Pions (conclusion)

$$S_\chi\left[U = e^{\frac{i 2 \pi^a(x) T^a}{F_\pi}}\right] = \int \left(\frac{1}{F_\pi^2} \mathrm{tr}\, \partial^\mu U \partial_\mu U^\dagger + \dots\right)$$

$$+ \underline{\underline{N_c}}\, W_3[U]$$

$N_c \in \mathbb{Z}$

WZW term is just like:

$$\ddot{x} = q B \dot{x}$$

$$\mathcal{L} = \int \frac{\dot{x}^2}{2} + q \int \underline{\underline{A_\mu(x)}}\, \dot{x}^\mu$$

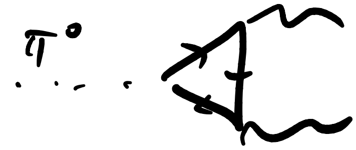
$$= \dots + q \oint_\gamma A$$

$$= \dots + q \int_D F$$

$\partial D = \gamma$.

To see that the coeff is N_c: couple to the photon by demanding gauge inv $U(1)$

$\rightsquigarrow \frac{N_c e^2}{16\pi^2}\pi^0 \, F\wedge F$



- $\mathcal{A}(\pi^+\pi^- \xrightarrow{W} K^0 K^+ K^-) = N_c \in \mathbb{Z}$.
- $\mathcal{W}$ breaks $(-1)^{N_b}$: $\pi \to -\pi$.

Solitons & Topological terms:

S_χ has a soliton sol'n

$U(\vec{x}, t) = U(\vec{x})$ const in time

$U(|x| \to \infty) = \mathbb{1}$

$$U : (\underbrace{\text{space} \cup \infty}_{\sim S^3}) \longrightarrow G/H$$

$$= SU(3)\times SU(3)/SU(3)$$

$$[U] \in \pi_3(G/H) = \mathbb{Z}$$

The integer is a winding #

$$\frac{1}{24\pi^2}\int_{\substack{\text{space}\\ \sim S^3}} \mathrm{tr}\, U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU \in \mathbb{Z}$$

generalization of :

$S^1 \to U(1)$

$\theta \mapsto e^{ni\theta} = u$

$\frac{1}{2\pi}\oint u^{-1}du = \frac{1}{2\pi}\int u^{-1}\frac{\partial u}{\partial\theta}d\theta$

$= n.$

<u>CLAIM</u> :

The baryon # symmetry current is :

$$B_\mu = \frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2}\,\mathrm{tr}\, U^{-1}\partial_\nu U\, U^{-1}\partial_\alpha U\, U^{-1}\partial_\beta U .$$

<u>Pf</u> : 1 Goldstone-Wilczek current :

start w/ QCD

do an axial transf param. by U.

2 Couple to $SU(2)_{EW}$ gauge fields

& demand $U(1)_B\, SU(2)_{EW}^2$ anomaly

Baryon # $= \int_{\text{space}} B_0 =$ winding #.

<u>claim #2</u>: If N_c is $\frac{\text{odd}}{\text{even}}$, this particle is a $\frac{\text{fermion}}{\text{boson}}$!

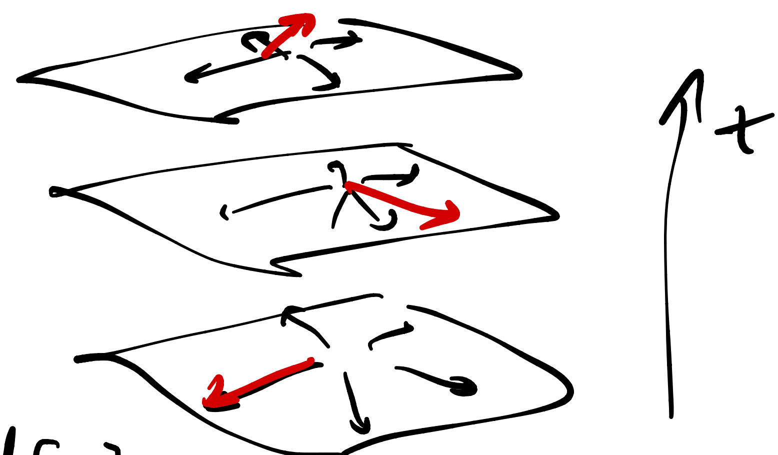
[Baryon in $SU(N_c)$ QCD is

$$B = \epsilon_{\alpha_1 \dots \alpha_{N_c}} q_{\alpha_1} \dots q_{\alpha_{N_c}}$$

is a boundstate of N_c fermions.]

$W_3\left[\begin{array}{l} U(\tilde{x},t) \text{ describe the baryon-\#-1 soliton} \\ \text{doing a } 2\pi \text{ rotation} \end{array}\right]$

$= \pi.$

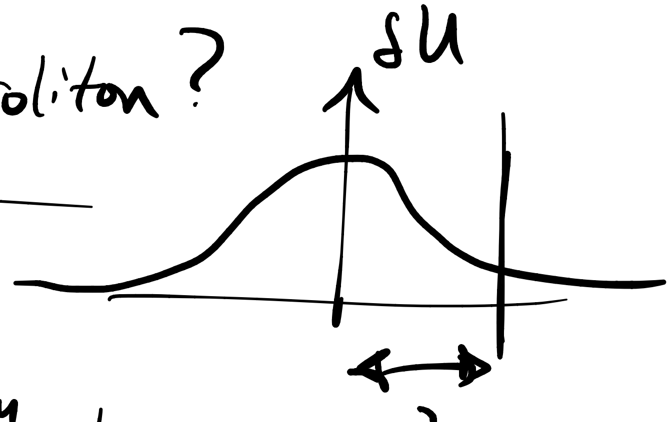


$\Rightarrow$ amplitude $\propto e^{i N_c W_3[\dots]} = (-1)^{N_c}.$

Conclusion: The soliton is a baryon!!

[Skyrme 1960] → "skyrmion".

Q: How big is the soliton?



A: depends on $L_{1,2}\ldots$ coeff of $(DU)^4$ terms.

Derrick's Thm: No stable (static) solitons in a scalar field thy w/ at most 2 derivs. in $D > 2+1$

$$E[\phi] = \int d^d x \left(g(\phi) (\nabla\phi)^2 + V(\phi) \right)$$

$$= I_2 + I_0$$

(e.g. $L = \mathrm{tr}(U^{-1} dU)^2$ has $I_0 = 0$.)

Suppose $\underline{\phi}$ is stationary.

$$0 = \left.\frac{\delta E}{\delta \phi}\right|_{\phi = \underline{\phi}} \qquad \underline{\phi}(x) \xrightarrow{x \to \infty} \phi_0 .$$

Consider a dilated config:

$$\phi_\lambda(x) \equiv \underline{\phi}(\lambda x) \qquad \lambda \in \mathbb{R}.$$

$$E[\phi_\lambda] = \frac{I_2[\phi]}{\underline{\underline{\lambda^{d-2}}}} + \frac{I_0[\phi]}{\lambda^d}$$

$$\Rightarrow \quad 0 = \partial_\lambda E[\phi_\lambda]\Big|_{\lambda=1} = (2-d) I_2 - d I_0$$

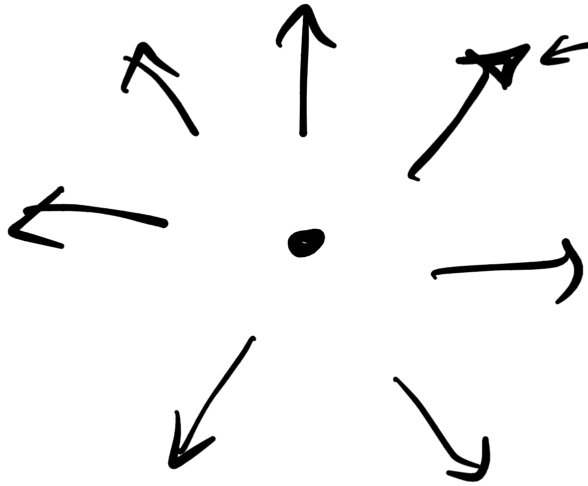
$$\Rightarrow I_0 = \frac{2-d}{d} I_2 \quad \circledast$$

$$\partial_\lambda^2 E[\phi_\lambda]\Big|_{\lambda=1} = (2-d)(1-d) I_2 + d(d+1) I_0$$

$$\overset{\circledast}{=} -2(d-2)\underbrace{I_2}_{\geq 0} < 0 .$$

$\left(\overset{!}{\geq} 0 \text{ for stability} \right)$

$\longrightarrow$ shrinks until the higher-deriv terms in $S_{eff}[\phi]$ matter.



$\text{Isospin} \times SO(3)_{ROT}$

$\longrightarrow SU(2)_{diag}.$

$\Longrightarrow$ Soliton has a collective coord which is an isospin rotor.

$$S \ni \int dt \, \frac{I\dot{\theta}^2}{2}$$

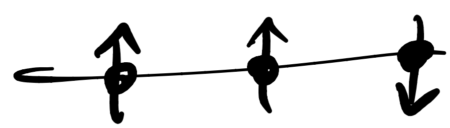
$\Longrightarrow$ solitons come in reps. of isospin $\underline{2}$

4 Field theory of spin systems

0. As a lattice discretization of any (bosonic) CFT.

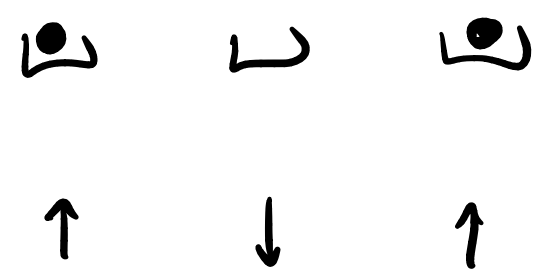
1. Spinful fermions (e^-) on a lattice "at half-filling" ← one fermion per site
with a charge gap.

i.e. magnetic insulator.



2. Spinless fermions hopping on a lattice
or hardcore bosons

→ 2 states per site.



hardcore ≡ states w 2 bosons cost too much energy

4.1 (ferro & antiferro) magnets

$D \geq 1+1$. Lattice of spins $s \in \mathbb{Z}/2$

$$H = H_{\text{Heisenberg}} = \sum_{\langle ij \rangle} J \, \vec{S}_i \cdot \vec{S}_j$$

preserves $SU(2)$ $\quad S^a_j \to \hat{R} S^a_j \hat{R}^{-1} \quad \forall j$

$$= R^a{}_b \, S^b_j$$

For $J<0$ this is ferromagnetic

wants to maximize total spin $\quad \langle \vec{S}_j \rangle \approx s\check{z}$.

$J>0$ this is antiferromagnetic

wants to MINIMIZE total spin

$\langle \vec{S}_j \rangle \sim (-1)^j s \check{z}$.

$\cdots \downarrow \uparrow \downarrow \cdots$

eg $\quad \vec{S}_1 \cdot \vec{S}_2$

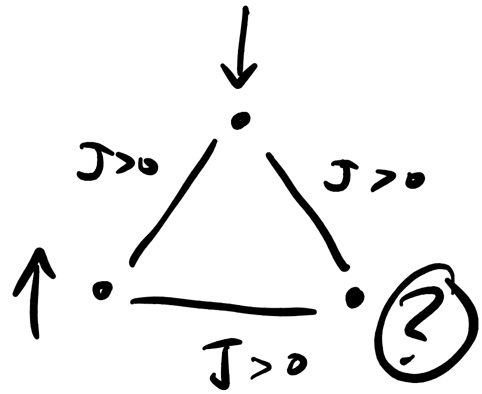
$$= \frac{1}{2}\left((\vec{S}_1 + \vec{S}_2)^2 - \text{const} \right)$$

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 0 \oplus 1$$

Q: Do the spins order?

A : Hard

eg: (1)



frustration.

maybe: $\langle \vec{S}_i \rangle = 0$

$\rightsquigarrow$ "spin liquid"

(2) In $D \leq 1+1$ Coleman Hohenberg Mermin Wagner

thm: can't break a continuous symmetry.

Pf: Imagine you did.

fluctuations of would-be goldstones

In $D=1+1$: $\langle \phi(x)\, \phi(0) \rangle \sim \log(x)$.

To study QFT in $D = d+1$ at finite T

make $\tau \sim \tau + 1/T$. $\longrightarrow$ QFT in d dims.

$\Rightarrow$ no continuous SSB at $T > 0$

in $D = 2+1$.

Write a field theory that respects the spin $SU(2)$ sym:

$$\mathcal{H} = \bigotimes_j \mathcal{H}_s^{(j)} = \text{span}\{ \ldots \otimes |\check{n}_j\rangle \otimes |\check{n}_{j'}\rangle \ldots \}$$

$$\check{n}_j \cdot \vec{S}_j |\check{n}_j\rangle = s|\check{n}_j\rangle$$

$\rightsquigarrow L = is \sum_j z_j^\dagger \partial_t z_j - J s^2 \sum_{\langle jj' \rangle} \check{n}_j \cdot \check{n}_{j'}$.

$(\check{n}_j = z_j^\dagger \vec{\sigma} z_j)$

Spin waves in ferromagnets: $\check{n} \sim \hat{z}$

$$\delta \check{n}_j = \check{n}_j - \check{z}$$

Roughly: $z^\dagger \dot{z} \quad - \sum_{\langle ij \rangle} n \cdot n$

$$\rightsquigarrow \quad \omega\, \delta n \quad - \quad k^2 \delta n = 0.$$

$$\longrightarrow \quad \omega \sim k^2.$$

$$\check{n}_j^2 = 1 \implies \delta n_z(k) = 0$$

$$\Downarrow$$

Pf: $\check{n}_j \cdot \delta \check{n}_j = 0 \quad \forall j$

$$\implies 0 = \sum_j e^{ikja}\, n_j \cdot \delta n_j$$

$$= \sum_j e^{ikja} (\check{z} + \delta n_j) \cdot \delta n_j$$

$$= \delta n_k^z + O(\delta n^2)$$

$$0 = \frac{\delta S}{\delta \check{n}_j(t)} = s\, \check{n}_j \times \dot{n}_j - s^2 \sum_{\langle j l \rangle} \check{n}_l$$

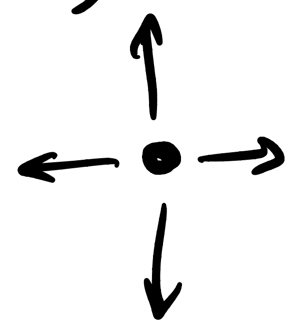
$$\frac{1}{s}\,\vec{n}_j \times (\text{BHS}) \Longrightarrow$$

$$0 = -\dot{\check{n}}_j + \check{n}_j \times \underline{\sum_{\langle jl \rangle} sJ\,\check{n}_l}$$

$\delta n^2 = 0$, FT $\Longrightarrow$

$$0 = \begin{pmatrix} h(k) & -\frac{i\omega}{2} \\ \frac{i\omega}{2} & h(k) \end{pmatrix} \begin{pmatrix} \delta n_x(k) \\ \delta n_y(k) \end{pmatrix} \quad *$$

$$h(k) = \sum_{\substack{\text{lattice} \\ \text{directions} \\ \check{e}}} \left(1 - e^{ika\check{e}}\right)$$



$$\overset{\text{square lattice}}{=} \underline{2 - 2\cos ka}$$

$$\overset{\text{small } k}{\sim} k^2$$

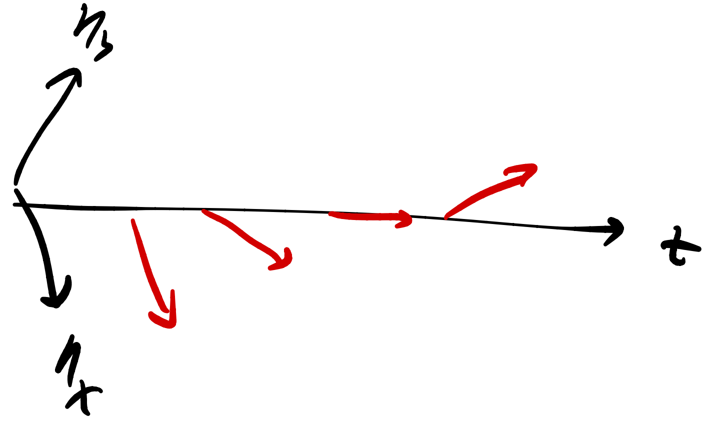
sol'ns of * have $\underline{\omega \sim k^2}$.

$z = 2$ dispersion

preserved by $\begin{cases} \omega \to \lambda^z \omega \\ k \to \lambda k \end{cases}$

$\delta n_x^{(1)} = i\,\delta n_y^{(1)}$ #

precession about $\hat{z}$.



Blackbody spin wave spectrum:

$F \propto L^d \qquad [F] = \text{energy} \qquad [T] = [L^{-z}]$

$$\Longrightarrow \quad F = c\, L^d\, T^{\frac{d+z}{z}} .$$

Ferromagnet is weird: The order param

$Q^z = \sum_i S_i^z$ is conserved

$[H, Q^z] = 0.$

groundstate breaks two symmetries Q^y, Q^x $\qquad [Q^x, Q^y] = iQ^z$

but only one goldstone w/ $z=2$.

<u>Next</u>: AFM

$\underline{\omega \sim k}$.

as if $\mathcal{L} = \dot{n}^2 - (\nabla n)^2$