5. DUALITY

	description 1	weakly coupled	strongly coupled
nonlocal		$\longrightarrow$	coupling
	descr. 2	strongly coupled	weakly coupled

5.1 Transverse - Field Ising Model.

$$\mathcal{H} = \bigotimes_j \mathcal{H}_j$$

QUBIT

arrange in a lattice Γ

$$H_{TFIM} = -J\left(\sum_{\langle jl \rangle} Z_j Z_l + g \sum_j X_j \right)$$

neighbors on Γ

has a $\mathbb{Z}_2$ (Ising) sym: $Z \to -Z$

generated by $S = \prod_j X_j$. $0 = [H_{TFIM}, S]$.

Assume Γ is bipartite $J > 0$ WLOG.

(If $J < 0$, $Z_j \to (-1)^j Z_j$.)

Competition: $\boxed{g \gg 1}$ $\quad -\sum X$ wants

$$|\Rightarrow\rangle = \bigotimes_j |\rightarrow_j\rangle$$

trivial paramagnet

(no entanglement)

$$S|\Rightarrow\rangle = |\Rightarrow\rangle.$$

$\boxed{g \to 0}$ $\quad \sum ZZ$ wants

$$a|\uparrow\uparrow\uparrow\rangle + b|\downarrow\downarrow\downarrow\rangle$$

$$= a|\Uparrow\rangle + b|\Downarrow\rangle.$$

ferromagnet

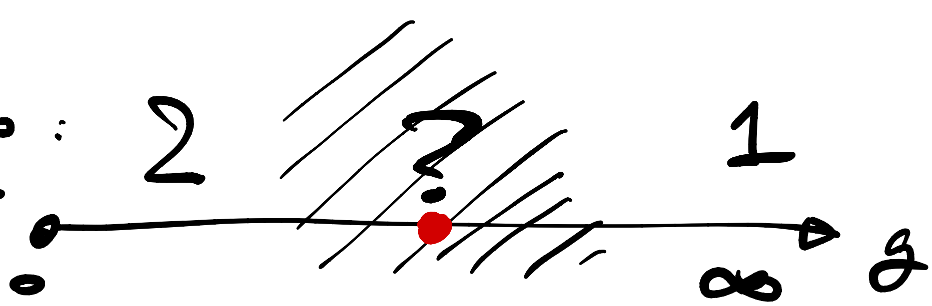
(SSB)

$$S|\Uparrow\rangle = |\Downarrow\rangle$$

vs: $\quad H_{\text{boring}} = -\left(\cos\theta \sum_j Z_j + \sin\theta \sum_j X_j\right) \quad \theta \in [0, \tfrac{\pi}{2}]$

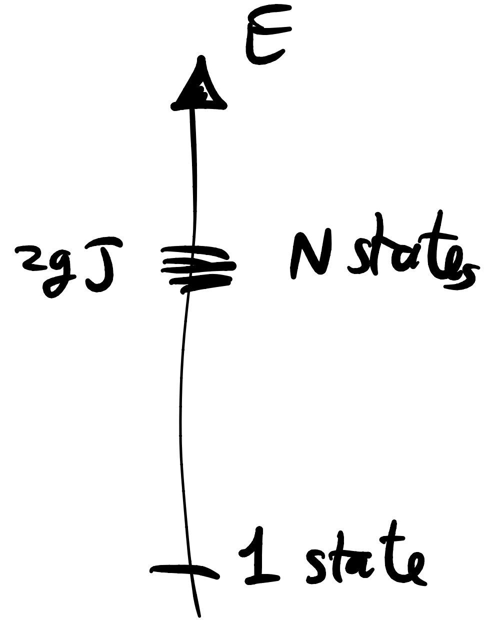
gs is $\quad |\theta\rangle = \bigotimes_j \left(\cos\tfrac{\theta}{2}|\uparrow\rangle + \sin\tfrac{\theta}{2}|\downarrow\rangle\right)$

# of gs: $\quad 2 \qquad ? \qquad 1$



QUASIPARTICLES:

$\boxed{g=\infty}$: $\quad H_{g\to\infty} = -gJ\sum_j X_j$



$$|gs\rangle = \bigotimes_j |\to\rangle \equiv |\Rightarrow\rangle$$

$$|n\rangle = |\to \dots \to \underset{\substack{\uparrow \\ \text{nth site}}}{\leftarrow} \to \dots \to\rangle$$

$$(H_\infty - E_0)|n\rangle = 2gJ\,|n\rangle \qquad \forall\, n = 1\dots N$$

In the subspace span$\{|n\rangle\}$ $\quad (Z_j|\to_j\rangle = |\leftarrow_j\rangle)$

$$Z_j Z_{j+1}|\to_j \leftarrow_{j+1}\rangle = |\leftarrow_j \to_{j+1}\rangle$$

kinetic term for the reversed spin
(hopping)

$$\langle n\pm 1|\sum_j Z_j Z_{j+1}|n\rangle = 1.$$

$$\Rightarrow H_{eff}|n\rangle = -J\big(|n+1\rangle + |n-1\rangle\big) + (E_0+2gJ)|n\rangle.$$

PBC : $\mathcal{H}_{j+N} = \mathcal{H}_j$.

$$|n\rangle \equiv \frac{1}{\sqrt{N}} \sum_j e^{-ikx_j} |k\rangle$$

$x_j = ja$

$k = \frac{2\pi n}{Na}$

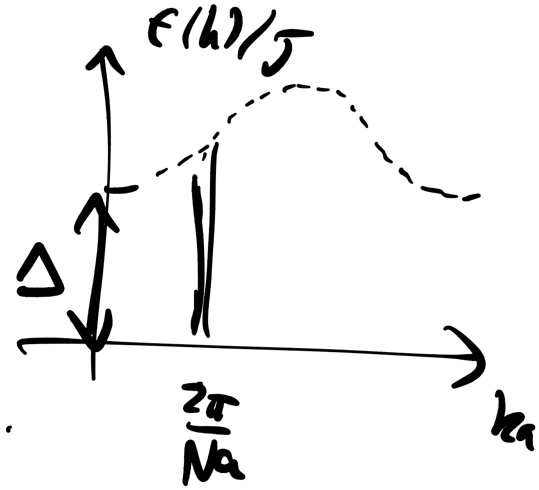
$n = 1..N$

$$(H - E_0)|k\rangle = (\underbrace{-2J\cos ka + 2gJ}_{\equiv\, \epsilon(k)})|k\rangle$$

$$\epsilon(k) \overset{k\to 0}{=} \Delta + J(ka)^2 + \dots$$

$$\Delta = 2J(g-1) \quad \left.\right\} = E_{rest} + \frac{k^2}{2m_{inertial}} + \dots$$

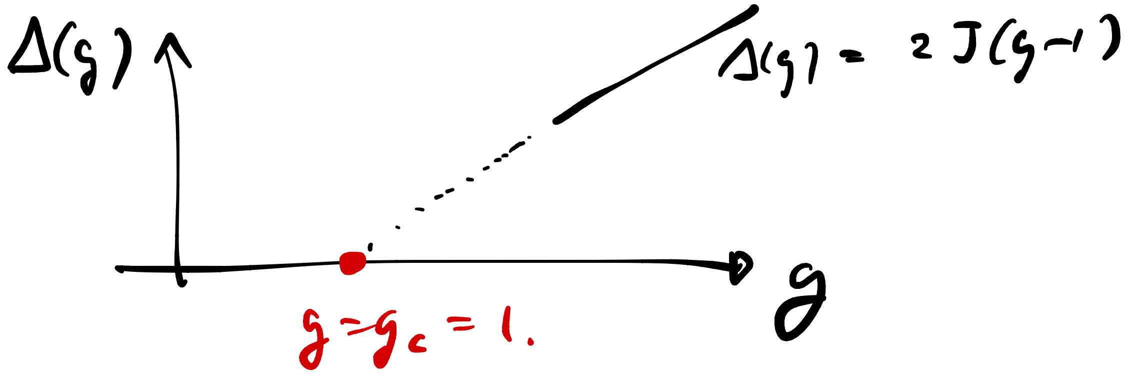
<u>massive</u> particles.

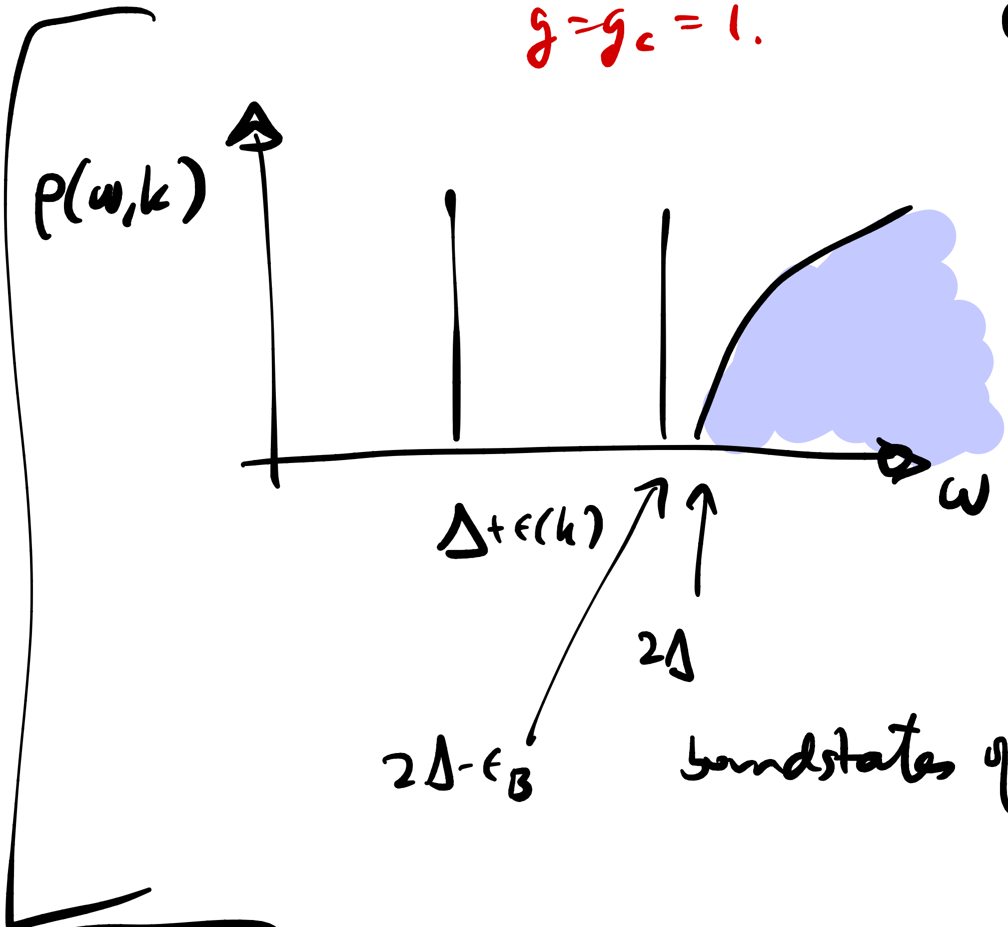


$$|n\rangle = Z_n\, |gs_{g=\infty}\rangle \qquad Z_n^2 = 1$$

creation operator

$$\#\text{ of particles} = \sum_j (-X_j) + N \quad \left(\begin{matrix}\text{conserved}\\ \text{mod } 2\end{matrix}\right)$$





spectral density of QFT w/ massive scalar

$\langle Z_n Z_0 \rangle$

boundstates of spin flips.

$\boxed{g \gg 1}$ $\quad |\Uparrow\rangle = |\uparrow \cdots \uparrow\rangle$

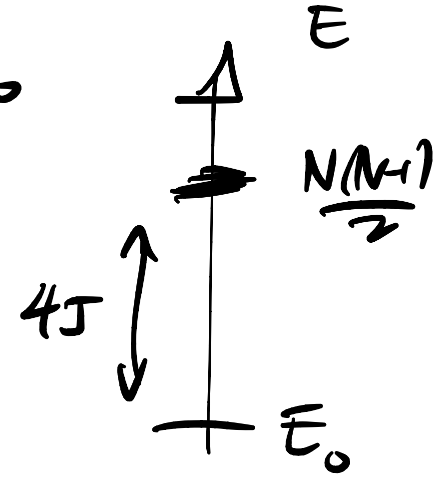
an eigenstate of $H_0 = -J \sum_j Z_j Z_{j+1}$

w/ $E_0 = -JN$.

$| \cdots \uparrow \cdots \uparrow \downarrow \uparrow \cdots \uparrow \cdots \rangle$ costs 4J

$| \cdots \uparrow \cdots \uparrow \downarrow \downarrow \downarrow \uparrow \cdots \uparrow \cdots \rangle$ costs 4J exc. one <u>domain walls.</u>

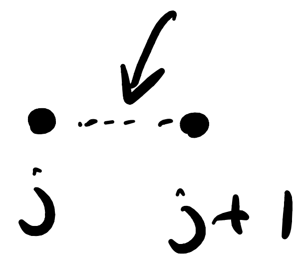
at $g=0$



$$X_{j+1}\underbrace{|\cdots\uparrow\cdots\uparrow_j\downarrow_{j+1}\downarrow\downarrow\cdots\rangle}_{\equiv|J\rangle}$$

$$=\underbrace{|\cdots\uparrow\cdots\uparrow_j\uparrow_{j+1}\downarrow_{j+2}\downarrow\cdots\rangle}_{\equiv|J+1\rangle}$$

<u>$J\equiv j+\frac{1}{2}.$</u>



$$(H_{eff}-E_0)|J\rangle$$

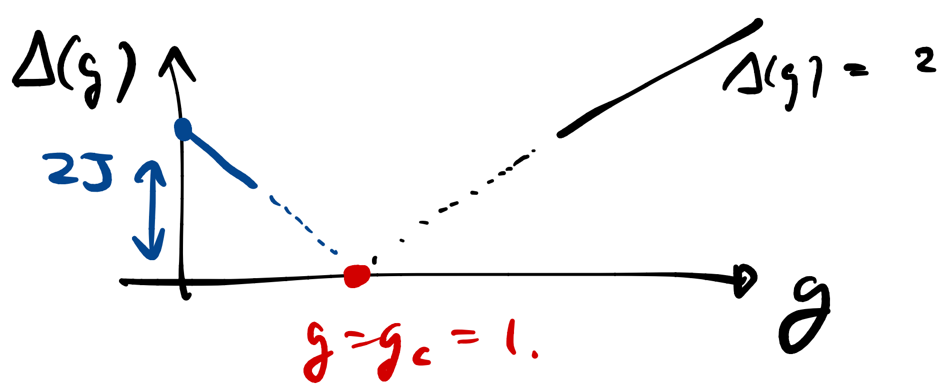
$$=-gJ\left(|J+1\rangle+|J-1\rangle\right)$$

$$+2J|J\rangle.$$

$$\epsilon_{one\text{-}DW}(k)=2J(1-g\cos ka)$$

$$=\Delta_{DW}+\#k^2+\dots$$

$$\Delta_{DW}=2J(1-g)$$



- Mean field theory also predicts a (continuous) transition

- <u>About SSB</u>:

$$|🐱_\pm\rangle \equiv \frac{|\Uparrow\rangle \pm |\Downarrow\rangle}{\sqrt{2}}$$

are also eigenstates of H_0

" " " $S = \prod X_j$.

at $N < \infty$, $|🐱_+\rangle$ is the g.s.

BUT

the <u>splitting</u> is: $T \sim \dfrac{g^N}{\Delta^N} \langle \Uparrow | X_1 X_2 \cdots X_N | \Downarrow \rangle$

$$\sim e^{-N}$$

the right way: $\Delta H = -\sum_j h_j Z_j$

✓ $\lim_{h\to 0} \lim_{N\to \infty} (\quad) \neq \lim_{N\to\infty} \lim_{h\to 0} (\quad)$

Duality betw. D.W. & spin flips :

$\epsilon_{\text{one DW}}(k)$ & $\epsilon_{\text{spin flip}}(k)$

are related by $g \to 1/g$, $J \to Jg$.

suppose Open B.C.

$\underset{j=1}{\cdot} \cdot \cdot \cdot \cdot \quad \cdot \cdot \cdot \cdot \cdot \underset{j=N}{\cdot}$

- $\tau^x_J \equiv Z_{J-\frac{1}{2}} Z_{J+\frac{1}{2}} = \begin{cases} +1 & \text{if } Z_{J-\frac{1}{2}} = Z_{J+\frac{1}{2}} \\ -1 & \text{if } Z = -Z \end{cases}$

  $= (-1)^{\text{disagreement}}$

detects a DW at J.

$$\underline{(\tau^x_J)^2 = 1} \quad , \quad \underline{(\tau^x_J)^\dagger = \tau^x_J} .$$

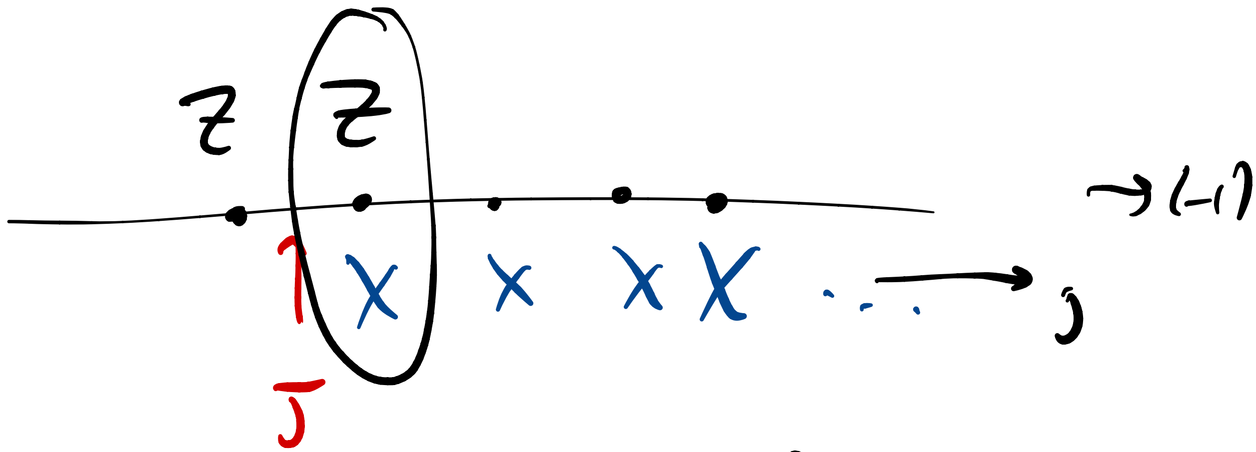
- $\tau^z_J \equiv X_{J+\frac{1}{2}} X_{J+\frac{3}{2}} \cdots \equiv \prod_{j>J} X_j$

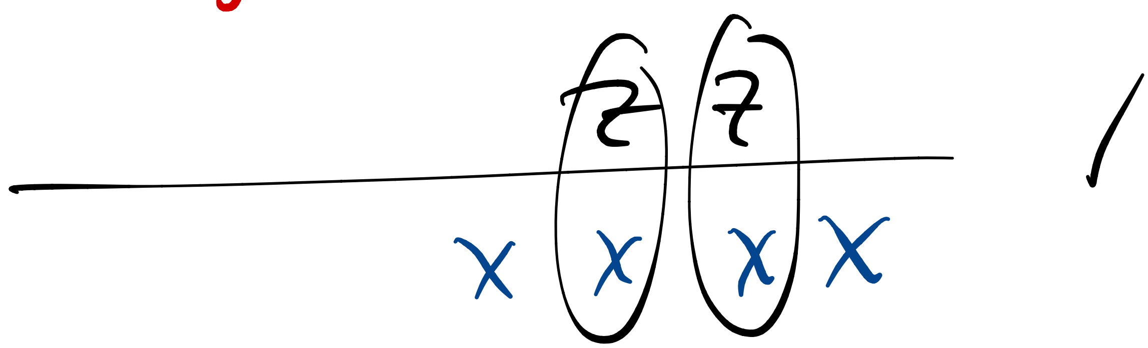
creates a DW at J.
or annihilates

also
- squares to 1
- is hermitian

$$\tau^x_J \, \tau^z_{J'} = (-1)^{\delta_{JJ'}} \tau^z_{J'} \, \tau^x_J$$

(just like $X_j Z_{j'} = (-1)^{\delta_{jj'}} Z_{j'} X_j$.)





$$X_j = \tau^z_{j-\frac{1}{2}} \, \tau^z_{j+\frac{1}{2}}$$

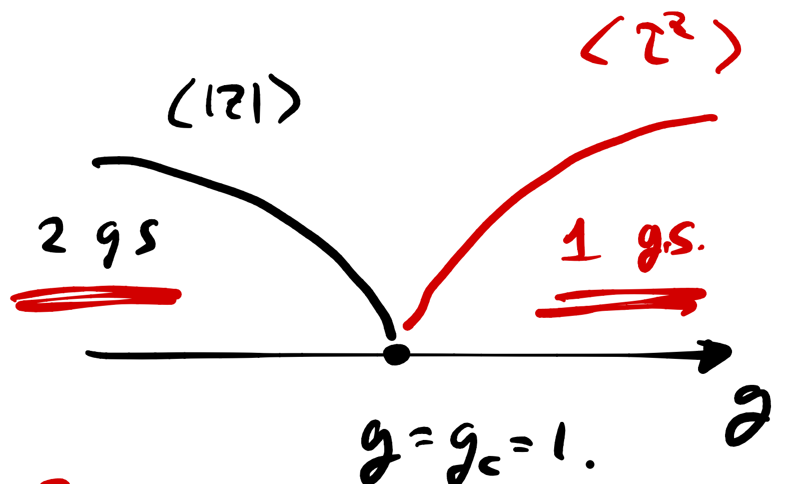
$$\to H_{TFIM} = -J \sum_j \left(g X_j + Z_j Z_{j+1} \right)$$

$$= -J \sum_J \left(g \, \tau^z_J \tau^z_{J+1} + \tau^x_J \right)$$

$$= H_{TFIM} \left(g \to 1/g, \; J \to gJ \right)$$

This is $_\wedge$ Kramers-Wannier duality

related to by Q-C correspondence.



In the para phase

DWs are condensed:

$$\langle \tau^z_j \rangle = \langle gs_{g=\infty} | \tau^z_{\bar{j}} | gs_{g=\infty} \rangle$$

$$= \langle gs_\infty | \prod_{j > \bar{j}} X_j | gs_{g=\infty} \rangle$$

$$= 1.$$

Sol'n by Jordan-Wigner.

Jordan-Wigner in 0+1 dims: $c^2 = 0$, $\{c, c^\dagger\} = 1$

$c|\to\rangle = 0$, $c^\dagger|\to\rangle = |\leftarrow\rangle$, $c^\dagger|\leftarrow\rangle = 0$

$c|\leftarrow\rangle = |\to\rangle.$

$$\begin{cases} Z = c + c^\dagger \\ X = \mathbb{1} - 2c^\dagger c \\ Y = \frac{1}{i}(c - c^\dagger). \end{cases}$$

$$\begin{pmatrix} c^\dagger c\,|\rightarrow\rangle = 0 \\ c^\dagger c\,|\leftarrow\rangle = |\leftarrow\rangle. \end{pmatrix}$$

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = c^\dagger$$

$$\sigma^- = \qquad = c.$$

In $D > 0+1$: $\{c_\alpha, c_\beta\} = 0$. for $\alpha \neq \beta$

vs $[X_\alpha, Z_\beta] = 0$ " .

	ORDERED		DISORDERED
CORRECT VARS:	$\tau^z_j = \prod X_j$ creates a DW	$\xrightarrow[g=g_c]{\times} g$	Z_j creates a spin flip
condensate:	$\langle Z\rangle \neq 0$		$\langle \tau^z_j\rangle \neq 0$.

CORRECT VARS everywhere: (attach a spin to a DW)

$$\begin{cases} \chi_j \equiv Z_j \tau^z_{j+\frac{1}{2}} = Z_j \prod_{j'>j} X_{j'} \\ \tilde{\chi}_j \equiv Y_j \tau^z_{j+\frac{1}{2}} = -i Z_j \prod_{j' \geq j} X_{j'} \end{cases}$$

ordered phase: $\hat{Z} = \langle Z \rangle + \underbrace{\delta \hat{Z}}_{\text{small}}$

$g \ll 1$

$$\chi = Z\tau^z = \langle Z \rangle \tau^z + \text{small} \overset{g \ll 1}{\cong} \tau^z$$

disordered phase: $\tau^z = \langle \tau^z \rangle + \underbrace{\delta\tau}_{\text{small}}$

$g \gg 1$

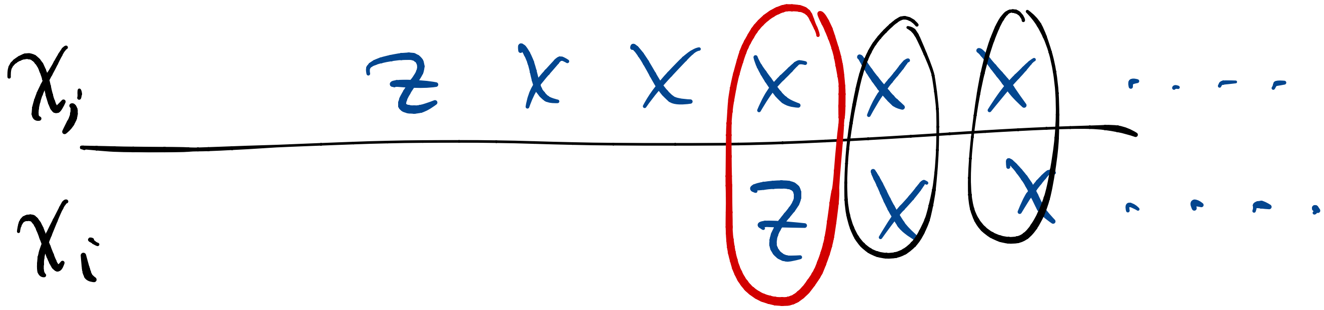
$$\chi = Z\tau^z = Z\langle \tau^z \rangle + \text{small} \overset{g \gg 1}{\sim} Z$$

<u>algebra of χs</u>: • <u>Real</u> $\begin{cases} \chi_j^\dagger = \chi_j \\ \tilde{\chi}_j^\dagger = \tilde{\chi}_j \end{cases}$

• <u>Fermions</u> for $i \neq j$

<u>claim</u>: $\chi_i\chi_j + \chi_j\chi_i = \{\chi_i, \chi_j\} = 0$.

$\{\chi_i, \tilde{\chi}_j\} = 0$, $\{\tilde{\chi}_i, \tilde{\chi}_j\} = 0$.



$$\Rightarrow \chi_i \chi_j = - \chi_j \chi_i .$$

χ_j $\tilde{\chi}_j$ are majorana fermion.

$$c_j = \frac{1}{2}(\chi_j - i\tilde{\chi}_j) \Rightarrow c_j^\dagger = \frac{1}{2}(\chi_j + i\tilde{\chi}_j)$$

$$\Rightarrow \left[\begin{array}{ll} \{c_i, c_j^\dagger\} = \delta_{ij} & \{c_i, c_j\} = 0 \\ & \{c_i^\dagger, c_j^\dagger\} = 0 . \end{array} \right.$$

$$X_j = -i \tilde{\chi}_j \chi_j = -2 c_j^\dagger c_j + 1$$

$$= (-1)^{c_j^\dagger c_j}$$

counts spin flips. = # of fermions (mod 2)

Pf: $\tilde{\chi}_j = i \mathbf{X}_j \chi_j$

$$Z_j Z_{j+1} = i \tilde{\chi}_{j+1} \chi_j .$$

$$\Longrightarrow H_{TFIM} = -J \sum_j \left(i \tilde{\chi}_{j+1} \chi_j + g\, i \tilde{\chi}_j \chi_j \right)$$

<u>Quadratic</u> in canonical fermions $\Longrightarrow$ <u>FREE</u>.

$$= \sum_{ij}^{N} \left(A_{ij} c_i^\dagger c_j + \Delta_{ij} c_i^\dagger c_j^\dagger + h.c. \right)$$

$\underline{N \times N}$

evals of

$$\begin{pmatrix} A & \Delta \\ \Delta^\dagger & A \end{pmatrix}$$

$2N \times 2N$

$\rightsquigarrow$ evals of $\underline{\underline{2^N \times 2^N \text{ matrix !!}}}$

$$S = \prod_j X_j = \prod_j (-1)^{c_j^\dagger c_j}$$

$$= (-1)^{\text{fermion \#}} .$$