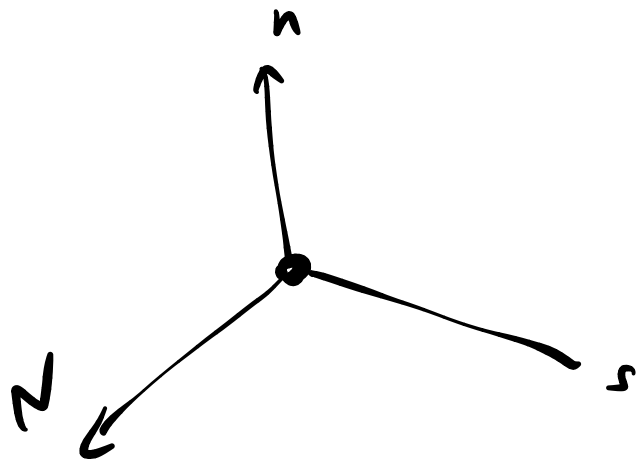
Large N :

small parameters : $n^{i=1..3} = z^\dagger_\alpha \sigma^i z_{\alpha=1,2}$

- \# of components of $n^{i=1..n}$ $\quad SO(3) \to SO(n)$
- Rep of $\vec{S}$ $\quad$ large s rep. of $\underline{SU(2)}$
- \# of components of $z_{\alpha=1..N}$

$$SU(2) \longrightarrow \underline{SU(N)}$$



$$\sum_{m=1}^{N} |z_m|^2 = N/2$$

$$n^A = z^\dagger T^A z$$

T^A: gen. of SU(N) $(N^2-1.)$

$$Z_{NLSM} = \int [D\vec{m}]\, e^{-\frac{\Lambda^{D-2}}{g^2}\int (\partial m)^2} \delta[m^2-1]$$

on $S^2 = \mathbb{C}P^1$

$$= \int [Dz Dz^\dagger DA D\lambda]\, e^{-\int d^Dx \left[\frac{2\Lambda^{D-1}}{g^2} |(\partial - iA)z|^2 - i\lambda(|z|^2-1)\right.}$$

z integral is gaussian!

$$Z_{\mathbb{CP}^{N-1}} = \int [dz\, dz^\dagger\, dA\, d\lambda]\, e^{-\int d^D x \left[\frac{2\Lambda^{D-1}}{g^2} |(\partial - iA) z|^2 - i\lambda \left(|z|^2 - \underline{\underline{N/2}} \right) \right]}$$

$$= \int [dA\, d\lambda]\, e^{-N\, S[A,\lambda]}$$

$$\overset{N \gg 1}{\simeq} e^{-N S[\underline{A}, \underline{\lambda}]}$$

$$S[A,\lambda] = \mathrm{Tr} \ln \left(-(\partial - iA)^2 + i\lambda \right) - \frac{\Lambda^{D-2}}{g^2} \int i\lambda$$

$$0 = \frac{\delta S}{\delta A} \Bigg| = \frac{\delta S}{\delta \lambda} \Bigg|_{\substack{A = \underline{A} \\ \lambda = \underline{\lambda}}}$$

$\underline{\text{Ansatz}}$: $\begin{cases} A = \underline{A}_\mu = 0 \\ \lambda = -i\underline{\lambda} \ \ \underline{\underline{\text{const}}} \end{cases}$

$$S[0, -i\underline{\lambda}] = V \int d^D k \ln(k^2 + \underline{\lambda}) - \frac{\Lambda^{D-2}}{g^2} V \underline{\lambda}$$

$$0 = \frac{\delta S}{\delta \underline{\lambda}} \iff \int \frac{d^D k}{k^2 + \underline{\lambda}} = \frac{\Lambda^{D-2}}{g^2}.$$

$\underline{\underline{\text{condition on } \underline{\lambda}.}}$

If $\underline{\lambda} \neq 0 \Rightarrow z$ massive $\Rightarrow$

no goldstones $\Rightarrow$ no SSB

$\boxed{D=1}$ $$\frac{1}{g^2\Lambda} = \int \frac{dk}{k^2+\underline{\lambda}} \underset{\text{scaling}}{=} \frac{1}{\sqrt{\underline{\lambda}}} \underbrace{\int \frac{dk}{\underline{k}^2+1}}_{1/2}$$

$$\Rightarrow \underline{\lambda} = \frac{g^4\Lambda^2}{4} . \Rightarrow \underline{\text{no SSB}} .$$

FOR $D=1$:

whence? Λ QM of a particle on $\mathbb{CP}^{N-1}$. (compact.)

$$H = -\frac{g^2\Lambda}{2} \underbrace{\Delta}_{\text{discrete spectrum}} \Rightarrow \underline{\text{gap} \sim g^2\Lambda}.$$

Laplacian

$$\langle z \mid gs \rangle = \Psi(z) = \frac{1}{\sqrt{\text{volume}}}$$

$\checkmark$

$\boxed{D=2}$

$$g^{-2} = \int \frac{d^2k}{k^2+\underline{\lambda}} = -\frac{1}{4\pi}\ln\frac{\underline{\lambda}}{\Lambda^2}$$

$$\Longrightarrow \quad \underline{\lambda} = \Lambda^2 e^{-4\pi/g^2}$$

(dimensional transmutation)

$$m_z = \sqrt{\underline{\lambda}} = \Lambda e^{-2\pi/g^2} \ll \Lambda$$

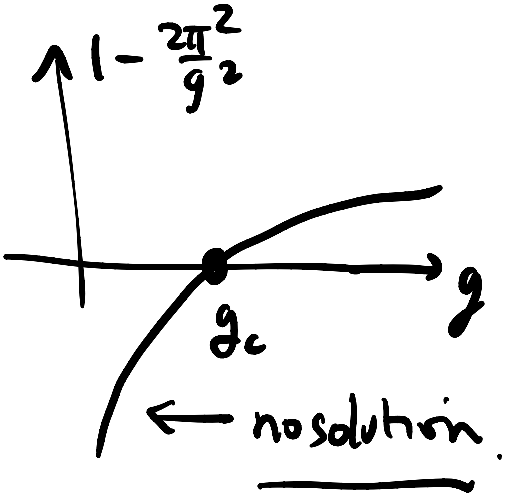
(Haldane gap.)

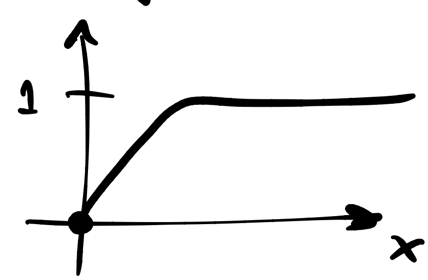
$\boxed{D=3}$

$$\frac{\Lambda}{g^2} = \int \frac{d^3k}{k^2+\underline{\lambda}} = \frac{1}{2\pi^2}\left(\Lambda - \sqrt{\underline{\lambda}}\arctan\frac{\Lambda}{\sqrt{\underline{\lambda}}}\right)$$

$$\Longrightarrow \quad \underline{1-\frac{2\pi^2}{g^2}} = \underbrace{x\arctan\frac{1}{x}}_{}\overset{\text{RHS}}{\geq 0}$$

$$x \equiv \frac{\sqrt{\lambda}}{\Lambda} > 0.$$





If no sol'n: $z \sim \int d\underline{\lambda} \; e^{-S_{\text{eff}}(\underline{\lambda})}$

minimized when $\underline{\lambda} = 0$.

$$g^2 < g_c^2 \Rightarrow \underline{\lambda} = 0 \Rightarrow \text{no mass}$$

$\Rightarrow$ SSB. z's are goldstones.

$D \geq 3$: $$g_c^{-2} = \int \frac{d^D k}{k^2} \sim \frac{\Lambda^{D-2}}{D-2}$$

$$m^2 \simeq \Lambda^2 \left(\frac{g^2 - g_c^2}{g_c^2} \right)^{\frac{2}{D-2}} \quad \text{as } g \to g_c^+.$$

universal exponent.

Correlation functions. $N=2$: $\langle S^+(0) S^-(x) \rangle \equiv S^{+-}(x)$

claim: $$\hat{S}^a = N_s \int d\check{n}^2 \, |\check{n}\rangle\langle\check{n}| \, n^a \quad \left(N_s = \frac{(S+1)(2S+1)}{4\pi}. \right)$$

$$S^{+-}(x) = \langle \underbrace{(n^x + i n^y)}(0)\,(n^x - i n^y)(x) \rangle$$

$$n^x + i n^y = z^\dagger \sigma^+ z = z_1^* z_2$$

<u>**general N:**</u>

$$\rightarrow S^{m\neq m'}(x) = \langle z_m^*(0)\, z_{m'}(0)\, z_m(x)\, z_{m'}^*(x) \rangle$$

$$\overset{N\gg 1}{\simeq} |G(x)|^2 + O\left(\frac{1}{N^2}\right)$$

$$G(x) = \langle z_m^\dagger(0)\, z_m(x) \rangle \qquad \text{ind. of } m$$

$$= \frac{1}{Z} \int [d^2 z]\, z^\dagger(0)\, z(x)\, e^{-\left[\frac{2\Lambda^{D-2}}{g^2}\int d^D k\, (|k|^2 + \underline{\lambda})\, z_k^\dagger z_k + \frac{NV\Lambda^{D-2}}{g^2}\underline{\lambda}\right]}$$

$$\propto \int d^D k\, \frac{e^{-ikx}}{|k|^2 + \underline{\lambda}} \simeq \frac{1}{|x|^{\frac{D-1}{2}}}\, e^{-|x|/\sqrt{\underline{\lambda}}}$$

$$\cong e^{-|x|/\xi}$$

$$\Rightarrow \xi = \frac{1}{\sqrt{\underline{\lambda}}}. \qquad D=1:\ \xi = \frac{1}{\Lambda g^2}. \qquad D=2:\ \xi = \Lambda^{-1} e^{\frac{2\pi}{g^2}}.$$

$\underline{D=3}$: $\quad 2\xi \simeq \Lambda^{-1}\left(\frac{2}{\pi} - \frac{4\pi}{g^2}\right)^{-1}$ for $g > g_c$

(correlations of $S^z = |z^m|^2 - |z^{m'}|^2$ have same behavior.)

Dynamical gauge field:

fluctuations

$$S_{eff}[A = \underline{\underline{0}} + a,\ \underline{\lambda} = \underline{\underline{m^2}} + v]$$

$$= W_0 + W_1 + \underline{\underline{W_2}} + \mathcal{O}(\delta^3)$$

indep of a, v — linear in a, v.

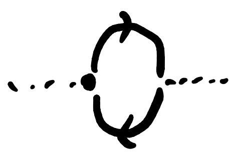
$$W_1 = \cancel{\left.\frac{\delta S}{\delta a}\right|_{a=0}} \cdot a = 0$$

$\to 0$

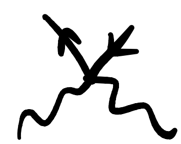
assume: $A = 0$, $\underline{\lambda} = m^2$ is a saddle point of S_{eff}.

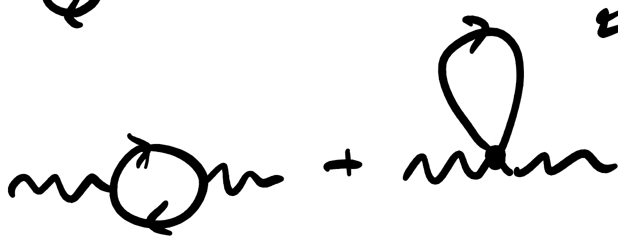
$$W_2 = \frac{N}{2}\int d^Dq\Big[v(q)\Pi(q)v(-q) + a_\mu(q)\Pi^{\mu\nu}(q)\,a_\nu(-q)\Big]$$

$$\Pi(q) = \int d^Dk\,\frac{1}{(k^2+m^2)((k+q)^2+m^2)}$$



$|(\partial - A)z|^2$

$\Rightarrow$ 

$\Pi^{\mu\nu}(q) =$ 

$$= \int d^Dk\,\frac{(2k+q)_\mu(2k+q)_\nu}{(k^2+m^2)((k+q)^2+m^2)} - 2g_{\mu\nu}\int\frac{d^Dk}{k^2+m^2}$$

Ward id $\Rightarrow$ $q^\mu\,\Pi_{\mu\nu}(q) = 0$

In $D=2$: $\Pi_{\mu\nu}(q) \overset{q\to 0}{\sim} \frac{c}{m^2}(q^2 g_{\mu\nu} - q_\mu q_\nu)$

$\Rightarrow W_2 \sim \frac{N}{m^2}\int d^2x\, F_{\mu\nu}F^{\mu\nu}$ + more derivatives

In $D = 1+1$: $\Delta S = \int \frac{\theta F}{2\pi}$ $F = da$.

total deriv. $\int \frac{F}{2\pi} \in \mathbb{Z}$ on closed mflds.

chern # of the gauge field config.

$$Z = \sum_{Q} e^{i\theta Q} Z_Q$$

NLSM on S^2 $\vec{m} : S^2 \longrightarrow S^2$.

Q = winding #

$\Longrightarrow$ for $N=2$ $\theta = 2\pi s$.

In fact: $F \propto \epsilon^{abc} n^a dn^b \wedge dn^c$.

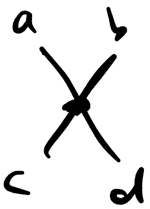
Large-N diagrams: $\vec{\varphi}$ vector of $O(N)$ in D dims.

$$L = \frac{1}{2} \partial\vec{\varphi} \cdot \partial\vec{\varphi} + \frac{g}{4N} (\vec{\varphi}\cdot\vec{\varphi})^2 + \frac{m^2}{2} \vec{\varphi}\cdot\vec{\varphi}.$$

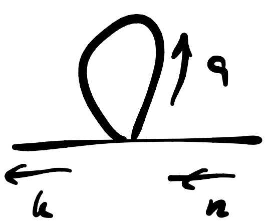
Pert thy in g.

$$\langle \varphi_b(x)\,\varphi_a(0)\rangle = \delta_{ab}\int d^Dk\,\frac{e^{-ikx}}{k^2+m^2}$$

$$= \delta_{ab}\int d^Dk\, e^{-ikx}\,\Delta_0(k)$$



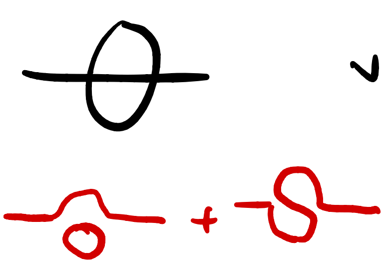
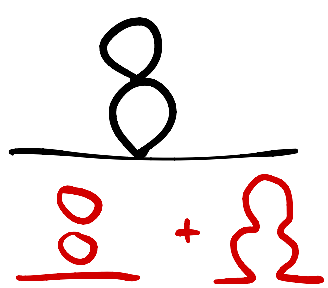
$$= -\frac{2g}{N}\left(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\right)$$

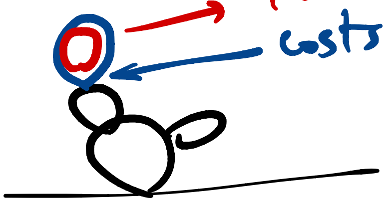
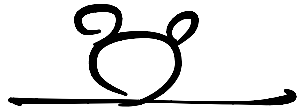


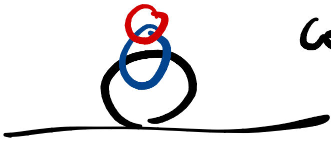
$$= -\frac{g}{4N}(4N+8)\,\delta_{ab}\int\frac{d^Dq}{q^2+m^2}$$

$$\overset{N\gg1}{\simeq} -g\,\delta_{ab}\int d^Dq\,\Delta_0(q)$$

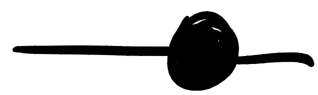
ind. of N. $\sim N^0$.

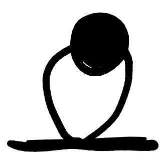
g:  vs  $\Rightarrow$ cactus wins!

 N costs g/N $\propto$  $\times\, g \cdot \underline{\underline{N^0}}$.

 $\leftarrow N$ costs $(g/N)^2$ $\propto \dfrac{g^2}{N} \longrightarrow 0$.

LARGE N fixed $g \equiv$ 't Hooft parameter.

$$\Delta_F(k) = \frac{1}{k^2+m^2+\Sigma(k)} \equiv$$ 

$$\Sigma(p) =$$  $$+\cancel{O\left(\tfrac{1}{N}\right)} = \underbrace{g\int d^Dk\, \Delta_F(k)}_{\text{indep of } p\,!} + \cancel{O\left(\tfrac{1}{N}\right)}$$

$\Rightarrow \Sigma(p) = \delta m^2$ - ind. of p.

$$\langle \varphi_a(x)\varphi_b(y)\rangle = \delta_{ab}\int d^Dk\, e^{-ik(x-y)}\Delta_F(k)$$

let $y^2 \equiv \left\langle \frac{\sum_a \varphi_a(x)\varphi_a(x)}{N} \right\rangle = \langle \frac{\varphi^2}{N} \rangle$.

indep. of x.

$y \to x$

$$y^2 = \int d^Dk\, \Delta_F(k) = g^{-1}\Sigma.$$

$$\int d^Dk \quad \left(\Delta_F(k) = \frac{1}{k^2+m^2-\Sigma} \right)$$

$$\longrightarrow \quad \int d^Dp\, \Delta_F(p) = \int d^Dp \frac{1}{p^2+m^2+\Sigma}$$

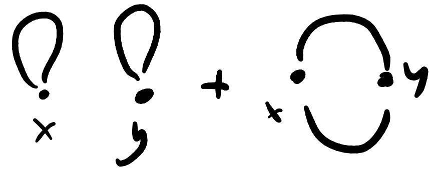
$$\boxed{y^2 = \int d^Dp \frac{1}{p^2+m^2+gy^2}}$$

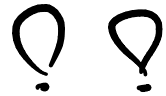
eqn determining y ~ mass for φ.

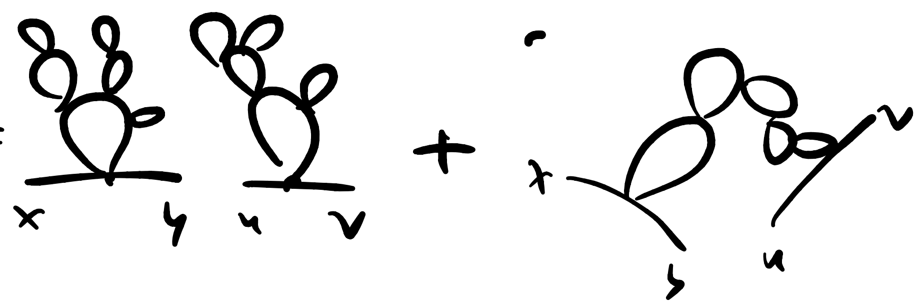
Large-N factorization:

claim: $\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle = \langle \mathcal{O}(x)\rangle\langle \mathcal{O}(y)\rangle + \mathcal{O}\!\left(\frac{1}{N}\right)$

$\mathcal{O}$ any invariant of the large-N group

eg: $\left\langle \frac{\varphi^2(x)}{N}\,\frac{\varphi^2(y)}{N}\right\rangle_{\text{free}} =$ 

$=$  $+\ \mathcal{O}\!\left(\frac{1}{N}\right)$

$\left\langle \frac{\varphi(x)\cdot\varphi(y)}{N}\,\frac{\varphi(u)\cdot\varphi(v)}{N}\right\rangle =$ 

$+$

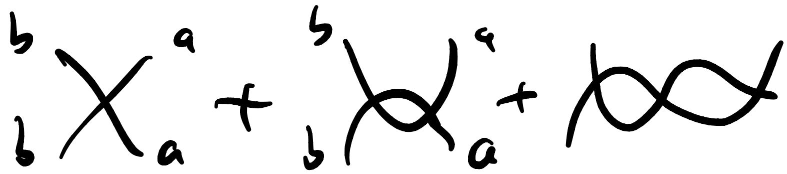
$\vdots$

$\Longrightarrow \quad = \left\langle \frac{\varphi(x)\cdot\varphi(y)}{N}\right\rangle\left\langle \frac{\varphi(u)\cdot\varphi(v)}{N}\right\rangle + \mathcal{O}\!\left(\frac{1}{N}\right)$

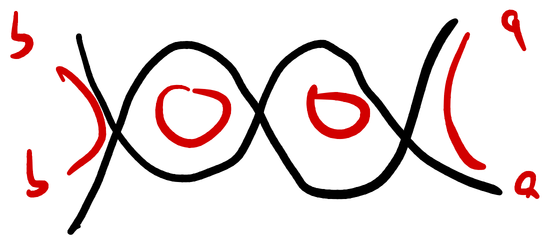
Mean field theory works for singlet operators

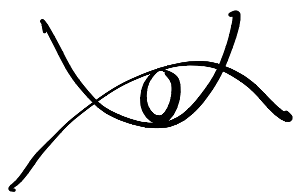
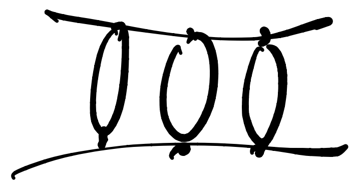
$$G_{4c}^{b\neq a} = \langle \varphi_b(p_4)\,\varphi_b(p_3)\,\varphi_a(p_2)\,\varphi_a(p_1)\rangle_c$$

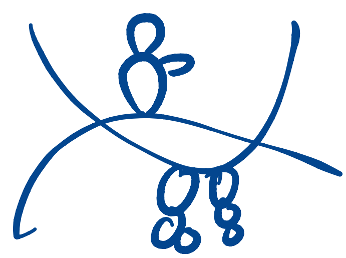
no sum.

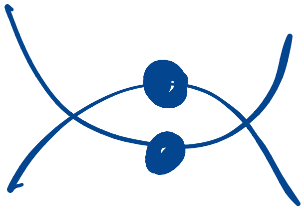
=  $+\ O(N^{-?})$

claim: chains of s-channel* bubbles win.



vs:   are down by $\frac{1}{N}$.

* decorated w cacti: 

= 

$\Delta_0 \to \Delta_F.$

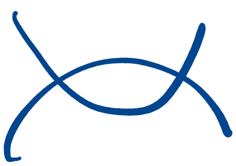
$$\left(\Sigma = \Sigma(\text{cacti})\right) = \frac{1}{k^2+m^2+\Sigma}$$

claim:

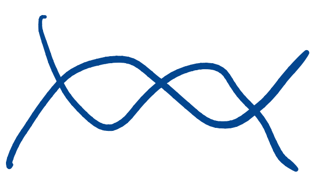
$$G^{b \neq a}_{4,c} = - \Delta_0(\text{external})^4 \, \frac{2}{N} \, \frac{g}{1 + g L(p_1 + p_2)} + O(N^{-2})$$

$$L(p) \equiv \int d^D k \, \Delta_F(k) \Delta_F(p+k) =$$  .

 $= \Delta_0^4 \left(\frac{g}{4N}\right) \cdot 2 \cdot 4$

 $= \Delta_0^4 \left(\frac{g}{4N}\right)^2 \cdot 2 \cdot 4 \cdot 8 \, \frac{1}{2!} \, L$

$$= \Delta_0^4 \, \frac{2}{N} \, g^2 L$$

 $= \Delta_0^4 \, \frac{2}{N} \, g^3 L$ . ...

<u>Why</u>: the bubble chain <u>is</u> the σ propagator!

$$e^{\frac{g}{N}\int (\vec{\varphi}\cdot\vec{\varphi})^2} = \int D\sigma \, e^{-\int \frac{\sigma^2}{N} + \int \sigma \vec{\varphi}\cdot\vec{\varphi}}$$

$$\xrightarrow[\uparrow]{\text{SDE}} \quad S_{eff}[\sigma] = \int \frac{\sigma^2}{g} + \lambda \ln(\partial^2 + m^2 + \sigma)$$

$$G_{4,c} = \langle \sigma_1 \ \sigma_2 \rangle = \left(\frac{\delta^2}{\delta\sigma_1 \delta\sigma_2} S_{eff}[\sigma] \right)^{-1}$$

$$= \left(\frac{1}{g^{-1} + \left(\frac{1}{\partial^2 + m^2 + \sigma}\right)^2} \right)^{-1} .$$

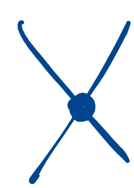
<u>Comments</u>: ① Large-N vs. <u>Wilsonian PoV</u>.

② $\sigma \sim \varphi^2$ is a composite operator

but $\langle \sigma \ \ \sigma \rangle$ has poles

at $p^2 = m^2$.

$$G_{4,c} = -\Delta_0^4 \frac{2}{N} g_{eff}(p_1 + p_2) + O(N^{-2})$$



$$g_{eff}(p) = \frac{g}{1 + g \int d^D k \, \Delta_F(k) \Delta_F(k+p)}$$

$\tilde{\lambda} = \frac{\lambda}{2s}$

$$\mathrm{tr}\log(2s\,\partial_\tau + i\lambda) + i\lambda$$

$$\underline{\underline{2s}}\,(\partial_\tau + i\tilde{\lambda})$$

$$= \mathrm{tr}\log(\underline{\partial_\tau + i\tilde{\lambda}}) + i\,\tilde{\lambda}\cdot 2s$$

$$+ \underline{\mathrm{tr}\log(2s)}$$