Coherent-State Path Integral for Spin Systems

$\mathcal{H}_{\frac{1}{2}} = \text{span}\{|\uparrow\rangle, |\downarrow\rangle\}$

<u>Define</u>: spin coherent state $|\check{n}\rangle$: $\quad \check{n}\cdot\check{n} = 1$

$$\vec{\sigma}\cdot\check{n}\,|\check{n}\rangle = |\check{n}\rangle$$

$$\left(\ \underline{\text{note}}:\quad \vec{\sigma}\cdot\check{n}\,|-\check{n}\rangle = -|-\check{n}\rangle\ \right)$$

$$|\check{n}\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle \qquad \vec{\sigma}\cdot\check{n} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{i\psi/2}\begin{pmatrix} e^{i\varphi/2}\cos\theta/2 \\ e^{-i\varphi/2}\sin\theta/2 \end{pmatrix}$$

$$\check{n} \equiv (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

<u>Note</u>:
- $|z_1|^2 + |z_2|^2 = 1$.
- $\underline{\underline{\check{n} = z^\dagger\vec{\sigma} z}}$. (check)
- ψ doesn't affect $\check{n}$.

This is the Hopf map: $S^3 \longrightarrow S^2$ ← Bloch sphere.

$$\underbrace{\{(z_1,z_2)\ |\ |z_1|^2+|z_2|^2=1\}}_{S^3} \longrightarrow \underbrace{\{(z_1,z_2)\ |\ |z_1|^2+|z_2|^2=1\}/(z_\alpha \sim e^{i\chi}z_\alpha)}_{S^2\,=\,\{\check{n}\}}.$$

The states $|\check{n}\rangle$ are <u>not</u> orthogonal:

- $\langle \check{n} | \check{n}' \rangle = z^\dagger z' = (z_1^\dagger, z_2^\dagger)\begin{pmatrix} z_1' \\ z_2' \end{pmatrix}$
- $\mathbb{1}_{2\times 2} = \int \frac{d^2\check{n}}{2\pi} |\check{n}\rangle\langle\check{n}|$ $\ast$

"overcompleteness rel'n"

<u>Comment</u>: for general spin $s > 1/2$

$$\vec{S}\cdot\check{n}\, |\check{n}\rangle = s\, |\check{n}\rangle$$ maximal-spin eigenstate along $\check{n}$.

(for $s = 1/2$, $\vec{S} = \frac{\vec{\sigma}}{2}$.)

Use to make path integral:

$$iG(\vec{n}_f, \vec{n}_0, t) \equiv \langle \check{n}_f | e^{-iHt} | \check{n}_0 \rangle$$

eg: $H = 0$.
$\Rightarrow U = \mathbb{1}$

$$= \int \prod_{\ell=1}^{M = t/\Delta t} \frac{d^2\check{n}(t_\ell)}{2\pi} \langle \check{n}_f | \check{n}(t_M)\rangle\langle \check{n}(t_M) | \check{n}(t_{M-1})\rangle \cdots \langle \check{n}(t_1) | n(0)\rangle$$

$\check{n}_f = \check{n}(t)$
$\check{n}_0 = \check{n}(0)$

$$\langle \check{n}(t+dt) | \check{n}(t)\rangle = z^\dagger(t+dt)\, z(t)$$

$$= 1 - z^\dagger(t+dt)\big(z(t+dt) - z(t)\big)$$

$$\langle \check{n}(t+dt) | \check{n}(t) \rangle \approx \exp\left(-z^\dagger \partial_t z \, dt\right)$$

$$\Rightarrow iG(\check{n}_f, \check{n}_0, t) = \left[\int_{\check{n}(0)=\check{n}_0}^{\check{n}(t)=\check{n}_f} [D\check{n}] \, e^{i S_B[\check{n}(t)]} \right]$$

w/ $$\underline{S_B[\check{n}] = \int_0^t dt \; i z_\alpha^\dagger \dot{z}_\alpha}$$

Berry phase term. $\pi_z = i z^\dagger$.

S is 1st order in time derivs $\Rightarrow H = 0$.

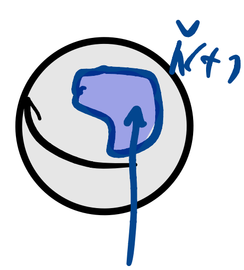
The phase space $= S^2 = \{ (z_1, z_2) \,|\, |z_1|^2 + |z_2|^2 = 1 \}$

gauge redundancy $\longrightarrow$ $z_\alpha(t) \sim e^{i\chi(t)} z_\alpha(t)$

$$S_B[\check{n}] = S_B[\theta(t), \varphi(t)] = \int dt \; \frac{1}{2}\left(\cos\theta \dot{\varphi} + \dot{\psi}\right)$$

choose $\psi = 0$

$$= 4\pi s \, W_0[\check{n}] \Big|_{s=1/2}$$



$=$ area swept out by the trajectory $\check{n}$.

$A_t \equiv z_\alpha^\dagger \partial_t z_\alpha$ is like the time component of a gauge field.

$$\begin{cases} z \to e^{i\chi(t)} \\ A \to A + i\, d\chi \end{cases}$$

why "Berry"? $S_B[\check{n}]$ is geometric:

depends only on $\{\check{n}(t)\}$ and not on the parametrization.

$$S_B[\check{n}] = \int dt\, z^\dagger \partial_t z = \int_{\substack{\text{path} \\ \text{in } S^2}} z^\dagger dz$$

= The phase acquired by a spin follows the instantaneous g.s. $|\Psi_0(t)\rangle$ of

$$H(\check{n}(t)) \equiv -h\, \check{n}(t) \cdot \vec{S} \qquad h>0.$$

$$S_B[n] = -\lim_{\left(\substack{\text{arbitrary} \\ \text{slowness}}\right)} \int dt\, \mathrm{Im} \langle \Psi_0(t) | \partial_t | \Psi_0(t) \rangle.$$

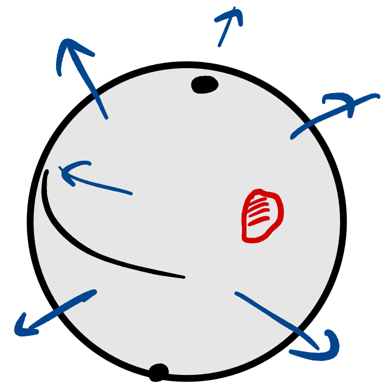
Ex: Find A_μ on S^2 s.t.

$$S_B[\check{n}] = \oint dt\, \dot{n}_a A^a = \oint_\gamma A \overset{\text{if } \gamma = \partial D}{=} \int_D F$$

$\gamma = \{\check{n}(t)\}$ $\qquad F = dA.$ $\qquad$ = magnetic flux coming out of $S^2 \subset \mathbb{R}^3$

$$\vec{\nabla} \times \vec{A} \cdot \check{n} = \epsilon^{abc} \partial_{n^a} A^b n^c = 1$$

= magn. field of a magn. monopole at $\vec{r} = 0$.



$$A^{(1)} = -\cos\theta\, d\varphi \quad \text{OR}$$

$$A^{(S)} = (1 - \cos\theta)\, d\varphi \quad \text{OR}$$

$$A^{(N)} = (-1 - \cos\theta)\, d\varphi.$$

differ by $A^{(1)} = A^{(2)} + d\chi$.

[Another physical realization: a charged particle of mass m restricted to $S^2 \subset \mathbb{R}^3$ w/ a magnetic monopole of magn. charge $2S$ in the center. w/ $m \to 0$.

$$\frac{eB}{mc} = \omega_c$$

keeps only the lowest Landau level.

$\Longrightarrow 2S+1$ states
$\in 2\mathbb{Z}$

- CLAIM: For spin $\frac{1}{2} \to$ spin s $\quad S_B \longrightarrow s S_B$.

- Suppose $\underline{\underline{\hat{H} = -\vec{h}\cdot\vec{S}}}$

$$S \rightsquigarrow S_B + S_h \qquad S_h = \int dt \; s\, \vec{h}\cdot\check{n}.$$

- Deep statement: WZW or Berry phase term enforces the commutation rel'ns

$$[S^i, S^j] = i\epsilon^{ijk} S^k .$$

Application: semiclassical spectrum.

$$G(n_t, n_0; E) \equiv -i\int_0^\infty dt\, \underline{\underline{G(n_t, n_0, t)}}\, e^{i(E+i\epsilon)t}$$

let $\Gamma(E) \equiv \int \frac{d^2 n_0}{2\pi} \underline{\underline{G(n_0, n_0; E)}}$

$$\boxed{\begin{array}{l} \mathrm{tr}(\dots) \\ = \int \frac{d^2 n_0}{2\pi} \langle \check{n}_0 | \dots | \check{n}_0\rangle \end{array}} = -i\int_0^\infty dt\, e^{i(E+i\epsilon)t} \underbrace{\int \frac{d^2 n_0}{2\pi} \langle n_0 | e^{-iHt} | n_0\rangle}_{\mathrm{tr}\, e^{-iHt}}$$

$$= \mathrm{tr}\frac{1}{E - H + i\epsilon}. \qquad \text{"Resolvent of } H\text{"}$$

$$\frac{1}{\pi}\mathrm{Im}\,\Gamma(E) = \sum_\alpha \delta(E - E_\alpha) = \rho(\epsilon) \quad \text{density of states!}$$

$$\vec{\Gamma(E)} = -i \int dt \oint D\check{n}\, e^{-i((E+i\epsilon)t + s\, S[\check{n}])}$$

$\oint$ means PBC.

$$S[\check{n}] = S_B[n] - \int^t dt' \frac{H_{cl}[n]}{s}$$

$$H_{cl}(\check{n}) \equiv \langle \check{n} | \hat{H} | \check{n} \rangle.$$

At large s we can use stationary phase:

- $0 = \frac{\delta S}{\delta \check{n}(t)} \propto \dot{n} \times n - \partial_n H_{cl}$

Keep classical sol'ns w/ period $t = nT$

T: basic period of the sol'n.

if $n=1$ the orbit is "prime"

- $0 = \frac{\partial}{\partial t}(\exp)$
$= E + \partial_t S[n]$
$= E - H_{cl}[n]$

S_B is geometric

Configs that contribute are ~~fixed~~ periodic sol'ns $\check{n}^E$ to EOM w/ energy $E = H_{cl}[n^E]$

$$\Rightarrow \Gamma(E) \sim \sum_{\substack{\text{prime orbits}\\ \check{n}_1^E}} \sum_{m=0}^{\infty} e^{i m s S_B[\check{n}_1^E]}$$

$$= \sum_{n_1^E} \frac{e^{i s S_B[n_1^E]}}{1 - e^{i s S_B[n_1^E]}} \leftarrow$$

[Gutzwiller trace formula]

locations of poles of $\Gamma(E)$ for real E

= eigenvalues of H.

poles at : $\boxed{S_B[\check{n}_1^{E_k}] = \frac{2\pi}{s} k}$ $k \in \mathbb{Z}$

evals of H are $E^k = E^k_{sc} + O(\frac{1}{s})$

eg: 1d particle in a potential. $\begin{cases} H_{cl} = p^2 + V(x) \\ S_B = \int p\, dx \end{cases}$

$$2\pi k = \overbrace{\oint_{x^{E_k}} p(x)\, dx}^{\text{Berry phase term!}} = 2\int_{\text{turning pt}}^{\text{turning pt'}} \sqrt{E_k - V(x)}\, dx$$

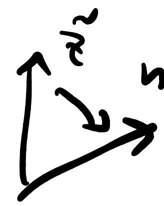
(Bohr-Sommerfeld quantization)

Pf of claim that $S_B^{(s)}[\check{n}] = 2s\, S_B^{(s=1/2)}[\check{n}]$.

$|\check{n}\rangle$ is the evec of max $\underline{\vec{S}\cdot\check{n}}$.

$$|\check{n}\rangle = R(\underline{\chi,\theta,\varphi})\,|s,s\rangle$$

$\nwarrow$ max eval of S^z



<u>Schwinger Bosons</u>. <u>two</u> SHOs
$$\begin{cases} [a,a^\dagger] = 1 = [b,b^\dagger] \\ [a,b] = 0 = [a,b^\dagger] \end{cases}$$

$$\boxed{S^+ = \underline{a^\dagger b} \qquad S^- = \underline{b^\dagger a} \qquad S^z = \underline{\tfrac{1}{2}(a^\dagger a - b^\dagger b)}} \quad *$$

satisfy the SU(2) algebra

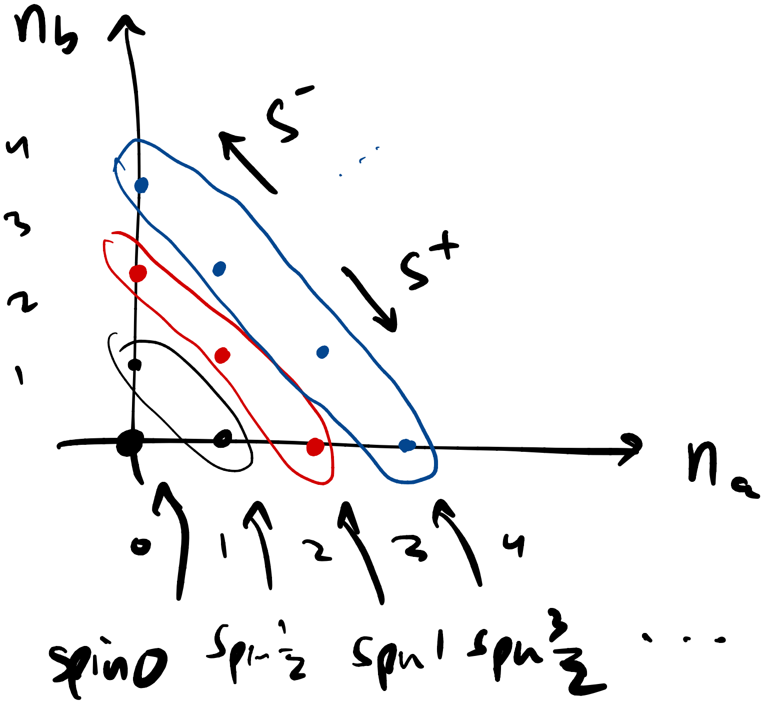
$$\begin{cases} [S^+, S^-] \propto i S^z \\ [S^\pm, S^z] = \pm i S^z \end{cases}$$

$|0\rangle$ is a singlet.

$$\begin{pmatrix} a^\dagger|0\rangle \\ b^\dagger|0\rangle \end{pmatrix} = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}|0\rangle$$ is a doublet.

$$\mathcal{H}_s \equiv \text{span}\{|n_a, n_b\rangle \mid n_a + n_b = 2s\}$$ form a spin-s rep. of SU(2)

$$a^\dagger a |n_a n_b\rangle = n_a |n_a n_b\rangle \ldots$$



$$\vec{S}^2 P_s = s(s+1) P_s$$

$$|s,m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} |0\rangle$$

$$|s,s\rangle = \frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}} |0\rangle$$

$\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ is a doublet.

$$\mathbb{1} = R^\dagger R$$

$$|\vec{n}\rangle = R|s,s\rangle = R \frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}} |0\rangle$$

$$\left(R \frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}} R^\dagger \right) \underbrace{R|0\rangle}_{|0\rangle} = \frac{(a^{\dagger\prime})^{2s}}{\sqrt{(2s)!}} |0\rangle$$

$$= \frac{(z_1 a^\dagger + z_2 b^\dagger)^{2s}}{\sqrt{(2s)!}} |0\rangle .$$

$z_{1,2}$ as above.

$$\langle \check{n} | \check{n}' \rangle = \frac{1}{(2s)!} \underbrace{\langle 0 | (z_1^* a + z_2^* b)^{2s} \, (z_1' a^+ + z_2' b^+)^{2s} | 0 \rangle}$$

$$\underset{\text{wick}}{=} (2s)! \left([z_1^* a + z_2^* b, \; z_1' a^+ + z_2' b^+] \right)^{2s}$$

$$= (z_1^* z_1' + z_2^* z_2')^{2s}$$

$$= (\underbrace{z^+ \cdot z'}_{\text{spin } \frac{1}{2} \text{ answer.}})^{2s}$$

$$\Rightarrow \; S_B^{(s)}[n] = 2s \, S_B^{(\frac{1}{2})}(n) = 4\pi s \, W_0[n].$$

$\blacksquare$

3.4 Topological terms from Integrating out Fermions [Abanov-Wiegmann]

Consider $D = 0+1$ ~ spinful fermion c_α $\alpha = \uparrow, \downarrow$.

coupled to a spin s, $\vec{S}$.

w/ $H_K = M \, (c^\dagger_\alpha \vec{\sigma}_{\alpha\beta} c_\beta) \cdot \vec{S}$

SU(2) symmetric   'K' is for Kondo.

$M > 0$ is an antifferomagnetic interaction

$M < 0$ " " fero " ".

$$Z = \int [D\psi D\bar{\psi} D\check{n}] \, e^{-S_0[\check{n}] - \int_0^T dt \, \bar{\psi}(\partial_t - M\check{n}\cdot\vec{\sigma})\psi}$$

$$S_0[n] = 4\pi s \, W_0[\check{n}].$$

$$\int [D\phi] \cdots$$

$$= \sum_{Q \in \mathbb{Z}} \int [D\phi]_Q \cdots$$

$$\sim \prod_\ell d\phi_\ell^{(Q)}$$

$\vee \quad \mathbb{1}_{\mathcal{H}} = \mathbb{1}_a \otimes \mathbb{1}_b$

on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$

$$\{c_a , c_b\} = 0$$

$$\Longrightarrow \underline{\{\psi_a , \psi_b\} = 0} \, .$$

$$c_a |\psi\rangle = \psi_a |\psi\rangle$$