3. Geometric & Topological terms in QFT actions

$$\text{operator or Hamiltonian} \;\underset{\text{resolution of } \mathbb{1}.}{\overset{\text{canonical quantization 215A}}{\rightleftarrows}}\; \text{path integral or Lagrangian}$$

- non-unique: each choice of basis for $\mathcal{H}_x$ $\longleftrightarrow$ different path integral.

3. Coherent state path integral for bosons.

Bosons: $\mathcal{H} = \bigotimes_k \mathcal{H}_k$ k labels a boson **mode**

$$\mathcal{H}_k = \text{span}\left\{ |0\rangle_k ,\ a_k^\dagger |0\rangle_k ,\ \frac{(a_k^\dagger)^2}{\sqrt{2!}} |0\rangle \dots \right\}$$

$$= \text{span}\left\{ |n_k\rangle_k \quad n_k = 0, 1, 2 \dots \right\}$$

is an SHO Hilbert space.

$$[a_{\tilde{k}}, a^{\dagger}_{\tilde{k}'}] = \delta^d(k-k')$$

eg: $H = H_0 \equiv \sum_k (\epsilon_{\tilde{k}} - \underline{\mu})\, a^{\dagger}_k a_k$

$\epsilon_k - \mu$ is energy of $a_k^{\dagger}|0\rangle$.

μ shifts the energy by an amount $\propto$

$$\left\langle \sum_k a^{\dagger}_k a_k \right\rangle = N \qquad \text{\# of bosons.}$$

or $H = H_0 + V$

$$V = \sum_{xy} V_{xy}\, a^{\dagger}_x a^{\dagger}_y a_y a_x$$

$$= \sum_{xy} n_x V_{xy} n_y$$

$$\left(a_x \equiv \int d^dk\, e^{ikx} a_k \right)$$

+ quadratic terms

For one mode a coherent state is $|\phi\rangle$

$a|\phi\rangle = \phi|\phi\rangle$

$\langle\phi| a^{\dagger} = \langle\phi|\phi^*$

$$|\phi\rangle = N e^{a^{\dagger}\phi}|0\rangle$$

$$= N \sum_{n=0}^{\infty} \phi^n \frac{(a^{\dagger})^n}{n!}|0\rangle$$

$$\langle \phi | = \langle 0 | e^{+\phi^* a} N \longleftarrow$$

$$\langle \phi_1 | \phi_2 \rangle = e^{\phi_1^* \phi_2} |N|^2$$

<u>choose $N = 1$.</u>

$$\mathcal{H}_a = \text{span}\{ |\phi\rangle , \phi \in \mathbb{C} \}$$

but it is over complete:

- $$\mathbb{1}_a = \sum_{n=0}^{\infty} |n\rangle\langle n| = \int \frac{d\phi d\phi^*}{\pi} e^{-|\phi|^2} |\phi\rangle\langle\phi|$$

- $$\operatorname{tr} A = \int \frac{d\phi d\phi^*}{\pi} e^{-|\phi|^2} \langle \phi | A | \phi \rangle$$

- assume H is normal-ordered

$$: a_k a_\ell^\dagger : = : a_\ell^\dagger a_k : = a_\ell^\dagger a_k .$$

$$\langle \phi' | \prod_k (a_k^\dagger)^{M_k} (a_k)^{N_k} | \phi \rangle = \prod_k (\phi_k'^*)^{M_k} (\phi_k)^{N_k} .$$

First: a single mode

$$Z = \mathrm{tr}_{\mathcal{H}_a} e^{-H/T}$$

$$= \underline{\underline{\int \frac{d\phi\, d\phi^*}{\pi}}} e^{-|\phi|^2} \langle \phi | \quad e^{-H/T} \quad | \phi \rangle$$

$$e^{-\Delta\tau H} \wedge e^{-\Delta\tau H} \cdots \wedge e^{-\Delta\tau H}$$

$$\underline{\mathbb{1} = \int \frac{d^2\phi_\ell}{\pi} e^{-|\phi_\ell|^2} |\phi_\ell\rangle\langle\phi_\ell|}$$

$$\underline{\underline{M \Delta\tau = \frac{1}{T}}}.$$

$$\downarrow$$

$$= \int \prod_{\ell=0}^{M-1} \frac{d^2\phi_\ell}{\pi} \left(e^{-|\phi_\ell|^2} \underbrace{\langle \phi_{\ell+1} | e^{-\Delta\tau H} | \phi_\ell \rangle} \right)$$

tr $\Rightarrow$ periodic bcs in "imaginary time": $\underline{\phi_M = \phi_0}$.

$$\langle \phi_{\ell+1} | e^{-\Delta\tau H} | \phi_\ell \rangle = \langle \phi_{\ell+1} | \left(1 - \Delta\tau H(a^\dagger, a)\right) | \phi_\ell \rangle + O(\Delta\tau^2)$$

$$= \langle \phi_{\ell+1} | \left(1 \underbrace{- \Delta\tau H(\phi^*_{\ell+1}, \phi_\ell)}_{\#}\right) | \phi_\ell \rangle + O(\Delta\tau^2)$$

$$= e^{-\phi^*_{\ell+1}\phi_\ell} \left(1 - \Delta\tau H(\phi^*_{\ell+1}, \phi_\ell)\right) + O(\Delta\tau^2)$$

$$\langle \phi_{l+1} | e^{-\Delta\tau H} | \phi_l \rangle = e^{-\phi^*_{l+1}\phi_l - \Delta\tau H(\phi^*_{l+1}, \phi_l)} + O(\Delta\tau^2)$$

$$\Longrightarrow$$

$$Z = \int_{\phi_M = \phi_0} \underbrace{\prod_{l=0}^{M-1} \frac{d^2\phi_l}{\pi}}_{= D^2\phi} e^{-\sum_{l=0}^{M-1} \left[\phi^*_{l+1}(\phi_{l+1} - \phi_l) - \Delta\tau H(\phi^*_{l+1}, \phi_l)\right]}$$

$$= \int_{\phi(0) = \phi(1/T)} [D\phi] \, e^{-\int_0^{1/T} d\tau \left(\phi^* \partial_\tau \phi - H(\phi^*, \phi)\right)}$$

$$\left\{ \begin{array}{ll} \phi(\Delta\tau l) \equiv \phi_l & \text{defining the continuum field.} \\ \phi_{l+1} - \phi_l = \Delta\tau \, \partial_\tau \phi + O(\Delta\tau^2) & \\ \Delta\tau \sum_{l=0}^{M} = \int_0^{1/T} d\tau & \end{array} \right.$$

<u>Note</u>: $Z = \int Dx \, e^{-\int_0^{1/T} d\tau \left((\partial_\tau x)^2 - V(x)\right)}$

Many modes:

$$Z = \int \underline{\underline{[D^2 a]}}\, e^{\int d\tau \sum_k \left[\frac{1}{2}(a_k^* \dot{a}_k - a_k \dot{a}_k^*) - (\epsilon_k - \mu) a_k^* a_k \right] + V(a^*, a)}$$

$$a_{\tilde{k}} = \int d^{D-1}x\, e^{i\tilde{k}\cdot\tilde{x}}\, \Phi(x)$$

Choose : $\epsilon_k - \mu = -\mu + \frac{k^2}{2m}$

$$\Rightarrow Z = \int [D^2\Phi]\, e^{\int d^d x\, d\tau \left[\frac{1}{2}(\Phi^* \partial_\tau \Phi - \Phi \partial_\tau \Phi^*) - \frac{\vec{\nabla}\Phi \cdot \vec{\nabla}\Phi}{2m} + V(\Phi) \right]}$$

<u>note</u>: $\int d\tau\, \Phi^* \partial_\tau \Phi \overset{IBP}{=} \int d\tau\, \frac{1}{2}(\Phi^* \partial_\tau \Phi - \partial_\tau \Phi^* \Phi)$.

is called the Berry phase term.

<u>Real Time</u> :

$$\langle \Phi_f, t_f | e^{-itH} | \Phi_0, t_0 \rangle = \int_{\Phi(t_0)=\Phi_0}^{\Phi(t_f)=\Phi_f} [D^2\Phi]\, e^{\frac{i}{\hbar}\int_{t_0}^{t_f} dt \left[i\hbar \Phi^* \partial_t \Phi - H(\Phi^*, \Phi) \right]}$$

Notice: $S_{\substack{\text{Real}\\\text{time}}}[\Phi] = i\hbar\, \Phi^* \partial_t \Phi - H(\Phi^*, \Phi)$

!!

Berry phase term is: • complex in real time

• geometric. given some history $\Phi(t)$

$$\int dt\, \Phi^*(t)\, \dot{\Phi}(t) = \int_{\substack{\text{image}\\\text{of } \Phi(t)}} \Phi^* d\Phi$$

index of parametrization of the trajectory.

Like: $\int A_\mu(x(t)) \frac{dx^\mu}{dt} = \int_{\text{path}} A_\mu dx^\mu$

$= \int A$

$$\mathcal{L}(\Phi) = i\Phi^* \partial_t \Phi - V(\Phi^*, \Phi)$$

$\underline{eg}$: $V(\Phi^*, \Phi) = \int dx\, \mu\, \Phi^* \Phi(x) +$

$$\int dx \int dy\; \Phi^* \Phi(x)\, V_{xy}\, \Phi^* \Phi(y)$$

this is the NR. SF action!

Φ^* is the eigenvalue of a^+!

if $\langle \Phi \rangle \neq 0$ $\underline{\text{bosons are condensed}}$

$\equiv \langle a^+ \rangle \neq 0.$

$\boxed{i\epsilon \text{ prescription:}}$

$$\tau = e^{i\left(\frac{\pi}{2} - \epsilon\right)} t$$

3.2 Coherent state path integral for fermions

mode: $\{c_i, c_j\} = 0$, $\{c_i^+, c_j^+\} = 0$, $\{c_i, c_j^+\} = 1.\delta_{ij}$

one mode: represented by

$$\mathcal{H} = \text{span}\{|0\rangle, |1\rangle = c^\dagger|0\rangle\}$$

most general $H = c^\dagger c\,\underline{(\omega_0 - \mu)} + \cancel{c_0^\dagger}$ ^{0}

$$Z = \text{tr}\, e^{-H/T} = \langle 0| e^{-H/T} |0\rangle + \langle 1| e^{-H/T} |0\rangle = 1 + e^{-\frac{(\omega_0-\mu)}{T}}$$

but: $H = \sum_k c_k^\dagger c_k (\omega_k - \mu) + \sum c_x^\dagger c_y^\dagger V_{xy} c_x c_y$

normal ordered

Coherent states for fermionic ops:

$$c|\psi\rangle = \psi|\psi\rangle \qquad \langle\psi|c^\dagger = \langle\psi|\psi^*$$

$$c(c|\psi\rangle) = \psi^2|\psi\rangle \Rightarrow \psi^2 = 0$$

$$= c^2|\psi\rangle = 0.$$

ψ is a grassmann #!

$$\psi_1 \psi_2 = - \psi_2 \psi_1. \qquad \psi c = - c \psi.$$

$$\psi \, 3 = 3 \psi \qquad \psi c^\dagger c = c^\dagger c \psi.$$

$$\{ c_1, c_2 \} = 0 \Rightarrow \{ \psi_1, \psi_2 \} = 0$$

$$c_1 | \psi_1, \psi_2 \rangle = \psi_1 | \psi_1 \psi_2 \rangle$$

$$c_2 | \psi_1 \psi_2 \rangle = \psi_2 | \psi_1, \psi_2 \rangle$$

$[c_1, c_2] \neq 0$ is ok because the <u>evals</u> anticommute.

$$|\psi\rangle = |0\rangle - \psi |1\rangle = |0\rangle - \psi c^\dagger |0\rangle$$

(same as bosonic coherent state!) $\quad = e^{-\psi c^\dagger} |0\rangle.$

$$c|\psi\rangle = \underbrace{c|0\rangle}_{\to 0} - c \psi |1\rangle = + \psi c |1\rangle = \psi |0\rangle \checkmark$$

$$\psi |\psi\rangle = \psi (|0\rangle - \psi |1\rangle) \underset{\psi^2 = 0.}{=} \psi |0\rangle$$

$$\langle \bar\psi | c^\dagger = \langle \bar\psi | \bar\psi$$

$$\langle \bar\psi | = \langle 0 | - \langle 1 | \bar\psi = \langle 0 | + \bar\psi \langle 1 |$$

$|\psi\rangle \in$ <u>enlarged Hilbert space</u>

$$\bar\psi \neq (\psi)^*$$

$$\langle \bar\psi | \psi \rangle = 1 + \bar\psi\psi = e^{\bar\psi\psi}.$$

- $$\mathbb{1} = \int d\bar\psi\, d\psi\, e^{-\bar\psi\psi} |\psi\rangle\langle\bar\psi|$$
- $$\mathrm{tr}\, A = \int d\bar\psi\, d\psi\, e^{-\bar\psi\psi} \langle -\bar\psi | A | \psi\rangle$$

(A is grassman even)

$$Z = \mathrm{tr}\, e^{-\beta H} = \int d\bar\psi_0\, d\psi_0\, e^{-\bar\psi_0\psi_0} \langle -\bar\psi_0 | \underbrace{e^{-H/\!_{4}}}_{} |\psi_0\rangle$$

$$M\Delta\tau = \frac{1}{T}. \qquad (1 - \Delta\tau H)\ldots\ldots$$

$$Z = \int \prod_{l=0}^{M-1} \left(d\bar{\psi}_l d\psi_l \; e^{-\bar{\psi}_l \psi_l} \langle \bar{\psi}_{l+1} | (1 - \Delta\tau H) | \psi_l \rangle \right)$$

$$\bar{\psi}_M = -\bar{\psi}_0 \qquad \psi_M = -\psi_0 .$$

ANTI PERIODIC B.C.'s.

so that $\langle -\bar{\psi}_0 | = \langle \bar{\psi}_M |$

$$\langle \bar{\psi}_{l+1} | (1 - \Delta\tau H(c^\dagger, c)) | \psi_l \rangle$$

$$= \langle \bar{\psi}_{l+1} | (1 - \Delta\tau H(\bar{\psi}_{l+1}, \psi_l)) | \psi_l \rangle$$

$$\simeq e^{\bar{\psi}_{l+1} \psi_l - \Delta\tau H(\bar{\psi}_{l+1}, \psi_l)}$$

$$Z = \int \prod_{l=0}^{M-1} \left(d\bar{\psi}_l d\psi_l \; e^{-\bar{\psi}_l \psi_l + \bar{\psi}_{l+1} \psi_l - \Delta\tau H(\bar{\psi}_{l+1}, \psi_l)} \right)$$

$$= \int \prod_l d\bar{\psi}_l d\psi_l \; e^{+\Delta\tau \underbrace{\frac{\bar{\psi}_{l+1} - \bar{\psi}_l}{\Delta\tau}}_{= \partial_\tau \bar{\psi}} \psi_l - H(\bar{\psi}_{l+1}, \psi_l)}$$

$$Z = \int_{\substack{\psi(\tau+1/T) = -\psi(\tau) \\ \bar\psi(\tau+1/T) = -\bar\psi(\tau)}} [D\bar\psi D\psi] \exp \int_0^{1/T} d\tau \; \bar\psi(\tau)(-\partial_\tau - \omega_0 + \mu)\psi(\tau)$$

(Specific choice H.)

$$\begin{cases} \psi(\tau_\ell = \Delta\tau\ell) \equiv \psi_\ell \\ \bar\psi(\tau_\ell = \Delta\tau\ell) \equiv \bar\psi_\ell \end{cases}$$

$$H(\psi_{\ell+1}, \psi_\ell) \simeq H(\bar\psi_\ell, \psi_\ell) + O(\Delta\tau)$$

$$\psi(\tau + \tfrac{1}{T}) = -\psi(\tau)$$

$$\Rightarrow \begin{cases} \psi(\tau) = T \sum_n \psi(\omega) e^{-i\omega_n \tau} \\ \bar\psi(\tau) = T \sum_n \bar\psi(\omega) e^{i\omega_n \tau} \end{cases}$$

where $\omega_n = (2n+1)\pi T \quad n \in \mathbb{Z}$

Matsubara freqs.

$$\sum_{\ell=0}^{M-1} (\bar\Psi_{\ell+1} - \bar\Psi_\ell)\Psi_\ell = \sum_{\ell=0}^{M-1} \bar\Psi_{\ell+1}\Psi_\ell - \sum_\ell \bar\Psi_\ell \Psi_\ell$$

$$= \sum_{\underline{\underline{\ell'=\ell-1}}} \bar\Psi_{\ell'}\Psi_{\ell'-1} - \sum_\ell \bar\Psi_\ell\Psi_\ell$$

$$= -\sum_\ell \bar\Psi_\ell (\Psi_\ell - \Psi_{\ell-1}) \qquad \text{IBP}$$

- $$\int d\tau \; \partial_\tau \bar\Psi \; \Psi = -\int d\tau \, \bar\Psi \, \partial_\tau \Psi + \bar\Psi\Psi\Big|_0^{1/T} = 0.$$

$$\Rightarrow S[\bar\Psi, \Psi] = \int d\tau \left(\bar\Psi \partial_\tau \Psi + \underset{\sim}{H(\bar\Psi, \Psi)} \right)$$

Continuum Limit Warning:

$$S_{\text{Berry}} = \sum_{\ell=0}^{M-1} \bar\Psi_{\ell+1} (\Psi_{\ell+1} - \Psi_\ell)$$

$$= T \sum_{\omega_n} \bar\Psi(\omega_n) \left(\underline{\underline{1 - e^{i\omega_n \tau}}} \right) \Psi(\omega_n)$$

$$\simeq i\omega_n \tau \qquad \underline{\text{if}} \; \underline{\omega_n \tau \ll 1}.$$

Reassurance: If we use a reasonable

$$H = H_{quadratic} + H_{int}$$

then reasonable quantities

$$Z, \ \langle O^\dagger O \rangle$$

are dominated by $\omega_n \ll \tau^{-1}$.

eg: $\langle \hat{N} \rangle = \frac{1}{Z} \operatorname{tr} e^{-H/T} c^\dagger c$.

$\hat{N} \equiv c^\dagger c$.

Freq. space:

$$D\bar{\psi}(\tau) D(\tau) = \prod_n d\bar{\psi}(\omega_n) d\psi(\omega_n)$$

$$= D\bar{\psi}(\omega) D\psi(\omega)$$

$$Z = \int D\bar{\psi}(\omega) D(\omega) \exp T \sum_{\omega_n} \bar{\psi}(\omega_n)(i\omega_n - \omega_0 + \mu)\psi(\omega_n)$$

$$T \sum_{\omega_n} \xrightarrow{T \to 0} \oint d\omega \equiv \int \frac{d\omega}{2\pi}.$$

$$Z \xrightarrow{T\to 0} \int D\bar{\psi}(\omega) D\psi(\omega) \; \exp \int_{-\infty}^{\infty} d\omega \, \bar{\psi}(\omega)(i\omega - \omega_0 + \mu) \psi(\omega)$$

$$\langle \bar{\psi}(\omega_1) \psi(\omega_2) \rangle = \underbrace{\frac{\delta_{\omega_1 \omega_2}}{T}}_{\xrightarrow{T\to 0} \delta(\omega_1 - \omega_2)} \frac{2\pi}{i\omega_1 - \omega_0 + \mu}$$

$$\langle \bar{\psi}(\omega) \psi(\omega) \rangle = \frac{2\pi / T}{i\omega - \omega_0 + \mu} \qquad \left(\delta(\omega = 0) = \frac{1}{T} \right)$$

$$\langle N \rangle = \frac{1}{Z} \int \prod_{l=0}^{M-1+1} \left(d\bar{\psi}_l \, d\psi_l \, e^{-\bar{\psi}_l \psi_l} \right)$$

$$\prod_{l=1}^{M-1} \langle \bar{\psi}_{l+1} | \left(1 - \Delta\tau \, H(c_1^\dagger c) \right) | \psi_l \rangle$$

$$\langle \bar{\psi}_{N+1} | \, c^\dagger c \, | \psi_N \rangle$$

$$= \bar{\psi}_{N+1} \psi_N \to \bar{\psi}_N(\tau_N + \Delta\tau) \psi(\tau_N)$$

for any $N \in 0 \ldots M-1$

$$\langle N \rangle = \frac{1}{Z} \int D\bar{\psi} D\psi \, e^{-S[\bar{\psi},\psi]} \, \bar{\psi}(\tau_N + \Delta\tau)\psi(\tau_N)$$

$$= \langle \bar{\psi}(\tau_N + \Delta\tau)\psi(\tau_N) \rangle$$

$$= T^2 \sum_{\omega_{n,m}} e^{i(\omega_n - \omega_m)\tau + i\omega_n \Delta\tau} \underbrace{\langle \bar{\psi}(\omega_n) \psi(\omega_m) \rangle}_{\to \frac{\delta_{nm}}{T} \frac{2\pi}{i\omega_n - \omega_0 + \mu}}$$

$\langle N \rangle = \langle c^\dagger c \rangle$

$$= T \sum_n \frac{e^{i\omega_n \Delta\tau}}{i\omega_n - \omega_0 + \mu}$$

$$\xrightarrow{T \to 0} \int_{-\infty}^{\infty} d\omega \, \frac{e^{i\omega\Delta\tau}}{i\omega - \omega_0 + \mu} \quad \Delta\tau > 0 \quad = \theta(\mu - \omega_0) \; \checkmark$$

$$\underline{\underline{\langle \psi_{l+1}|}}\left(1 - \Delta\tau\, H\right)|\psi_l\rangle = \langle \psi_{l+1}|\psi_l\rangle$$

$$=$$

$$\left(1 - \Delta\tau\, \bar{\psi}^{(1)}_{l+1} \bar{\psi}^{(2)}_{l+1} \psi^{(3)}_{l} \psi^{(4)}_{l}\right)$$

$$H = \underline{\underline{\hat{\psi}^{\dagger}_1 \hat{\psi}^{\dagger}_2}}\ \underline{\underline{\hat{\psi}_3 \hat{\psi}_4}}$$

$$\sum_{n_1 \ldots n_4} \underline{f\left(\sum \omega_n\right)}\ e^{i(\omega_{n_1}+\omega_{n_2})\Delta\tau} \bar{\psi}_{\omega_{n_1}} \bar{\psi}_{\omega_{n_2}} \psi_{\omega_{n_3}} \psi_{\omega_{n_4}}$$

$$\sum_l e^{i l \left(\sum \omega_n\right)}$$