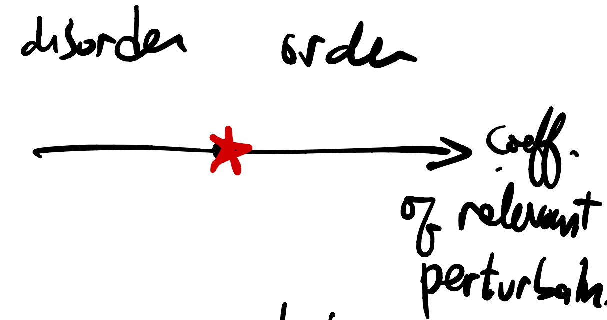
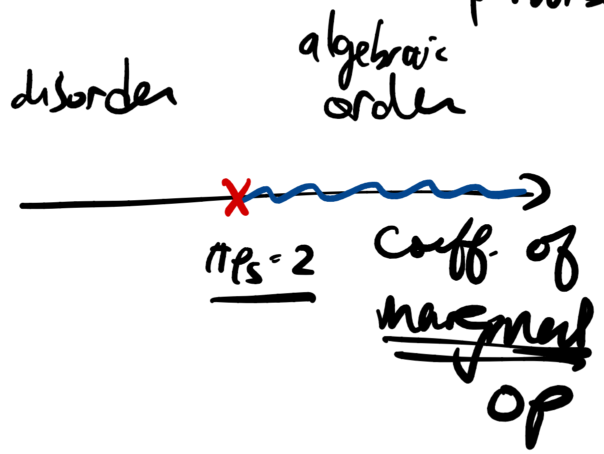
# Last Time:

usual phase trans




KT transition :




$$\mathcal{L} = \frac{1}{2(2\pi)^2 \rho_s} (\partial \Theta)^2 - \lambda \cos \Theta$$

$\Theta = \phi_L - \phi_R$ Q: when is $\lambda$ relevant?

$$\langle e^{i\Theta(x)} e^{-i\Theta(0)} \rangle = \frac{c}{x^{2\pi \rho_s}}$$

$$\Delta(\cos\theta) = \pi \rho_s$$

A: when $\pi \rho_s < 2$.

$$\Rightarrow \text{eg} \quad \xi \sim e^{-\frac{\#}{\sqrt{\rho_s - \rho_s^c}}}$$

# Bosonization : $D=1+1$ compact scalar $\longleftrightarrow$ $D=1+1$ Dirac fermion.

## part I : counting

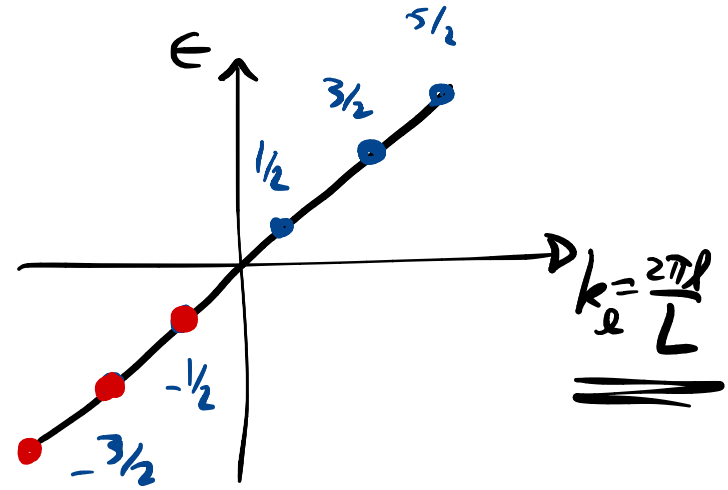
$$\Psi = (L, R)$$

on a circle
$x \cong x + L$

w/ APBCs :

$$\begin{cases} L(x+L) = -L(x) \\ R(x+L) = -R(x) \end{cases}$$

gs :




$$R(x) = \frac{1}{\sqrt{L}} \sum_{\ell \in \mathbb{Z}+\frac{1}{2}} R_\ell \, e^{\frac{2\pi i \ell x}{L}}$$

$$\{ R_\ell, R^\dagger_{\ell'} \} = \delta_{\ell, \ell'}$$

vac. energy

$$H_R = \frac{2\pi v_F}{L} \sum_{\ell \in \mathbb{Z}+\frac{1}{2}} \ell \underbrace{R^\dagger_\ell R_\ell}_{= n^R_\ell} - E_0$$

$H_R |gs) = 0$. symmetric under $\ell \leftrightarrow -\ell$

$$Z_R(T) \equiv \mathrm{tr}_{\mathcal{H}_R}\, e^{-H_R/T} \qquad q \equiv e^{-\frac{2\pi v_F}{LT}}$$

$$= \prod_{\ell\in\mathbb{Z}+\frac{1}{2}} \left( \sum_{n_\ell^R=0,1} q^{|\ell|\, n_\ell^R} \right)$$

$$= \prod_{\ell\in\mathbb{Z}+\frac{1}{2}} \left( 1 + q^{|\ell|} \right)$$

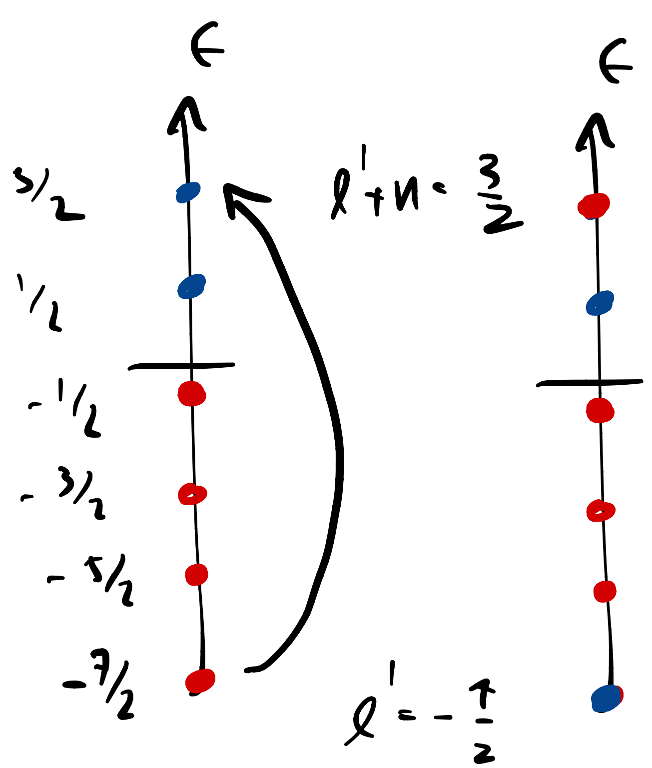
$$= \prod_{n=1}^{\infty} \left( 1 + q^{n-1/2} \right)^2$$

elliptic theta function.

$$\Delta\ell = \ell' + n - \ell' = n \in \mathbb{Z}$$

$$\Delta\epsilon = v_F \Delta k = \frac{2\pi v_F\, n}{L}$$

<u>indep. of $\ell'$.</u>




Particle + hole both move at velocity $+v_F$.

created by

$$\rho_n^\dagger \equiv \sum_{\ell'} R^\dagger_{\ell'+n} R_{\ell'}$$

bosonic

raises the momentum by $n$ units. & energy by $\frac{2\pi n}{L} v_F$.

$$\rho(x) = R^\dagger(x) R(x)$$
$$= n_R(x).$$

---

$$|F\rangle = \prod_{\substack{\ell>0 \\ \in F}} R^\dagger_\ell \prod_{\substack{\ell'<0 \\ \in F}} R_{\ell'} |gs\rangle$$

If this has charge $Q_R = \sum_\ell :R^\dagger_\ell R_\ell: \in \mathbb{Z}$

$$|F\rangle = \begin{pmatrix}\text{particle-} \\ \text{hole} \\ \text{exc.}\end{pmatrix} |Q_R\rangle \begin{pmatrix} Q_R |gs\rangle = 0 \\ [Q_R, H] = 0. \end{pmatrix}$$

w/ $|Q_R\rangle = R^\dagger_{Q_R - \frac{1}{2}} \cdots R^\dagger_{\frac{3}{2}} R^\dagger_{\frac{1}{2}} |gs\rangle$ lowest E state w/ charge $Q_R$

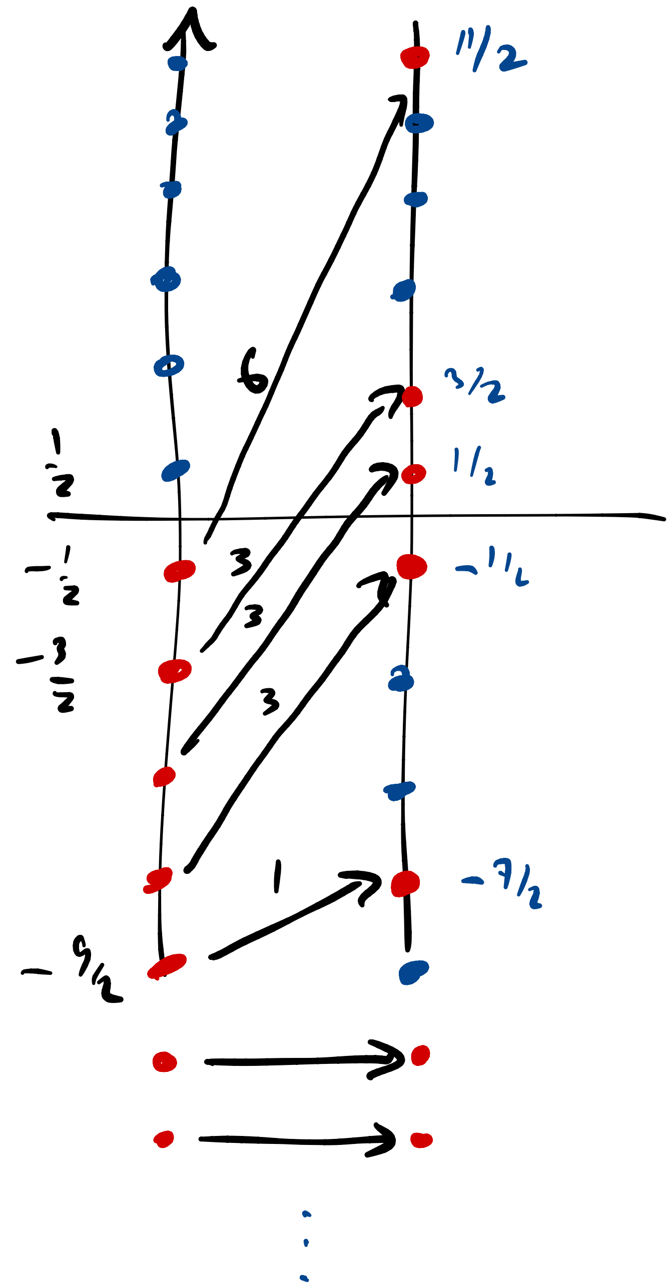
(for $Q_R > 0$)

$$H_R|Q_R\rangle = E_0(Q_R)|Q_R\rangle$$

$$E_0(Q_R) = \frac{2\pi v_F}{L}\sum_{l=1/2}^{|Q_R|-1/2} l = \frac{\pi v_F}{L} Q_R^2$$

focus on $|gs\rangle = |Q_R=0\rangle$.

$|63331; Q_R=0\rangle =$




$$(R^\dagger_{-7/2}R_{-9/2})(R^\dagger_{-1/2}R_{-7/2})$$
$\quad\quad\quad\quad\quad 1 \quad\quad\quad 3$

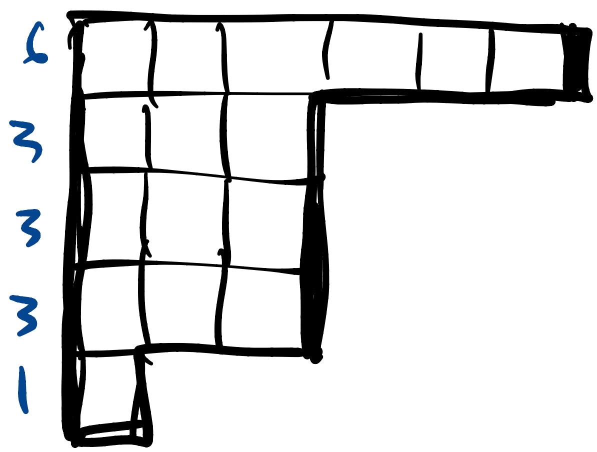
$$(R^\dagger_{1/2}R_{-5/2})(R^\dagger_{3/2}R_{-3/2})(R^\dagger_{11/2}R_{-1/2})|gs\rangle$$
$\quad 3 \quad\quad\quad 3 \quad\quad\quad 6$

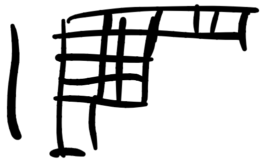
$|r_1 r_2 \ldots ; Q_R\rangle$

$r_1 \geq r_2 \geq r_3 \ldots \geq r_{n_*}$

$$= \rho^\dagger_{r_{n^*}} \cdots \rho^\dagger_{r_2}\rho^\dagger_{r_1}|Q_R\rangle.$$

$\{r_n\} = \{6\ 3\ 3\ 3\ 1\}$



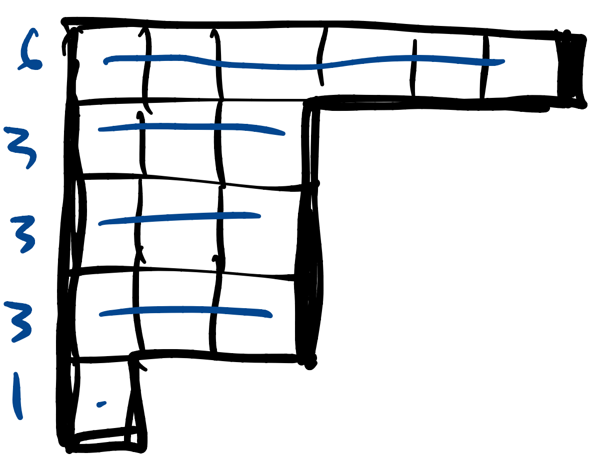

$|$  $, Q_R = 0\rangle$

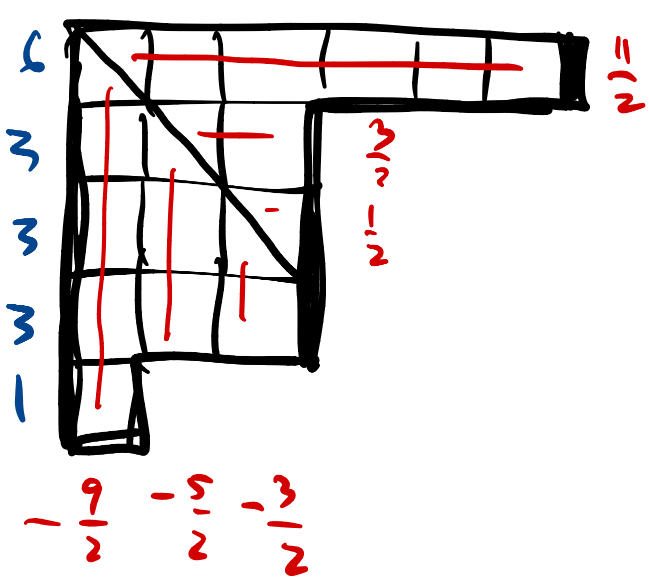
$= \rho_1^\dagger \underbrace{\rho_3^\dagger \rho_3^\dagger}_{\text{Bosonic}} \rho_3^\dagger \rho_6^\dagger |Q_R=0\rangle.$

Bosonic $(\rho_3^\dagger)^2 \neq 0.$

$= (R^\dagger_{-7/2} R_{-9/2})(R^\dagger_{-1/2} R_{-7/2})(R^\dagger_{1/2} R_{-5/2})(R^\dagger_{3/2} R_{-3/2})(R^\dagger_{11/2} R_{-1/2})|gs\rangle$

$= (R^\dagger_{11/2}\ R^\dagger_{3/2}\ R^\dagger_{1/2})(R_{-3/2} R_{-5/2} R_{-9/2})|gs\rangle$



$$H^R = \frac{\pi v_F}{L} Q_R^2 + \frac{2\pi v_F}{L} \sum_{n=l+\frac{1}{2}=\frac{1}{2}}^{\infty} n \, \rho_n^+ \rho_n$$

chiral boson $n > 0$.

$$Z^R_{(\text{bosonic})}(T) = \mathrm{tr}_{\mathcal{H}_B} \, e^{-H^B/T}$$

$$= \sum_{Q_R=-\infty}^{\infty}{}' \, q^{Q_R^2/2} \prod_{n=1}^{\infty} \left( \sum_{\substack{m_n = \rho^+_n \rho_n \\ = 0,1,2 \ldots \infty}}{}' q^{n m_n} \right)$$

$$= \frac{\sum_{Q_R} q^{Q_R^2/2}}{\prod_{n=1}^{\infty} (1-q^n)}$$

[same elliptic $\theta$-f'n.]

Part II: dictionary

$j_L = L^+ L = \frac{1}{2\pi} \partial_z \phi$ $\quad (z = x + i\tau)$

$j_R = R^+ R = \frac{1}{2\pi} \partial_{\bar{z}} \phi$

If $T = \frac{1}{\pi}$ $\left( \mathcal{L}_\phi = \frac{T}{2} (\partial \phi)^2 \right)$

CLAIM:

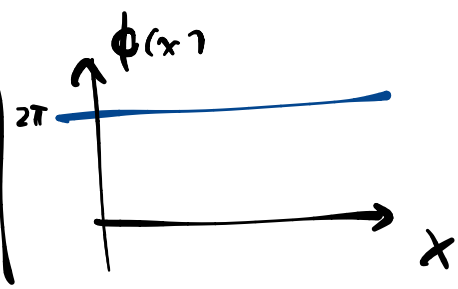
$$\begin{cases} :e^{i\phi_L(z)}: = L(z)^\dagger \\ :e^{i\phi_R(\bar{z})}: = R(\bar{z})^\dagger \end{cases}$$

total fermion # density $= L^\dagger L + R^\dagger R = \frac{1}{2\pi} \partial_x \phi$

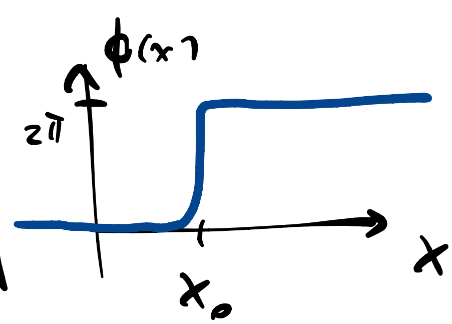
$= $ winding # density.

Refine: $\underline{L^\dagger L = \frac{1}{2\pi} \partial_x \phi_L} = j_L$

$\phi = \phi_L + \phi_R$.

$|$no fermions$\rangle = \Big|$  $\Big\rangle$



$L^\dagger(x_0) |$no fermions$\rangle = \Big|$  $\Big\rangle$

$$e^{-i\hat{p}a}\,\psi(x) = \psi(x+a)$$

$$\Rightarrow L(x_0) \sim e^{2\pi i \int_{-\infty}^{x_0} dx\, \pi_L(x)}$$

$$\hat{\pi} = T\dot{\phi}\,, \quad \dot{\phi}_L = \partial_x \phi_L\,. \Longrightarrow$$

$$[\partial_x \phi_L(x), \phi_L(0)] = \frac{2i}{T}\delta(x) \quad *$$

$$\Longrightarrow \begin{cases} \pi_L(x) = \frac{T}{2}\partial_x \phi_L(x) \\ \pi_R(x) = -\frac{T}{2}\partial_x \phi_R(x) \end{cases}$$

when $T = \frac{1}{\pi}$

$$L(x_0) = e^{i\int_{-\infty}^{x_0} dx\, \partial_x \phi_L}$$

$$\overset{\text{FTC}}{=} e^{i\phi_L(x_0)}$$

$$\{L(x), L(x')\} = 0 \text{ for } x \neq x'.$$

$$\int(*) \overset{T=1/\pi}{\Longrightarrow} [\phi_L(x), \phi_L(y)] = i\pi\, \mathrm{sign}(x-y).$$

$$e^{A} e^{B} = e^{B} e^{A} e^{-[A,B]} \quad \text{if } [A,B] \text{ is a c-\#}$$

$$\Rightarrow :e^{i\phi_L(x)}: \; :e^{i\phi_L(0)}:$$

$$= \underbrace{e^{-i\pi\,\mathrm{sign}(x)}}_{=-1} :e^{i\phi_L(0)}: \; :e^{i\phi_L(x)}: \qquad \forall x \; \checkmark$$

$$:e^{in\phi_L(x)}: \; :e^{im\phi_L(0)}:$$

$$= (-1)^{nm} :e^{im\phi_L(0)}: \; :e^{in\phi_L(x)}:$$

$$e^{i\int_{-\infty}^{x_0} dx\, j_L(x)} = \text{JW string}.$$

$$= \overline{\#\text{ of fermions to the left}}.$$

Multiple fermions: $N$ complex fermions $\psi^{a=1..N}$

$$\psi^a(z) \sim C_a e^{i\phi^a(z)} \qquad a=1..N$$

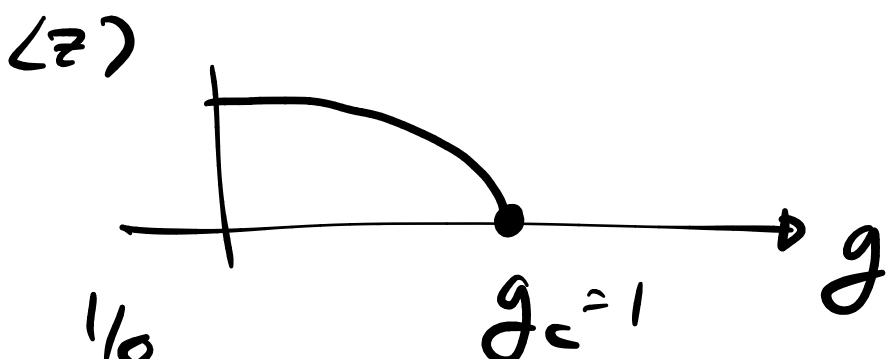
$$C_a \equiv (-1)^{\sum_{b<a} N_b}$$ 'Klein factor'.

## Spin field in (two copies of) the Ising Model

Critical Ising Model : one Majorana fermion (real)

$2\times($ " $)$ : one Dirac fermion

to show:




$$\langle Z \rangle \sim (g_c - g)^{1/8}$$

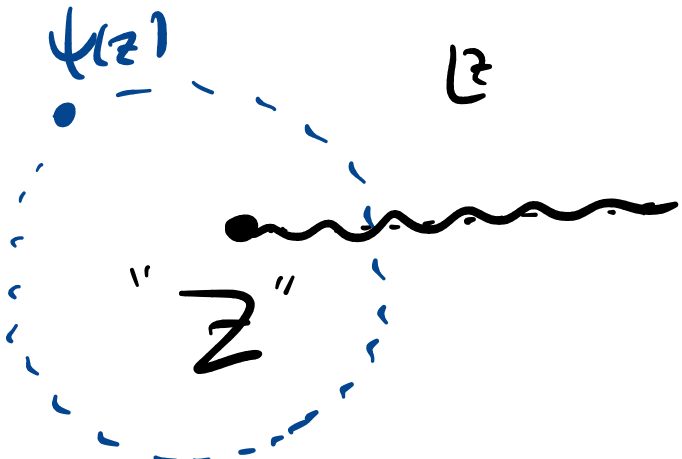
$$\psi(z) = c(z) = \frac{1}{\sqrt{2}}\left(\chi_1(z) + i\,\chi_2(z)\right)$$

$$\sim e^{i\phi(z)}$$

<u>Recall</u>: fermions are DW operators

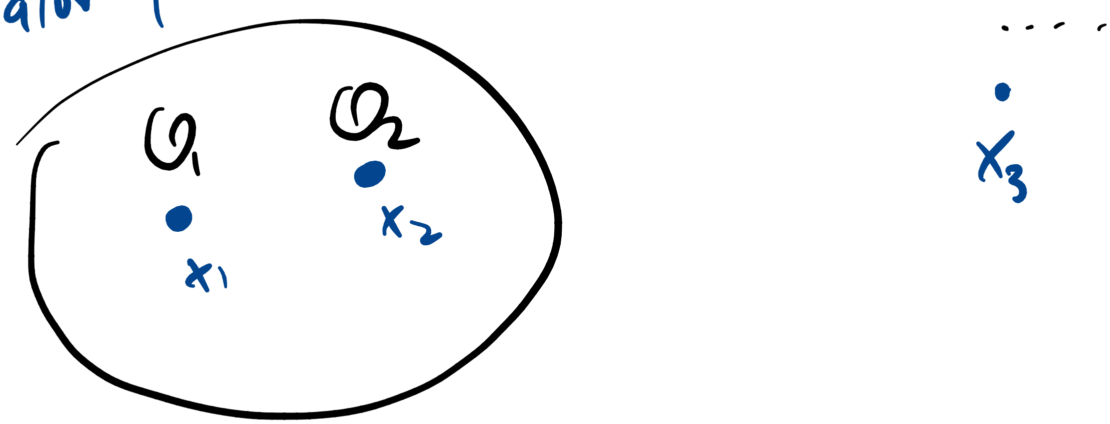
$$Z_\ell\, c_j = -\,c_j\, Z_\ell \quad \text{if } \ell > j.$$

$Z_\ell$ creates a <u>branch cut</u> for $c_j$.




$$\psi(z)\,\sigma(0) \sim z^{-1/2}\mu(0)$$

Operator Product Expansion (OPE).



$$\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2) = \sum_{a} \mathcal{O}_a(x_1)\, C^a_{12}(x_1 - x_2)$$

$\in$ all possible local ops.

conformal / ~~scale~~ inv: organize by dimension $\Delta_a$

$$= \sum_a \mathcal{O}_a(x_1) \frac{c^a_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2 - \Delta_a}}$$

as $\Delta_a$ grows $\sim (x_1 - x_2)^{+\Delta_a}$

is less singular as $\underline{x_1 \to x_2}$.

---

claim: $\sigma(w) = e^{-\frac{i}{2}\phi(w)}$ $\qquad \mu(w) = e^{+\frac{i}{2}\phi(w)}$

$$\psi(z)\,\sigma(0) = :e^{i\phi(z)}:\;:e^{-\frac{i}{2}\phi(0)}:$$

$$= :e^{\frac{i\phi}{2}(0)}:\, z^{-1/2} + \text{regular}$$

$\langle \phi(z)\phi(0)\rangle \sim \# \log z$

$$\psi(z)\,\mu(0) = :e^{\frac{3}{2}i\phi(0)}:\, z^{1/2} + \cdots$$

What is $\Delta(\sigma)$ ?

$$e^{\alpha i\phi(z)}\, e^{-i\alpha\phi(0)} \sim \frac{1}{z^{\alpha^2}}$$

For $\alpha = \pm\frac{1}{2} \implies \Delta_\sigma = \Delta_\mu = \left.\frac{\alpha^2}{2}\right|_{\alpha=\pm\frac{1}{2}} = \frac{1}{8}.$

Claim: $\Delta_\sigma$ is additive in the # of fermions for which $\sigma$ produces a branch cut.

Illustration:

$$\psi^a(z) \sim c_a e^{i\phi^a(z)} \qquad a = 1\ldots N$$

$$\sigma_s(z) \equiv e^{is^a\phi^a} \qquad \left\{s^a = \pm\tfrac{1}{2}\right\}$$

$$\psi^a(z)\,\sigma_s(0) \sim z^{s^a}\,\sigma_{s'}(0) + \dots \qquad \underline{\forall a}.$$

w/ $\begin{cases} (s')^a = s^a + 1 \\ (s')^{b\neq a} = s^b \end{cases}$

$$\Delta(\sigma_s) = \frac{1}{2}\left(s_1^2 + \dots s_N^2\right) = \frac{N}{8} \quad \blacksquare$$

Take $N = \frac{1}{2}$. $\Longrightarrow \Delta_L(z) = \frac{1}{16}$

$$\langle \sigma(z,\bar{z})\;\sigma(0,0)\rangle = \frac{\#}{z^{\Delta_L}\,\bar{z}^{\Delta_R}}$$

$$\Delta_z = \Delta_L + \Delta_R = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}.$$

'Derivation' of spin field:

$$\sigma_j^+ = (-1)^{\sum_{i<j} n_i} c_j^+$$

$$= Z_j + iY_j \qquad \sigma^+ + \sigma^- = 2Z.$$

Continuum

$$\underline{\underline{\sigma^+(x) = e^{i\pi \int_{-\infty}^{x} dx' j_0(x')} \psi^+(x)}}$$

low E

$$\Psi(x,\tau) = e^{ik_F x} R(x,\tau) + e^{-ik_F x} L(x,\tau)$$

$$e^{-i\pi \int_{-\infty}^{x} dx'\, j_0(x')} R^+$$

$$\boxed{j^0 = \frac{1}{2\pi}\partial_x \phi}$$

$$= :e^{i\pi \int_{-\infty}^{x} dy \frac{(\partial_y \phi_L + \partial_y \phi_R)}{2\pi}} :: e^{-i\phi_R(x)}:$$

$$\overset{\text{FTC}}{=} :e^{i\frac{(\phi_L(x) + \phi_R(x))}{2}} :: e^{-i\phi_R(x)}:$$

$$= :e^{i\frac{\phi_L(x) - \phi_R(x)}{2}}: + \text{regular}$$

$$\sigma^+(x) \sim e^{i\frac{\phi_L(x)-\phi_R(x)}{2}} e^{-ik_F x}$$

$$+ e^{-i\frac{\phi_L(x)-\phi_R(x)}{2}} e^{ik_F x}.$$

has $\Delta = \frac{1}{8}$.

---

Bosonization & the anomaly:

$$\partial_z \phi \propto \sum_n \rho_n z^{-n-1} \qquad \underline{z = e^{i(x+i\tau)}}$$

$$\xrightarrow{\text{ETCR}} [\rho_n, \rho_{-n'}] = \delta_{n,n'} n \qquad \rho_n^\dagger = \rho_{-n}.$$

$$\rho_n = \sum_\ell L_\ell^\dagger L_{\ell+n}$$

$$[\rho_n, \rho_{-n'}] = \sum_{\ell,\ell'} [L_\ell^\dagger L_{\ell+n}, L_{\ell'}^\dagger L_{\ell'-n'}]$$

$$= \sum_{\ell'} \left( L_{\ell'-n}^\dagger L_{\ell'-n'} - L_{\ell'}^\dagger L_{\ell'+n-n'} \right) \overset{?}{=} 0$$

$\ell' \to \ell' - n.$

# Sine Gordon $\longleftrightarrow$ Thirring

$$\mathcal{L}_\phi = \frac{T}{2}(\partial_\mu\phi)^2 + g\cos\phi \longleftrightarrow$$

$$\mathcal{L}_\psi = \bar\psi\not\partial\psi + \#\,\bar\psi\psi\bar\psi\psi$$

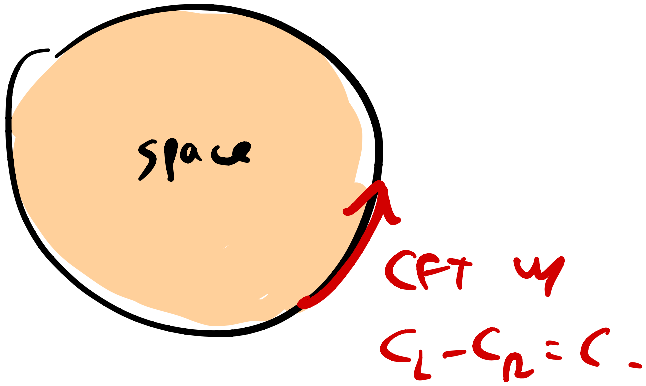
$\# \propto (T - \frac{1}{\pi}\cdot)$

$$+\, g\bar\psi\psi$$

[Luscher

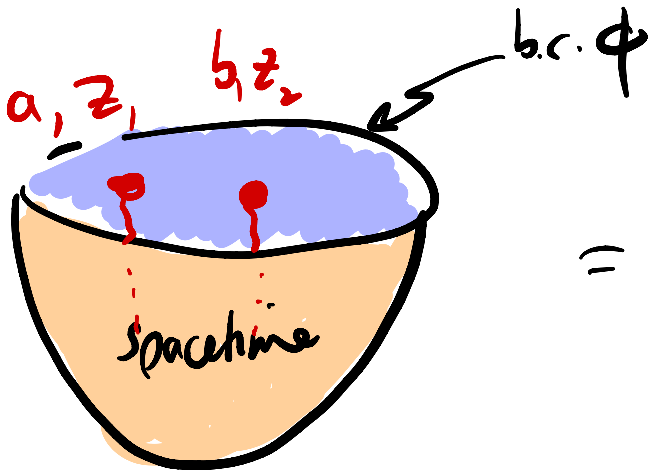
Coleman 70s]

chiral TO. $c_- > 0$.




anyon types in bulk $\longleftrightarrow$ conformal primaries of CFT

topological spin $\longleftrightarrow$ scaling dim $\Delta_a$




$$= \Psi_{gs}[\phi, \overset{a}{z_1}, \overset{b}{z_2} \dots]$$

$$= \langle O_a(z_1) O_b(z_2) \dots \rangle$$