frac{1}{2})}|n\right\rangle. \tag{5}$$

The phase $e^{-\mathbf{i}\hbar\omega(n+\frac{1}{2})} = e^{\mathbf{i}\phi}$ disappears in the probability.

Wait – how did I know that

$$e^{-\mathbf{iH}t}\mathbf{a}e^{\mathbf{iH}t} = e^{\mathbf{i}\hbar\omega t}\mathbf{a}, e^{-\mathbf{iH}t}\mathbf{a}^\dagger e^{\mathbf{iH}t} = e^{-\mathbf{i}\hbar\omega t}\mathbf{a}^\dagger \quad ? \tag{6}$$

Well, one way is to use the general fact that $e^{\mathcal{O}}\mathbf{a}e^{-\mathcal{O}} = e^{\mathrm{ad}_\mathcal{O}}\mathbf{a}$ where $\mathrm{ad}\mathcal{O}(\mathbf{a}) \equiv [\mathcal{O},\mathbf{a}]$. Or we could just Taylor expand in t and repeatedly use $[\mathbf{H},\mathbf{a}] = -\hbar\omega\mathbf{a}$.

So we just need to know

$$\langle q|D(z)|n\rangle. \tag{7}$$

Notice that $D(z)|0\rangle = |z\rangle$ is the normalized coherent state with $\mathbf{a}|z\rangle = z|z\rangle$. So for $n = 0$ the answer is just the wavefunction of the coherent state. To figure out (7), rewrite

$$D(z) = e^{z\mathbf{a}^\dagger - z^\star\mathbf{a}} = f(z,z^\star)e^{c(z-z^\dagger)\mathbf{Q}}e^{\mathbf{i}c'(z+z^\star)\mathbf{P}} \tag{8}$$

using the BCH identity

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad \text{if } [A,B] \text{ is a } c\text{-number}.$$

This gives $f(z,z^\star) = e^{-\frac{1}{4}\left(z^2 - z^{\star 2}\right)}$ and $c = c' = \frac{1}{\sqrt{2}}$. Then

$$\langle q|D(z)|n\rangle = f(z,z^\star)\left\langle q|e^{c(z-z^\dagger)\mathbf{Q}}e^{\mathbf{i}c'(z+z^\star)\mathbf{P}}|n\right\rangle \tag{9}$$

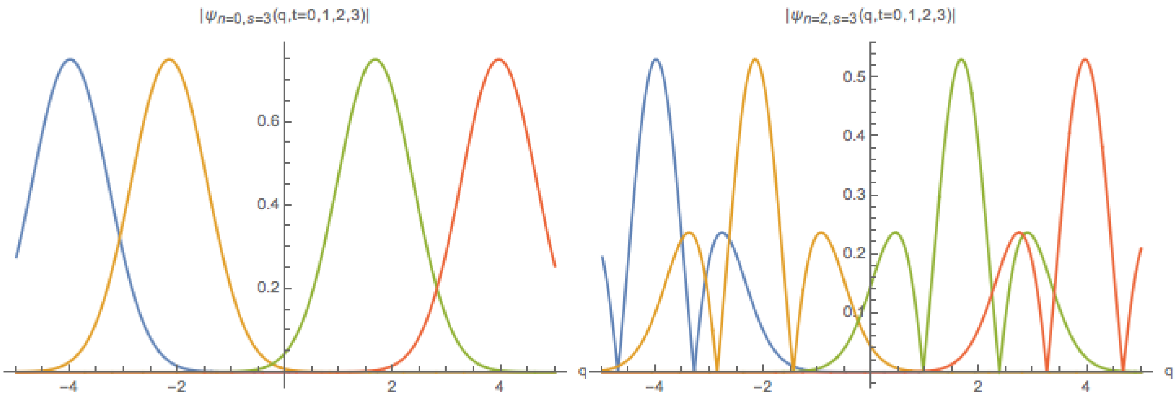
$$= f(z,z^\star)e^{c(z-z^\dagger)q}\left\langle q|e^{\mathbf{i}c'(z+z^\star)\mathbf{P}}|n\right\rangle \tag{10}$$

$$= f(z,z^\star)e^{c(z-z^\dagger)q}\langle q + c'(z+z^\star)|n\rangle \tag{11}$$

$$= f(z,z^\star)e^{c(z-z^\dagger)q}\psi_n(q + c'(z+z^\star)) \tag{12}$$

where $\psi_n(q) \equiv \langle q|n\rangle = \frac{1}{\sqrt{2^n n!}}\pi^{-1/4} H_n e^{-|q|^2}$ is just the wavefunction for the nth excited oscillator state.

So the wavefunction keeps its shape and sloshes back and forth. It looks like this for $n = 0$ (left) and $n = 2$ (right) at various t (smaller than the period, which I've set to 2π):



2. **Coherent states.**

Consider a quantum harmonic oscillator with frequency ω. The creation and annihilation operators $\mathbf{a}^\dagger$ and $\mathbf{a}$ satisfy the algebra

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1$$

and the vacuum state $|0\rangle$ satisfies $\mathbf{a}|0\rangle = 0$. Coherent states are eigenstates of the annihilation operator:

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle .$$

(a) Show that

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \mathbf{a}^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

is an eigenstate of $\mathbf{a}$ with eigenvalue α. ($\mathbf{a}$ is not hermitian, so its eigenvalues need not be real.)

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \to \hat{a}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{a}|n\rangle = e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle$

Where we've used the fact $\hat{a}$ annihilates the vacuum. Reshuffling the summand:

$\hat{a}|n\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^{n+1}}{\sqrt{n!}}|n\rangle = \alpha|\alpha\rangle$

(b) Coherent states with different α are not orthogonal. ($\mathbf{a}$ is not hermitian, so its eigenstates need not be orthogonal.) Show that $|\langle\alpha_1|\alpha_2\rangle|^2 = e^{-|\alpha_1-\alpha_2|^2}$.

$\langle\alpha_1|\alpha_2\rangle = e^{-|\alpha_1|^2/2}e^{-|\alpha_2|^2/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha_1^{*n}}{\sqrt{n!}}\frac{\alpha_2^m}{\sqrt{m!}}\langle n|m\rangle = e^{-|\alpha_1|^2/2}e^{-|\alpha_2|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_1^{*n}\alpha_2^n}{n!}$

Where in the last step we have used the orthogonality of $\{|n\rangle\}$. We recognize this sum as an exponential:
$\langle\alpha_1|\alpha_2\rangle = e^{-|\alpha_1|^2/2}e^{-|\alpha_2|^2/2}e^{\alpha_1^*\alpha_2} \to |\langle\alpha_1|\alpha_2\rangle|^2 = e^{-|\alpha_1-\alpha_2|^2}$

(c) Compute the expectation value of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$ in the coherent state $|\alpha\rangle$.
$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2\langle\alpha|\alpha\rangle = |\alpha|^2$

(d) Time evolution acts nicely on coherent states. The hamiltonian is $\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$. Show that a coherent state evolves into a coherent state with an eigenvalue $\alpha(t)$:

$$e^{-\mathbf{i}\mathbf{H}t}|\alpha\rangle = e^{-\mathbf{i}\omega t/2}|\alpha(t)\rangle$$

where $\alpha(t) = e^{-\mathbf{i}\omega t}\alpha$.
$|\alpha(t)\rangle = e^{-\mathbf{i}\hat{H}t}|\alpha_0\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}e^{-\mathbf{i}\omega(n+\frac{1}{2})t}|n\rangle$
We pull out the ground state contribution: $= e^{-\mathbf{i}\frac{\omega t}{2}}e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0 e^{-\mathbf{i}\omega t})^n}{\sqrt{n!}}|n\rangle$
Which by looking at the definition $|\alpha(t)\rangle = \left|e^{-\mathbf{i}\omega t}\alpha_0\right\rangle$ we have shown the result.

(e) Show that the coherent states can be used to resolve the identity in the form

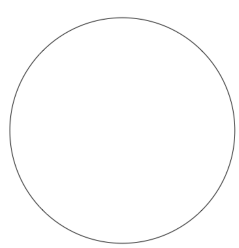
$$\mathbb{1} = \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|$$

where $d^2\alpha \equiv d\alpha_1 d\alpha_2$ in terms of the real and imaginary parts of $\alpha = \alpha_1 + \mathbf{i}\alpha_2$. One way to do this is to relate this expression to $\mathbb{1} = \sum_{n=0}^{\infty}|n\rangle\langle n|$.

The following three problems form a triptych, on the subject of resolving the various infinities involved in the quantum mechanics of a particle on the real line. There are two such infinities: one is the fact that the real line goes on forever; this is resolved in problem 3. The other is the fact that in between any two points there are infinitely many points; this is resolved in problem 4. In problem 5 we resolve both to get a finite-dimensional Hilbert space.

3. **Particle on a circle.**

Consider a particle which lives on a circle:



That is, its coordinate x takes values in $[0, 2\pi R]$ and we identify $x \simeq x + 2\pi R$.

(a) Let's assume that the wavefunction of the particle is periodic in x:

$$\psi(x + 2\pi R) = \psi(x) \ .$$

What set of values can its momentum (that is, eigenvalues of the operator $\mathbf{p} = -i\hbar\partial_x$) take?

$\langle x + 2\pi R|\psi\rangle = \langle x|\psi\rangle$

$\int \frac{dp}{2\pi} \langle x + 2\pi R|p\rangle \langle p|\psi\rangle = \int \frac{dp}{2\pi} \langle x|p\rangle \langle p|\psi\rangle$

$\int \frac{dp}{2\pi} e^{\mathbf{i}(x+2\pi R)p} \langle p|\psi\rangle = \int \frac{dp}{2\pi} e^{\mathbf{i}xp} \langle p|\psi\rangle$

For this to be true $e^{2\pi \mathbf{i} Rp} = 1$ thus quantizing $p = \frac{n}{R}$ for $n \in \mathbb{Z}$

To emphasize: $x \in S^1 \implies p \in \mathbb{Z}$

(b) Recall that the overall phase of the state vector is not physical data. This suggests the possibility that the wavefunction might not be periodic, but instead might acquire a phase when we go around the circle:

$$\psi(x + 2\pi R) = e^{i\varphi}\psi(x)$$

for some fixed φ. In this case what values does the momentum take?

The same logic of the above holds only now $e^{2\pi \mathbf{i} Rp} = e^{\mathbf{i}\phi}$ implying $p = \frac{n}{R} + \frac{\phi}{2\pi R}$

4. **Particle on a lattice.**

Now consider a particle which lives on a lattice: its position can take only the discrete values $x = na, n \in \mathbb{Z}$ where a is some unit of length and n is an integer. We'll call the corresponding position eigenstates $|n\rangle$. The Hilbert space is still infinite-dimensional, but at least we have in our hands a countably infinite basis.

In this problem we will determine: what is the spectrum of the momentum operator $\mathbf{p}$ in this system?

(a) Consider the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} e^{in\theta} |n\rangle \ .$$

Show that $|\theta\rangle$ is an eigenstate of the *translation operator* $\hat{T}$, defined by

$$\hat{T} = \sum_{n \in \mathbb{Z}} |n+1\rangle \langle n| \ .$$

Why do I want to call θ momentum?

$T|\theta\rangle = \sum_{n \in \mathbb{Z}} e^{in\theta} |n+1\rangle = e^{-\mathbf{i}\theta} |\theta\rangle$. The values of n shift along $\mathbb{Z}$.

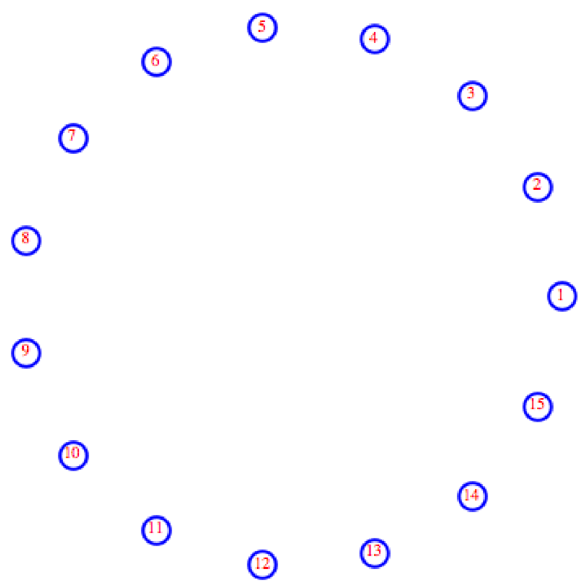
Recall that $T = e^{-\mathbf{i}\hat{p}a}$ so $e^{-\mathbf{i}pa} |\theta\rangle = e^{-\mathbf{i}\theta} |\theta\rangle$ implying $\theta = pa$.

(b) What range of values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

Since n is an integer $|\theta\rangle = |\theta + 2\pi\rangle$. We've found that for $x \in \mathbb{Z} \implies p \in S^1$ (this circle is called the *Brillioun zone*)!

5. **Discrete Laplacian.**

Consider again a particle which lives on a lattice, but now we'll wrap the lattice around a circle, in the following sense. Its position can take only the discrete values $x = a, 2a, 3a, ..., Na$ (where, again, a is some unit of length and again we'll call the corresponding position eigenstates $|n\rangle$). Suppose further that the particle lives on a circle, so that the site labelled $x = (N+1)a$ is the same as the site labelled $x = a$. We can visualize this as in the figure:



In this case, the Hilbert space has finite dimension N.

Consider the following $N \times N$ matrix representation of a Hamiltonian operator

(a is a constant):

$$H = \frac{1}{a^2}\left(\begin{array}{c}\overbrace{\underbrace{\begin{matrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{matrix}}_{N}}\end{array}\right\} N \Bigg)$$

(a) Convince yourself that this is equivalent to the following: Acting on an N-dimensional Hilbert space with orthonormal basis $\{|n\rangle, n = 1, \ldots, N\}$, $\hat{H}$ acts by

$$a^2 \hat{H} |n\rangle = 2|n\rangle - |n+1\rangle - |n-1\rangle \,, \qquad \text{with } |N+1\rangle \simeq |1\rangle$$

that is, we consider the arguments of the ket to be integers modulo N.

I will set $a = 1$ until needed. Recall that $H_{nm} \equiv \langle n| H |m\rangle$. Our claim above is compatible with $H_{nn} = 2$ by orthogonality as well as off diagonals $H_{n+1,n} = H_{n-1,n} = -1$.

The top right and left corners are compatible by: $H_{0,N} = 2\langle 0|N\rangle - \langle 0|N+1\rangle - \langle 0|N-1\rangle = -\langle 0|N+1=0\rangle = -1$ making use of the periodicity. The rest are appropriately 0.

(b) Show that $\hat{H}$ and $\hat{T}$ (where $\hat{T}$ is the 'shift operator' defined by $\hat{T} : |n\rangle \mapsto |n+1\rangle$) can be simultaneously diagonalized.

$HT|n\rangle = H|n+1\rangle = 2|n+1\rangle - |n+2\rangle - |n\rangle = T(2|n\rangle - |n+1\rangle - |n-1\rangle) = TH|n\rangle$ so there is a discrete translation invariance.

Consider again the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle \,.$$

(c) Show that $|\theta\rangle$ is an eigenstate of $\hat{T}$, for values of θ that are consistent with the periodicity $n \simeq n + N$.

See solutions to homework 1.

(d) What values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

Once again see homework 1. Specifically $\theta = \frac{2\pi k}{N}$ for $k \in \{0, 1, \cdots, N-1\}$

Recalling the relationship between p and θ we arrive at the punchline that for $x \in \mathbb{Z}_N \implies p \in \mathbb{Z}_N$

(e) Find the matrix elements of the unitary operator $\mathbf{U}$ which relates position eigenstates $|n\rangle$ to momentum eigenstates $|\theta\rangle$: $U_{\theta n} \equiv \langle n|\theta\rangle$.

$\langle n|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n'} e^{\mathbf{i}n'\theta} \langle n|n'\rangle = \frac{1}{\sqrt{N}} e^{\mathbf{i}n\theta}$ by orthogonality.

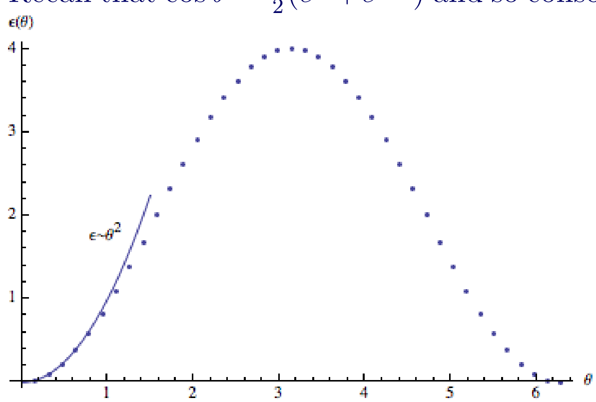
(f) Find the spectrum of $\hat{H}$.

Draw a picture of $\epsilon(\theta)$: plot the energy eigenvalues versus the 'momentum' θ.

Because $[H, T] = 0$ we can diagonalize them in the same basis.

$H|\theta\rangle = \frac{1}{\sqrt{N}} \sum_n e^{\mathbf{i}n\theta}(2|n\rangle - |n+1\rangle - |n-1\rangle) = 2|\theta\rangle - e^{-\mathbf{i}\theta}|\theta\rangle - e^{\mathbf{i}\theta}|\theta\rangle$

Recall that $\cos\theta = \frac{1}{2}(e^{\mathbf{i}\theta} + e^{-\mathbf{i}\theta})$ and so consolidating terms: $\epsilon(\theta) = 2 + 2\cos\theta$



(g) Show that the matrix above is an approximation to (minus) the 1-dimensional Laplacian $-\partial_x^2$. That is, show (using Taylor's theorem) that

$$a^2 \partial_x^2 f(x) = -2f(x) + (f(x+a) + f(x-a)) + \mathcal{O}(a)$$

(where "$\mathcal{O}(a)$" denotes terms proportional to the small quantity a).

Taylor Expansion: $f(x+a) = f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \cdots$ so we can write that $f'(x) = \frac{f(x+a)-f(x-a)}{2a}$

$f(x+a) = f(x) + \frac{1}{2}(f(x+a) - f(x-a)) + \frac{1}{2}a^2 f''(x)$

$a^2 f''(x) = f(x+a) + f(x-a) - 2f(x)$

Thus our form of H acting on $|n\rangle$ approximates a finite step differentiation.

(h) In the expression for the Hamiltonian, to restore units, I should have written:

$$\hat{H}|n\rangle = \frac{\hbar^2}{2m}\frac{1}{a^2}(2|n\rangle - |n+1\rangle - |n-1\rangle), \quad \text{with } |N+1\rangle \simeq |1\rangle$$

where a is the distance between the sites, and m is the mass. Consider the limit where $a \to 0, N \to \infty$ and look at the lowest-energy states (near $p = 0$); show that we get the spectrum of a free particle on the line, $\epsilon = \frac{p^2}{2m}$.

Based on the above we have that in the continuum limit $H = -\partial_x^2$ and thus with the inclusion of the appropriate factors becomes the kinetic energy operator $\frac{\hat{p}^2}{2m}$ in position space.

Since H has no potential terms this is the only contribution to the total energy.

The highest momentum states would be the one's associated with largest values of θ and to whom the details of our regularization would matter most.