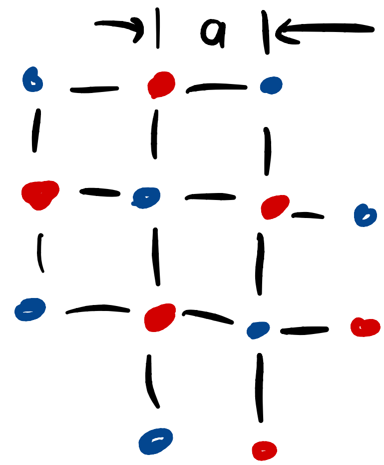
Last time: ferromagnet $\omega \sim k^2$ spin waves

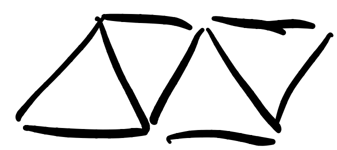
Today: Antiferromagnets $\supset$ Neel state.

on a bipartite lattice



$$\vec{S}(\vec{x}) = (-1)^{x} \vec{m}$$

$$\equiv \begin{cases} +1 & \text{blue} \\ -1 & \text{red} \end{cases}$$

NOT: 

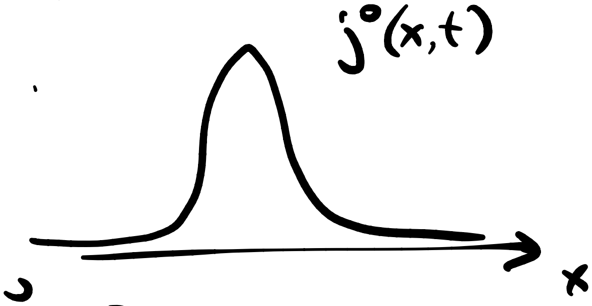
Let: $\vec{n}_j(t) = (-1)^j \vec{m}_j + a\,\vec{\ell}_j$

Neel vector ... ferro. fluctuation

$$n^2 = 1 \implies \vec{m}^2 = 1, \quad \vec{m}\cdot\vec{\ell} = 0.$$

why: a cognate of our analysis of S.F.

When asking "what are the low-energy dof?" the answer always includes conserved quantities.



$$\partial_t j^0 + \vec{\nabla}\cdot\vec{j} = 0$$

$\Rightarrow$ fluctuations of j^0 are always slow.

$\Rightarrow$

"hydrodynamics".

$$\sum_{\langle jj'\rangle} J\,\vec{S}_j\cdot\vec{S}_{j'}, \quad J>0 \rightsquigarrow S_J\left[n_j = (-1)^j m_j + a l_j\right]$$

$$\underset{aN \text{ fixed}}{\overset{a\to 0,\ N\to\infty}{=}} -aJs^2\int d^dx\,dt\left(\frac{1}{2}(\vec{\nabla} m)^2 + 2l^2\right)$$

$$\begin{cases} \vec{n}_i\cdot\vec{n}_j = \frac{1}{2}(n_i+n_j)^2 - 1 \\ \vec{n}_{2r} + \vec{n}_{2r-1} \simeq a\left(\partial_x m_{2r} + 2 l_{2r}\right) + O(a^2) \end{cases}$$

In $D=1+1$: $S_w[n] = 4\pi s \sum_{j=1}^{N} W_0\left[(-1)^j m_j + a l_j\right]$

$$\simeq \int dx\,dt \left(\frac{s}{2}\,\vec{m}\cdot\partial_t\vec{m}\times\partial_x\vec{m} + s\,\vec{l}\cdot(\vec{m}\times\partial_t\vec{m})\right) + \mathcal{O}(a^2)$$

Why: $\bullet$ $\delta W_0[n] = \frac{1}{4\pi}\int dt\, \delta\vec{n}\cdot(\vec{n}\times\partial_t\vec{n})$

$2r-1 \quad 2r$

$\cdots \bullet \quad \bullet \cdots$

$\bullet\ 4\pi W_0[\underbrace{n_{2r}}_{-n_{2r-1}+\Delta n}] + 4\pi W_0[n_{2r-1}] = 4\pi\left[W_0[-n_{2r-1}] + W_0[n_{2r-1}] + \int dt \frac{\delta W}{\delta n(t)}\Delta n(t)\right]$

$W_0(-n) = -W_0(n)$

$\Longrightarrow$ l is an auxiliary field (no $\dot{l}$ in $\mathcal{L}$)

$$0 = \frac{\delta S}{\delta \vec{l}} = -4aJs^2\vec{l} + s\,\vec{m}\times\dot{\vec{m}}$$

$$\Rightarrow S[m] = \int dx\, dt \left[\frac{1}{2g^2} \left(\frac{1}{v_s} (\partial_t \vec{m})^2 - v_s (\partial_x \vec{m})^2 \right) + \frac{\theta}{8\pi} \epsilon_{\mu\nu} \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m}) \right]$$

w/ $g^2 = \frac{2}{s}$, $v_s = 2aJs$, $\theta = 2\pi s$.

$$\Longrightarrow \quad 0 = \frac{\delta S}{\delta m} \propto (\omega^2 - v_s^2 k^2) \vec{m}.$$

2 linearly-dispersing Goldstones
($m^2 = 1$)

[actually: $\omega \sim |k - \pi/a|$.]

NLSM w/ target space S^2 + a θ term.

like $\int F \wedge F$

or $\int \dot{\phi}\, dt$

$\vec{m}$: spacetime $\longrightarrow S^2$

$\mathbb{R}^2 \cup \infty = S^2$

$\frac{1}{8\pi} \int m^a dm^b \wedge dm^c \epsilon^{abc} \in \underline{\underline{\mathbb{Z}}}$. $\underline{\text{winding \#}}$.

$$Z(\theta) = \int_{\text{on } S^2} [Dm]\, e^{iS} = \sum_{Q \in \mathbb{Z}} \int [Dm]_Q \, e^{i S_{\theta=0}[m]} \, e^{i\theta Q}$$

$$Z(\theta) = Z(\theta + 2\pi) \Longleftarrow \int \frac{m\, dm \wedge dm}{8\pi} = Q \in \mathbb{Z}.$$

<u>Haldane Conjecture</u>: when $2s$ is even the IR is gapped

when $2s$ is odd the IR is a CFT.

($SU(2)_1$ WZW model)

Pert. Thy in g:

$R = g^{-1} \sim$ size of the spin moment:

$$\mathcal{L} = \frac{1}{g^2} (d\vec{m})^2 = R^2 (d\vec{m})^2 = dM^2$$

w/ $M^2 = R^2 m^2 = R^2$.

weak-coupling $\leftrightarrow$ big sphere.

small curvature.

claim: $\frac{dg^2}{d\ell} = \beta_{g^2} = (D-2)g^2 + \underline{(n-2)K_D g^4} + O(g^5)$

$\boxed{\ell \to \infty \text{ is the IR.}}$

$\vec{m}^2 = \sum_{\alpha=1}^{n} m_\alpha^2 = 1.$

$K_D = \frac{\Omega_{D-1}}{(2\pi)^D}$ (loop factor).

- $(n-2) K_D g^4 \propto$ curvature of the target space
- for $n=2$ the target is S^1, which is flat. $ds^2 = d\varphi^2$.
- In $D=2$ $\beta > 0$ at small $g \Rightarrow$ ~~Asymptotic~~ freedom!

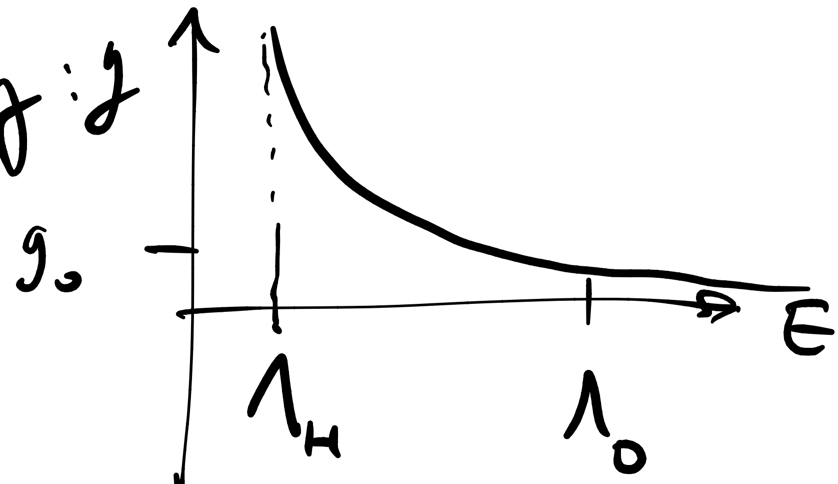
Antiscreening: short-wavelength fluctuations decrease the spin moment.

$$\left\langle \sum_j (-1)^j \vec{S}_j \right\rangle$$

$\vec{n} \to \lambda \vec{n}$ $\lambda < 1$

$g \to g/\lambda$ makes g bigger.

Infrared slavery:



↑ analog of Λ_{QCD}

'H' is for Haldane.

pert. thy $\Longrightarrow$

$$\Lambda_H \sim \Lambda_0 e^{-c/g_0^2}.$$

(dim'l transmutation)

other input: for $s \in \mathbb{Z}$ gap.
θ-term doesn't matter for bulk properties.

But: path integral w/ bdy cares about θ. either: $\Psi_{gs}[\vec{n}]$

or Z_{interval}

$s \in \mathbb{Z}_>$ this is an example of a SPT (symmetry-protected top.) phase.

signature: although the bulk is made of $SO(3)$ reps $(s \in \mathbb{Z})$, an edge hosts spin $\frac{\mathbb{Z}}{2}$ reps.

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for $s \in \mathbb{Z} + 1/2$: gapless bulk.

4.3 β-fn for 2d NLSM on S^{n-1}

$$S[n] = \int d^2x \; R^2 \, \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} = \int R^2 \, dn^2$$

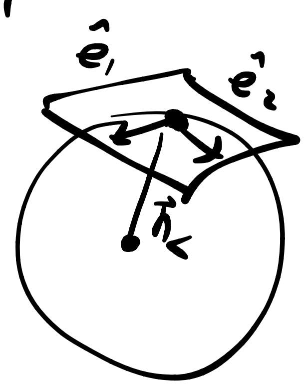
$$n^i(x) \equiv n^i_<(x) \sqrt{1-\phi_>^2} + \sum_{a=1}^{n-1} \phi_a^>(x) \, e_a^i(x)$$

slow

$n^2 = 1.$

Fast

$\vec{n}_<^{\,2} = 1.$



$$n_< \cdot \hat{e}_a = 0 \, , \quad \hat{e}_a \cdot \hat{e}_a = 1.$$

$$\phi_a^>(x) = \int_{\Lambda/s}^{\Lambda} dk \, e^{ikx} \phi_k$$

$$n_< \cdot dn_< = 0. \qquad n_< \cdot de_a + dn_< \cdot e_a = 0 \quad \forall a.$$

$$dn^i = dn^i_< (1-\phi^2)^{1/2} - n^i_< \frac{\phi \cdot d\phi}{\sqrt{1-\phi^2}}$$

$$+ d\phi \cdot e^i + \phi \cdot de^i$$

$$S = \int d^2x \, \mathcal{L} \qquad \mathcal{L} = \frac{1}{2g^2}(d\vec{n})^2$$

$$= \frac{1}{2g^2}\Big((dn_<^2)(1-\tfrac{1}{2}\phi^2) + \underline{d\phi^2} \qquad + O(\phi^3)$$

$$+ 2\phi_a d\phi_b \vec{e}_a \cdot d\vec{e}_b + \underbrace{d\phi_a dn_< \cdot e}_{\text{source for } \phi}$$

$$+ \phi_a \phi_b de_a \cdot de_b \Big)$$

$$e^{-S_{\text{eff}}[n^<]} = \int [D\phi_>]^{\Lambda}_{\Lambda/s} e^{-\int L}$$

$$= \int [D\phi_>]^{\Lambda}_{\Lambda/s} e^{-\frac{1}{2g^2}\int d\phi^2} (\ldots) = \langle \ldots \rangle_{Z_{>,0}}$$

$$L_{eff}[n^<] = \frac{1}{2g^2}(dn_<)^2\left(1 - \langle \phi^2\rangle_{>,0}\right.$$

$$\left. + \langle \phi_a \phi_b \rangle_{>,0}\, \underline{\underline{d\vec{e}_a \cdot d\vec{e}_b}} + \cdots \right)$$

$$\langle \phi_a \phi_b \rangle_{>,0} = \delta_{ab} \int_{\Lambda/s}^{\Lambda} \frac{d^2k}{k^2} = g^2 K_2 \log(s)\, \delta_{ab}$$

$$K_2 = 1/2\pi$$

$$d\vec{e}_a = \hat{n}_<\left(d\vec{e}_a \cdot \hat{n}_<\right) + \sum_{c=1}^{n-1} \vec{e}_c\left(d\vec{e}_a \cdot \vec{e}_c\right)$$

$$\Rightarrow de_a \cdot de_a = (dn_<)^2 + \sum_{c,a}(e_c \cdot de_a)^2$$

$$\Rightarrow L_{eff}[n] = \frac{1}{2g^2}(dn_<^2)\left(1 - ((N-1)-1)g^2 K_2 \log s + \ldots\right)$$

$$\simeq \frac{1}{2}\left(g^2 + \frac{g^4}{2\pi}(N-2)\log s + \ldots\right)^{-1} dn_<^2 + \ldots$$

Payoff: 2+1 d Neel state at $T>0$

spacetime = space $\times$ (S^1 of radius $1/T$)

$$Z = \int \mathcal{D}n \, e^{-\frac{1}{T}S}$$

"dimensional reduction" $\longrightarrow$ 2d NLSM w target S^2

w $\frac{1}{g^2} \propto \frac{1}{T}$.

$$\xi^{-1}(g) \sim \Lambda_H(g) \sim e^{-\#/g^2}$$

$$\xi^{-1}(T) \sim e^{-\#/T}.$$

4.3 $\mathbb{C}P^1$ representation & large N

$$n^a_{(\vec{x})} = z^\dagger_{(\vec{x})} \sigma^a z_{(\vec{x})} \qquad \sum_{\alpha=1}^{2} z^\dagger_\alpha z_\alpha = 1$$

(2 dofs) (3 dofs.)

<u>price</u>: (*) $z \to e^{i\chi(x)} z$ doesn't act on $\check{n}$.

$$S_B[\check{n}] = \int i z^\dagger \dot{z}$$

<u>Path Integral Method</u>: $\partial_\mu n^a \partial^\mu n^a =$

$$4\left(\partial_\mu z^\dagger \partial^\mu z - A_\mu A^\mu z^\dagger z\right)$$

$$\text{w/ } A_\mu = -\frac{i}{2}\left(z^\dagger \partial_\mu z - \partial_\mu z^\dagger z\right)$$

$$A_\mu \to A_\mu + \partial\chi \quad \text{under (*)}$$

To impose $z^\dagger z = 1$:

$$\delta[z^\dagger z - 1] = \int D\lambda \, e^{i\int d^Dx \, \lambda(x)\left(|z|^2 - 1\right)}$$

$$e^{c A_\mu \mathcal{A}^\mu} = \sqrt{\frac{c}{\pi}} \int dA_\mu \, e^{-c A_\mu^2 - 2c A_\mu \mathcal{A}^\mu}$$

$$A\big|_{\text{saddle}} = \mathcal{A} .$$

$$e^{-\# \int dn^2} = \int [dA] \, e^{-\# \int |(\partial - iA) z|^2}$$

- $\underline{\int d^3 n \, \delta(n^2 - 1)} = \dots = \text{cot} \underline{\int \prod_{\alpha=1}^{2} d^2 z_\alpha \, \delta(|z|^2 - 1)}$

$$\ni \int_0^{2\pi} d\chi = 2\pi$$

$$S^2 \cong \mathbb{C}P^1 = \underline{SU(2)} / \underline{U(1)}$$

$$z = R(\theta, \varphi, \chi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\in SU(2) \quad =$$

$$Z_{S^2} \cong \int [dz \, dz^\dagger dA \, d\lambda] \, e^{-\int d^D x \left(\frac{2\Lambda^{D-2}}{g^2} |(\partial - iA) z|^2 - i \lambda (|z|^2 - 1) \right)}$$

$$\text{w/ } \delta S = \int \frac{\lambda^2}{4K} \xrightarrow{\int [D\lambda]} V(|z|^2) = K (|z|^2 - 1)^2 .$$

$$Z_{S^2} = \int (dz\, dz^\dagger dA)\, e^{-\int d^Dx \left[\frac{2\Lambda^{\Delta-1}}{g^2} |(\partial - iA)z|^2 - K(|z|^2-1)^2\right]}$$

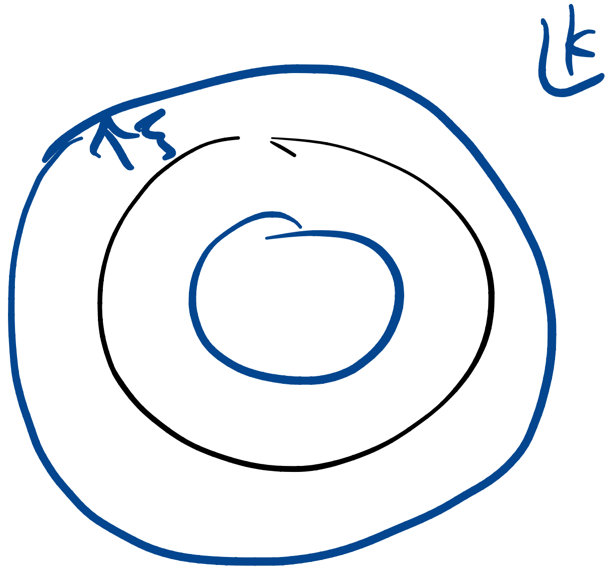
$$\vec{n} = z^\dagger \vec{\sigma} z$$ "parton ansatz"

"slave ~~particle~~ bosons"

gauge inv't kinetic terms for z require a gauge field A_μ.

$$V(z) = \text{cst} + |z|^2 + |z|^4 + \ldots$$

$$d^d k \sim \frac{d\xi \, d\Omega_{d-1}}{2\pi v_F} \qquad \xi = \frac{|k - k_F|}{v_F}$$



$$\sum_A (T^A)_\alpha{}^\beta (T^A)_\gamma{}^\delta = N \delta_\alpha^\delta \delta_\gamma^\beta - \delta_\alpha^\beta \delta_\gamma^\delta$$