# 5.1 TFIM, continued

Jordan-Wigner transf.:

$$\begin{cases} \chi_j \equiv Z_j \prod_{j'>j} X_{j'} \\ \tilde{\chi}_j \equiv Y_j \prod_{j'>j} X_{j'} \end{cases} \rightarrow \{\chi_j, \chi_{j'}\} = \delta_{jj'} = \{\tilde{\chi}_j, \tilde{\chi}_{j'}\}.$$

$$\begin{cases} X_j = -i\,\tilde{\chi}_j \chi_j \\ Z_j Z_{j+1} = i\,\tilde{\chi}_{j+1}\chi_j \end{cases} \Longrightarrow S = \prod_j X_j = \prod_j (-1)^{c_j^\dagger c_j} = (-1)^{\hat{F}}$$

counts fermions at $j$

$$H_{TFIM} = -J \sum_j \left( i\,\tilde{\chi}_{j+1}\chi_j + g\,i\,\tilde{\chi}_j \chi_j \right) \qquad \begin{cases} c_j = \frac{1}{2}(\chi_j - i\tilde{\chi}_j) \\ c_j^\dagger = \frac{1}{2}(\chi_j + i\tilde{\chi}_j) \end{cases}$$

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$$\begin{cases} X_j = 1 - 2c_j^\dagger c_j = (-1)^{c_j^\dagger c_j} \\ Z_j = -\left(\prod_{i>j} (-1)^{c_i^\dagger c_i}\right)(c_j + c_j^\dagger) \end{cases} \qquad F = \sum_j c_j^\dagger c_j .$$

$$c_k = \frac{1}{\sqrt{N}} \sum_j c_j\, e^{-ikx_j} \qquad x_j \equiv aj$$

$$H_{TFIM} = \sum_k \left[ \epsilon_1(k)\, c_k^\dagger c_k + \epsilon_2(k)\left(c_{-k}^\dagger c_k^\dagger + c_{-k} c_k\right) - g \right]$$

$\epsilon_1(k) = J(2g - \cos ka)$ $\qquad$ $\underline{\underline{\epsilon_2(k) = -J\sin ka.}}$

Bogoliubov <u>transformation</u>: let $\gamma_k \equiv u_k c_k - i v_k c^\dagger_{-k}$

demand canonical: $\{\gamma_k, \gamma^\dagger_{k'}\} = \delta_{kk'}$

$$\Longrightarrow \quad u_k = \cos(\phi_k/2) \quad v = \sin(\phi_k/2)$$

choose $\phi_k$ s.t. $H_{TFIM} \overset{!}{=} \sum_k \underline{\underline{\epsilon(k)}}\left(\gamma^\dagger_k \gamma_k - \frac{1}{2}\right)$

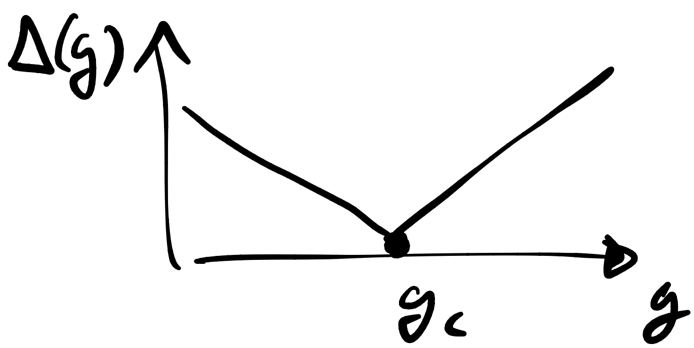
ie coeff of $\gamma_k \gamma_k$ is zero

$$\Longleftrightarrow \quad \tan\phi_k = \frac{\epsilon_2(k)}{\epsilon_1(k)}$$

$$\Rightarrow \quad \epsilon(k) = \sqrt{\epsilon_1^2(k) + \epsilon_2^2(k)}$$

$$= 2J\sqrt{1 + g^2 - 2g\cos ka}\,.$$

$$\Longrightarrow \quad \epsilon(k) \geqslant \epsilon(0) = 2J|1-g| = \Delta(g) \text{ gap}$$



one fermion exc = one domain wall

not allowed w/ PBC.

$$\underline{\mathcal{H}_{spins} = \mathcal{H}_{even\,\#}}$$

__Q:__ Where is the SSB degeneracy in terms of fermions?


$$\rightarrow \begin{cases} \kappa_{j+\frac{1}{2}} \equiv -\tilde{\chi}_{j+1} \\ \tilde{\kappa}_{j+\frac{1}{2}} \equiv \chi_j \end{cases} \qquad H_{TFIM} = -J \sum_{\bar{J} = j+\frac{1}{2}} \left( i \tilde{\kappa}_{\bar{J}} \kappa_{\bar{J}} + g\, i\, \tilde{\kappa}_{\bar{J}+1} \kappa_{\bar{J}} \right)$$

same form as $H(\chi, \tilde{\chi})$

w/ $\begin{cases} \chi \leftrightarrow \kappa \\ J \leftrightarrow Jg \\ g \leftrightarrow 1/g \end{cases}$

$\boxed{g \gg 1}$ $$H_{g\to\infty} = -Jg \sum_j i\chi_j \tilde{\chi}_j$$

$$= -Jg \sum_j (-1)^{c_j^\dagger c_j}$$

gs is $|0\rangle$ w $\underline{c_j|0\rangle = 0}$
$\forall j$.

no fermions.

$\boxed{g \ll 1}$ $$H_{g\to 0} = -J \sum_j i\tilde{\chi}_{j+1}\chi_j$$

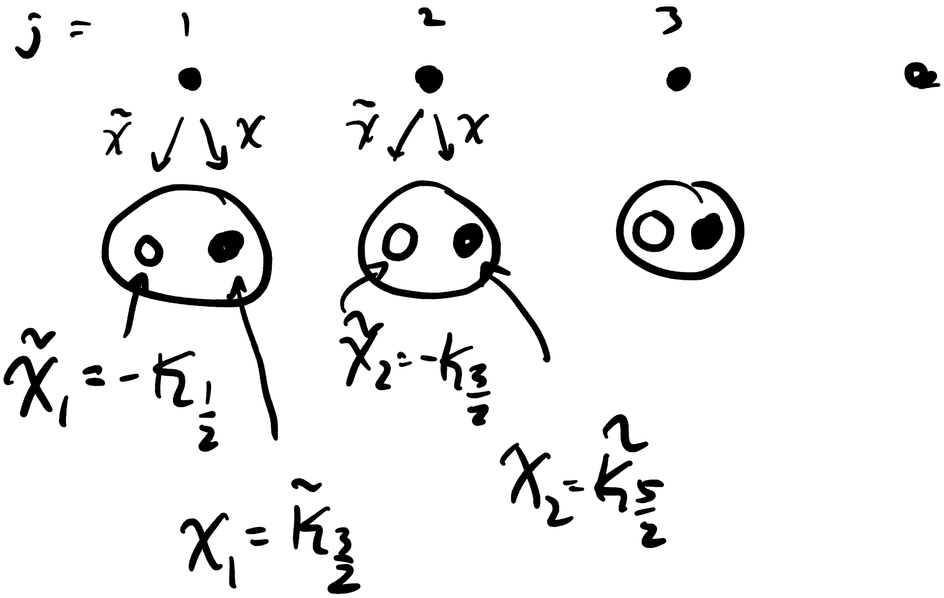
$$= -J \sum_{J=j+\frac{1}{2}} i\tilde{\kappa}_J \kappa_J$$

$$= -J \sum_J (-1)^{\check{c}_j^\dagger \check{c}_j}$$

$\left( \check{c}_j = \frac{1}{2}(\kappa_j + i\tilde{\kappa}_j) \right)$

gs is $|\check{0}\rangle$

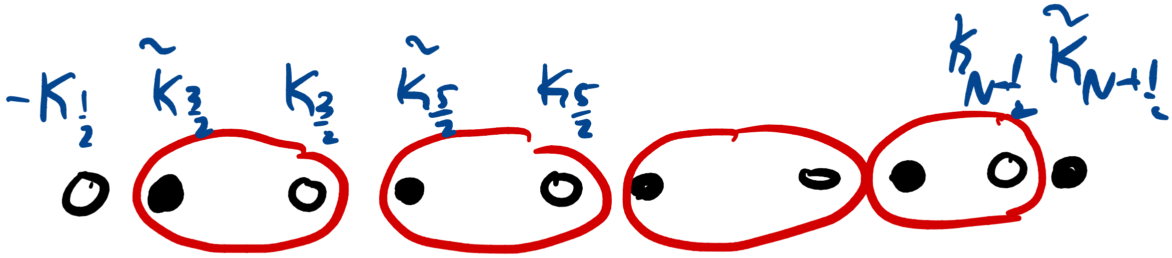
w $\check{c}_j|0\rangle = 0$ $\forall j$.




$g >> 1$

$$c_j = \frac{\chi_j + i\tilde\chi_j}{2}$$

gs is $c_j|0\rangle = 0$.




$g << 1$

$$\check c = \kappa_{\bar j} + i\tilde\kappa_{\bar j}$$

gs is $\check c_j|0\rangle = 0$.

extra fermion mode

$$a^\dagger = \tfrac12(i\tilde\chi_1 + \chi_N)$$

$$= \tfrac12(-i\kappa_{\frac12} + \tilde\kappa_{N+\frac12})$$

w/ $\{a, a^\dagger\} = 1 \rightsquigarrow a|0\rangle = 0, \quad a^\dagger|0\rangle = |1\rangle$

degeneracy.

$$\Delta H = \epsilon\, a^\dagger a = \epsilon\, i\, \tilde{\chi}_1 \chi_N$$

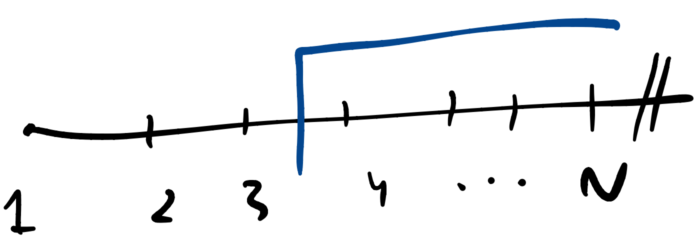
$$\epsilon \sim e^{-N/\xi}.$$

SSB degeneracy $\xleftrightarrow{\text{JW}}$ topological degeneracy from majorana edge zeromodes.

## PBC ?!

$$\tau^x_J = Z_{J-\frac{1}{2}} Z_{J+\frac{1}{2}}$$ detects DWs

$$\tau^z_J = \prod_{N \geq j > J} X_j$$ creates DWs at $J$ and at $\bar{N}$.




$$\Rightarrow \quad \tau^z_{N+\frac{1}{2}} = 1. \qquad \tau^z_{\frac{1}{2}} = \prod_j X_j = S.$$

$$1 = \tau^z_{N+\frac{1}{2}} = \begin{cases} \tau^z_{1/2} & \text{if } S = 1 \\ -\tau^z_{1/2} & \text{if } S = -1 \end{cases} = S\,\tau^z_{1/2}. \qquad [H_{TFIM}, S] = 0.$$

$S$ determines the BC.

BCs on fermions on a circle:

PBC on spins: $Z_{j+1} = Z_j$

$$\begin{cases} X_j = Z_j \prod_{N \geq i > j} X_i \\ \tilde{X} = Y_j \prod_{N \geq i > j} X_i \end{cases}$$

$$\Rightarrow Z_j = (c_j^\dagger + c_j) \prod_{N \geq i > j} (-1)^{c_i^\dagger c_i}$$

$$\Rightarrow Z_j Z_{j+1} = (c_j^\dagger + c_j)(-1)^{c_{j+1}^\dagger c_{j+1}} (c_{j+1}^\dagger + c_{j+1})$$

**but:** $$Z_N Z_1 = (c_N^\dagger + c_N)(c_1^\dagger + c_1) \underbrace{\prod_{N \geq j > 1} (-1)^{c_j^\dagger c_j}}_{= (-1)^F (-1)^{c_1^\dagger c_1}}$$

$$\Rightarrow \boxed{c_{N+1} = (-1)^F c_N}$$

$$\sum_j Z_{j+1} Z_j = \sum_j (c_j^\dagger + c_j)(c_{j+1}^\dagger + c_{j+1})(-1)^{c_{j+1}^\dagger c_{j+1}}$$

$(-1)^F = -1$ PBC $\quad k \in \frac{2\pi}{Na}\{1 \ldots N\}$

$(-1)^F = 1$ APBC $\quad k \in \frac{2\pi}{Na}\left(\frac{1}{2} + \{1 .. N\}\right).$

$$H = J \sum_k \left( c^\dagger_k c_k \, \epsilon_1(k) + \epsilon_2 \, c_k c_{-k} + h.c. \right)$$

$$= \sum_k h_k .$$

$\epsilon(k) = \epsilon(-k)$. if $k \neq 0$ or $\frac{\pi}{a}$

then $\exists$ a 2-fold degeneracy for $h_k$.

$k = 0, \frac{\pi}{a}$ only occurs for PBC.

for PBC: $\langle F \rangle_{gs} = \left\langle \sum c^\dagger c \right\rangle_{gs} = 2 \cancel{\sum_{k \neq 0, \frac{\pi}{a}} \langle c^\dagger_k c_k \rangle}$

$+ \langle c^\dagger_0 c_0 \rangle$

$+ \langle c^\dagger_\pi c_\pi \rangle.$

APBC: $\langle F \rangle = 2 \sum_k \langle c^\dagger_k c_k \rangle$

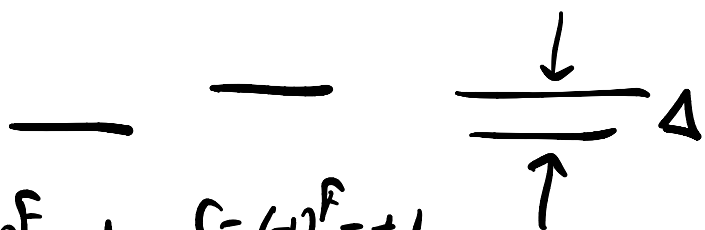
$h_{ka=\pi} = c^\dagger_\pi c_\pi (2g+2) > 0 \Rightarrow$ always empty in gs.

$$h_{ka=0} = c_0^\dagger c_0 \left(2g - 2\cos ka\right)\Big|_{k=0}$$
$$= c_0^\dagger c_0 \,(2g - 2)$$

- for $g > g_c = 1$ $\quad h_0 > 0 \implies (-1)^F = 1$
  APBC.

- for $\underline{g < g_c = 1}$ $\quad h_0 < 0 \implies (-1)^F = -1$
  PBC
  is the groundstate.

$\underline{g < g_c = 1}$ :




$\delta = (-1)^F = -1$     $\delta = (-1)^F = +1$

zm filled     no zeromode

PBC     APBC.

$|\Uparrow\rangle - |\Downarrow\rangle \qquad |\Uparrow\rangle + |\Downarrow\rangle$

<u>CLAIM:</u>

$$\boxed{\Delta \sim e^{-L/\xi} \quad \text{for } g < g_c.}$$

for $g = g_c$ :

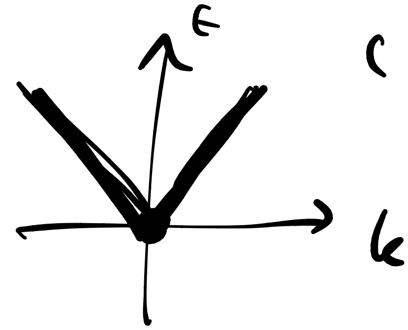
$$\Delta = \frac{1}{16}\cdot\frac{1}{L}.$$

<u>ex</u> : Prove the claim using the fermions.

## Critical Pt: At $g \to 1$

$$\epsilon(k) = 2J\sqrt{1+g^2-2g\cos ka} \overset{g=g_c=1}{=} c\,|k|.$$

$$c = 2Ja.$$




**Near** $g = g_c = 1$:

$$\epsilon(k) = c\sqrt{k^2 + \left(\frac{g-g_c}{a}\right)^2} + \dots = c\sqrt{k^2+m^2}$$

$$\xi = \frac{1}{m} = \frac{a}{|g-g_c|} \quad \text{diverging length scale.}$$

$$\xi \sim |g-g_c|^{-\nu} \quad (\nu = 1)$$

corr. length critical exponent.

$$\epsilon(k) \sim k^z \quad (z = 1)$$

**Continuum Limit:** $\Psi(x_j) = \frac{1}{\sqrt{a}} c_j.$

$$\{c_j, c_{j'}^\dagger\} = \delta_{jj'}$$

$$\longrightarrow \{\Psi(x), \Psi^\dagger(y)\} = \delta(x-y).$$

$$c_k = \int dx \, \frac{e^{-ikx}}{\sqrt{L}} \Psi(x)$$

$$J \sum_{k \text{ small}} (g - \cos ka) c_k^\dagger c_k \rightsquigarrow (g - g_c) \int dx \Psi^\dagger(x) \Psi(x) + \mathcal{O}(a \partial_x)$$

$$-iJ \sum_k \sin ka \, c_{-k}^\dagger c_k \rightsquigarrow \frac{c}{2} \int dx \Psi(x)^\dagger \partial_x \Psi(x)^\dagger$$

$$\Rightarrow H_{TFIM} \overset{g \to g_c}{\rightsquigarrow} \frac{c}{2} \int dx \left( \Psi^\dagger \partial_x \Psi^\dagger - \Psi \partial_x \Psi \right) + \Delta \int dx \, \Psi^\dagger \Psi$$

$$\Delta = 2J |g - 1|.$$

EOM: $\quad i\partial_t \mathcal{O} = [H, \mathcal{O}].$

$$H_{TFIM} = -iJ \sum_j \left( g \chi_j \tilde{\chi}_j - \chi_j \tilde{\chi}_{j+1} \right)$$

$$\begin{cases} i\partial_t \chi_j = iJ (g \tilde{\chi}_j - \tilde{\chi}_{j+1}) \\ i\partial_t \tilde{\chi}_j = iJ (-g \chi_j + \chi_{j-1}) \end{cases}$$

$$\chi(j+1) \simeq \chi(x_j) + a\,\partial_x \chi(x_j) + \mathcal{O}(a^2 \partial_x^2)$$

$$\Rightarrow \begin{cases} \frac{1}{aJ}\partial_t \chi = -\left(\frac{1-g}{aJ}\right)\tilde{\chi} - \partial_x \tilde{\chi} \\ \frac{1}{aJ}\partial_t \tilde{\chi} = \left(\frac{1-g}{aJ}\right)\chi - \partial_x \chi \end{cases}$$

$$\chi_\pm = \tfrac{1}{2}(\chi \mp \tilde{\chi}) \qquad \underline{t \equiv aJx^0.}$$

$$\begin{cases} \partial_0 \chi_+ = \partial_x \chi_+ + m\chi_- \\ \partial_0 \chi_- = -\partial_x \chi_- - m\chi_+ \end{cases}$$

$$m = \frac{1-g}{a}. \qquad \xrightarrow{m\to 0} \underline{\underline{(\partial_0 \mp \partial_x)\chi_\pm = 0.}}$$

$$0 = i\gamma^\mu \partial_\mu \chi + im\chi$$

$$\boxed{\text{Majorana fermion field.}}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \underline{\bar{\chi} = \chi^T\gamma^0}. \text{ i.e. } \underline{\chi^\dagger = \chi^T}.$$

$$H = \int dx\, h = \frac{c}{2}\int dx \left(\psi^\dagger \partial_x \psi^\dagger - \psi \partial_x \psi\right) + \Delta \int dx\, \psi^\dagger \psi$$

$$S[\psi, \psi^\dagger] = \int d\tau \int dx\, \mathcal{L}$$

grassmann vars.

$$\mathcal{L} = \bar{\Psi}\partial_\tau \Psi + h$$

$$= \bar{\Psi}\partial_\tau \Psi + \frac{c}{2}\left(\bar{\Psi}\partial_x \Psi - \Psi \partial_x \Psi\right) + \Delta \bar{\Psi}\Psi$$

let $\begin{cases} \Psi = \psi_+ + \psi_- + i(\psi_+ - \psi_-) \\ \bar{\Psi} = \psi_+ + \psi_- - i(\psi_+ - \psi_-) \end{cases}$

$$\rightarrow \mathcal{L} = \sum_\pm \psi_\pm (\partial_\tau \pm i\partial_x)\psi_\pm + \Delta \psi_+ \psi_- .$$

$$(\chi_\pm = \psi_\pm .)$$

scale inv't when $\Delta = 0$. $\begin{cases} x \to \lambda x \\ t \to \lambda t \end{cases}$ $\underline{\psi \to \lambda^{-1/2}\psi .}$

$\Rightarrow \langle \psi(x)^+ \psi(0) \rangle \sim \frac{1}{x}$ . (when $\Delta = 0$)

$\mathcal{O}(x) \longrightarrow \lambda^{-\delta} \mathcal{O}(\lambda x)$

$\longrightarrow \langle \mathcal{O}(x) \mathcal{O}(0) \rangle \sim \frac{1}{x^{2\delta}}$ .

$\int dx d\tau \, \bar{\psi}\psi \longrightarrow \lambda \int dx d\tau \, \bar{\psi}\psi$

Relevant.

$$\boxed{\xi \equiv \underline{a\lambda} \quad \text{s.t.} \quad \Delta = \Delta_0 \lambda^{\nu} \sim 1 .}$$

$\longrightarrow 1 = (\xi/a)^{\nu} \Delta_0 \Rightarrow \xi \sim \frac{1}{\Delta_0^{\nu}}$ .

Other Relevant ops?
Symmetric under $\psi \to -\psi$.

- $\bar{\psi}\psi\bar{\psi}\psi = 0$
- $\int dx dt \, \bar{\psi} \partial_x^2 \psi \sim \lambda^{-1}$
- $\int dx dt \, \bar{\psi} \partial_x \psi \bar{\psi} \partial_x \psi \sim \lambda^{-2}$

No.

What about $Z$ itself?

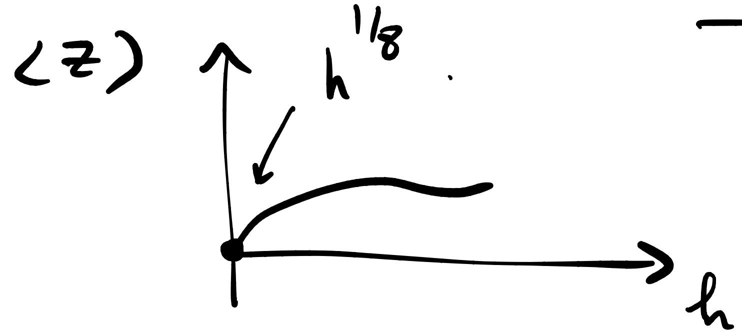
(odd under $Z \to -Z$)

makes a branch cut in $\psi(x)$

(creates a DW in the spins.)

"spin field operator" has dim $\delta = \frac{1}{8}$.

$\Rightarrow$




Compare to MFT:

$$S[\theta] \sim \int dx\,dt \left( (\partial_x \theta)^2 + (\partial_t \theta)^2 \right)$$

?