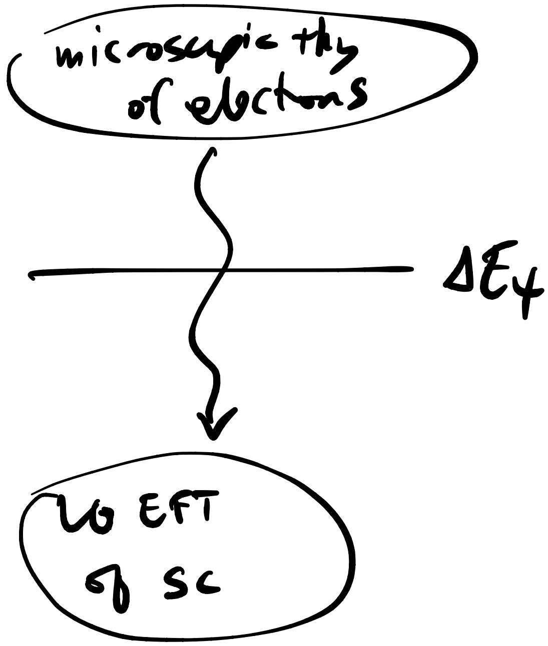
Superconductors & superfluids, cont'd



charge-1 spinor fermion ψ_α *

charge 2 scalar boson Φ ←

Simple case (s-wave): $\Phi \sim \psi_\alpha \psi_\beta \epsilon^{\alpha\beta} = \# \psi_\uparrow \psi_\downarrow$ Cooper pair.

eg: $S[\psi] = S_2[\psi] + \int dt\, d^dx \; u \; \psi^\dagger_\alpha(x) \psi_\alpha(x) \psi^\dagger_\beta(x) \psi_\beta(x) + h.c.$

$u > 0$ repulsive

w/ $S_2[\psi] = \int dt \underline{\underline{\int d^dk}} \; \psi_k^\dagger \left(i\partial_t - \epsilon(k) \right) \psi_k$

Mean Field Theory manoeuver:

$$S_{MFT}[\psi] = S_2[\psi] - \int dt\, d^dx \; u \langle \psi\psi \rangle \psi^\dagger \psi^\dagger + h.c.$$

$$= S_2[\psi] - \int dt\, d^dx \; u \, \Phi \, \psi^\dagger \psi^\dagger + h.c.$$

if $\Phi \neq 0$ ψ gets a mass.

$$\Rightarrow \Delta E_\psi = u \langle \Phi \rangle .$$

Why might this work?

Hubbard-Stratonovich transformation:

$$Z = \int_{-\infty}^{\infty} dx \, e^{iS(x)} .$$

eg 1 in $D = 0+0$.

$S \ni x^4$.

$$e^{-iux^4} = \frac{1}{\sqrt{\pi u i}} \int_{-\infty}^{\infty} d\sigma \, e^{-\frac{\sigma^2}{iu} - 2ix^2\sigma}$$

$$Z = \int_{\mathbb{C}} d^2x \, e^{iS[x,\bar{x}]}$$

eg 2 in $D = 0+1$

$$e^{-iux^2\bar{x}^2} = \frac{1}{i\pi u} \int_{\mathbb{C}} d^2\sigma \, e^{-\frac{|\sigma|^2}{iu} - ix^2\bar{\sigma} + i\bar{x}^2\sigma}$$

$x^2 \rightsquigarrow \psi\psi$.

$$Z = \int [D\psi D\bar\psi]\, e^{iS[\psi]}$$

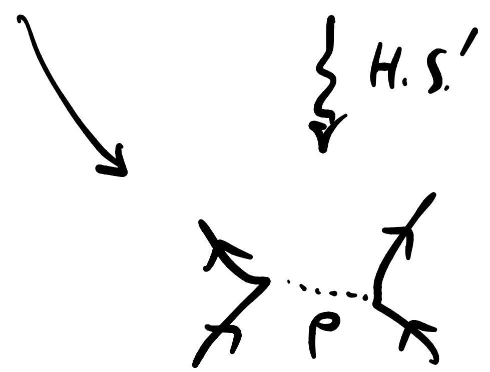
$$= \# \int [D\psi D\bar\psi D\sigma D\bar\sigma]\, e^{iS_2[\psi] + i\int d^Dx\, \bar\sigma \psi_\uparrow \psi_\downarrow + h.c.}\; e^{-\int d^Dx \frac{|\sigma|^2}{iu}} .$$

we "decoupled the ψ^4 term".

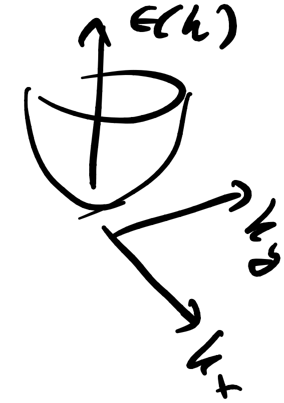
<u>notice</u>: At the saddle of σ integral,

$$\underline{\sigma} = u\, \psi_\uparrow \psi_\downarrow .$$

[Note that we made a choice to decouple in the "BCS channel" vs "density channel"



$$S[\rho, \psi) = \int \left(\rho \psi^\dagger \psi + \frac{\rho^2}{2u} \right)$$



$\int D\psi$ is now gaussian.

let's choose $\epsilon(k) = \frac{k^2}{2m} - \mu$

$$I_\psi[\sigma] \equiv \int (D\psi D\bar{\psi})\, e^{i\int dt\, d^d x\, L}$$

$$L = \psi^\dagger (i\partial_t - \epsilon(i\nabla))\psi + \bar{\sigma}\psi_\uparrow \psi_\downarrow + \sigma \psi_\uparrow^\dagger \psi_\downarrow^\dagger$$

$$= (\psi^\dagger \ \psi) \underbrace{\begin{pmatrix} i\partial_t - \epsilon(i\nabla) & \sigma \\ \bar{\sigma} & -(i\partial_t - \epsilon(-i\nabla)) \end{pmatrix}}_{M} \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix}$$

$$I_\psi[\sigma] = \det\left(M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \det M$$

$$= e^{\mathrm{Tr} \log M}$$

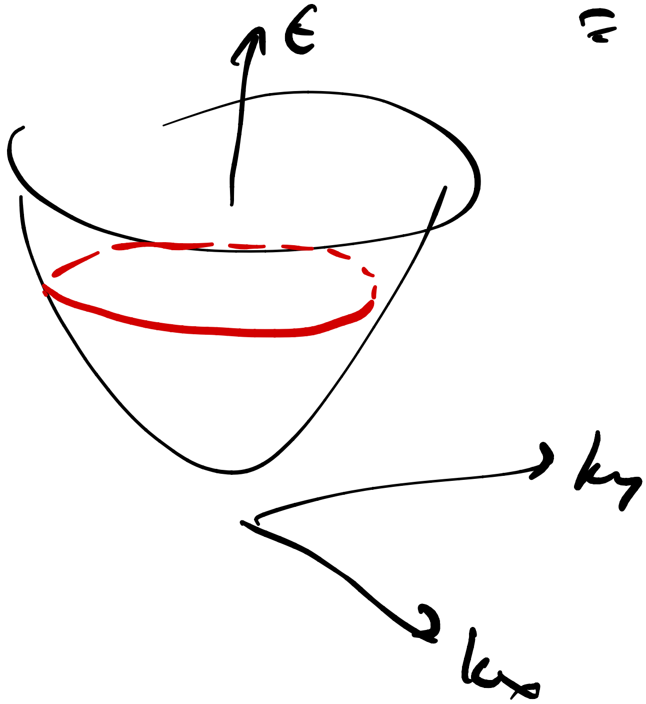
If σ is constant in space:

$$= e^{\int d^3k \log(\omega^2 - \epsilon_k^2 - |\sigma|^2) + O(\partial\sigma)}$$

$$Z = \int [D\sigma D\bar\sigma]\, e^{-\int d^Dx \frac{|\sigma|^2}{2u i} + \int d^Dk \log(\omega^2 - \epsilon_k^2 - |\sigma|^2)} + O(\partial\sigma)$$

Sometimes we can do this integral by saddle point

① one example: when the Fermi surface $= \{ k \mid \epsilon(k) = 0 \}$



$$\epsilon(k) = \frac{k^2}{2m} - \mu$$

② large N # of fermion species

$\alpha = \uparrow, \downarrow$

$\rightsquigarrow \alpha = 1 \ldots N.$

$$0 = \frac{\delta(\text{exponent})}{\delta \bar\sigma} = i\frac{\sigma}{2u} + \int d\omega\, d^dk \frac{2\sigma}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + i\epsilon}$$

do $\int d\omega$ by residues

$$= i\frac{\sigma}{2u} + \int d^dk \frac{2\sigma i}{2\sqrt{\epsilon_k^2 + |\sigma|^2}}$$

→ Gap equation:

$$1 = -2u \int d^d p \, \frac{1}{\sqrt{\epsilon(p)^2 + |\underline{\sigma}|^2}} \qquad (*)$$

value of $\underline{\sigma}$ determines ΔE_f.

Note: • a solution of $*$ requires $u < 0$.

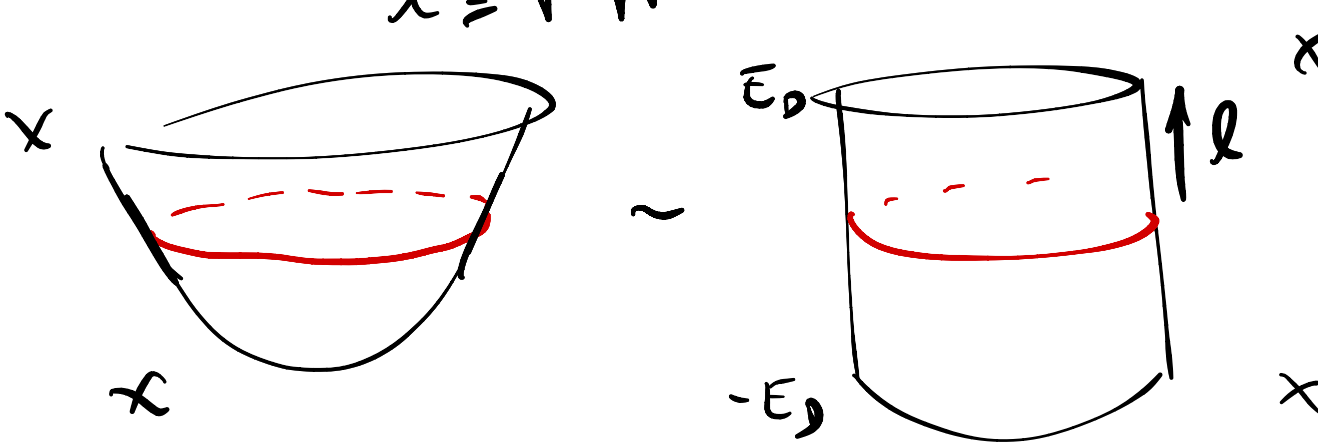
IF • $\Lambda^u \rightsquigarrow u(p, p')$

$$\sigma(\vec{p}) = -\frac{1}{2} \int d^d p' \, \frac{u(p, p')\, \sigma(p')}{\sqrt{\epsilon(p')^2 + |\sigma(p')|^2}}$$

• To do the integral: expand about the FS

$$\epsilon(p) \cong v_F \, \ell \equiv \varepsilon \qquad v_F = \left.\frac{\partial \varepsilon}{\partial p}\right|_{p_F}$$

$$\ell \equiv p - p_F.$$



$$\Rightarrow 1 = -2u \int \frac{d^d p}{\sqrt{\epsilon(p)^2 + |\sigma|^2}}$$

$$\simeq -2u \int_{FS} \frac{d^{d-1}k}{v_F} \int_{-E_D}^{E_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + |\sigma|^2}}$$

$$= N|u| \log\left(\frac{E_D + \sqrt{E_D^2 + |\sigma|^2}}{|\sigma|} \right) \simeq N|u| \log \frac{E_D}{d|\sigma|}$$

$E_D \gg |\sigma|$

$$\frac{E_D}{|\sigma|} = e^{\frac{1}{N|u|}}$$

$$N \equiv \int_{FS} \frac{d^{d-1}k}{v_F}$$

$=$ d.o.s at the FS

$$|\underline{\sigma}| = \frac{2E_D e^{\frac{1}{2Nu}}}{e^{\frac{1}{Nu}} - 1} \overset{\text{IF } Nu \ll 1}{\simeq} \underline{2E_D e^{-\frac{1}{N|u|}}}$$

- non-perturbative in u!!
- generates a hierarchy of scales.

Non-relativistic scalar fields:

$$S_{LG}[\Phi] = \int d^d x\, dt \left(\Phi^* i \partial_t \Phi - \frac{\vec\nabla\Phi \cdot \vec\nabla\Phi}{2m} - V(|\Phi|) \right)$$

D_0

$+ a\, \dot\Phi^* \dot\Phi$ is an irrelevant perturbation.

Number & Phase angle:

$$\frac{\partial L}{\partial \dot\Phi} = \Phi^* \quad (\text{w/o } \partial_t .)$$

In terms of $\Phi = \sqrt{\rho}\, e^{i\varphi}$

$$L = \frac{i}{2}\partial_t \rho - \rho\, \partial_t \varphi - \frac{1}{2m}\left(\rho (\nabla\varphi)^2 + \frac{1}{4\rho}(\nabla\rho)^2 \right) - V(\rho)$$

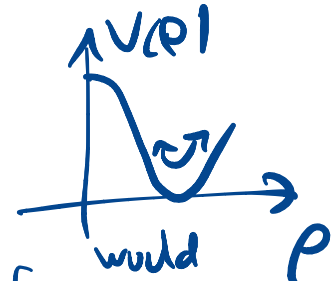
total deriv.

$$[\rho(x,t), \varphi(y,t)] = i\,\delta^d(\vec x - \vec y)$$

number

$\int d^d x\, \rho = $ # of particles.

phase of condensate.



$\delta\rho$ would be the Higgs.

$\Rightarrow$ no Higgs mode in vanilla NR SF or SC.

<u>Superfluids</u>: choose $V(\rho) = g^2(\rho - \bar\rho)^2$

expand $\sqrt{\rho} = \sqrt{\bar\rho} + h$. $h \ll \sqrt{\bar\rho}$

$$\Rightarrow L = -2\sqrt{\bar\rho}\, h\, \partial_t \varphi - \frac{\bar\rho}{2m}(\nabla\varphi)^2 - \frac{1}{2m}(\vec\nabla h)^2 - 4g^2\bar\rho h^2 + \mathcal{O}(h^3)$$

Do $\int \mathcal{D}h$

$$\rightsquigarrow \bar\rho\, \partial_t\varphi \frac{1}{4g^2\bar\rho - \frac{\nabla^2}{2m}} \partial_t\varphi - \frac{\bar\rho}{2m}(\vec\nabla\varphi)^2$$

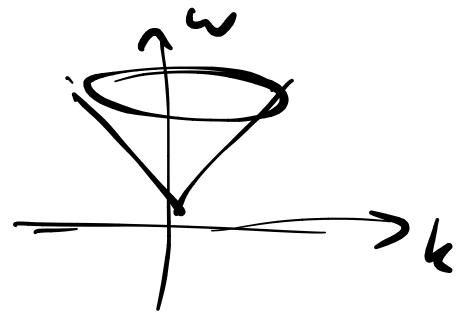
$$= \frac{1}{4g^2}(\partial_t\varphi)^2 - \frac{\bar\rho}{2m}(\nabla\varphi)^2 + \underbrace{\ldots}_{k \ll \sqrt{8g^2\bar\rho m}}$$

TRUE low-E EFT of SF

$$= \frac{\bar\rho}{2m}\partial^\mu\varphi\,\partial_\mu\varphi + \ldots$$ in the right units for t, x

XY model.

$\to \quad \omega^2 = \frac{2g^2\bar{\rho}}{m}\tilde{k}^2 .$ [Bogoliubov]

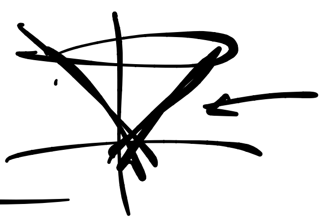


relativistic dispersion

w/ $V_s = g\sqrt{\frac{2\bar{\rho}}{m}}$.

vs: Relativistic theory $\quad V(\Phi) = g|\Phi|^4 + \dots$

$$L = \partial_\mu \Phi^* \partial^\mu \Phi - \kappa(\Phi^*\Phi - v^2)^2$$



← ind. of κ.

<u>Superflow</u>: If you start a current, it keeps going.

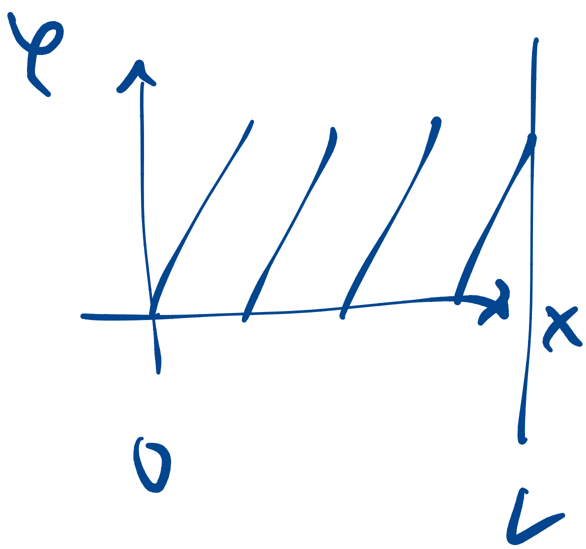
<u>Whence</u>: $\vec{J} = \frac{\rho}{m}\vec{\nabla}\varphi$ and $\varphi \cong \varphi + 2\pi$.

Consider $x \cong x + L$. set up a current J_x.

Requires $\varphi(x) = mvX \quad \to J_x = \rho v$

$$\left.\begin{array}{c} \varphi \simeq \varphi + 2\pi \\ \text{and} \\ x \simeq x + L \end{array}\right\} \Longrightarrow mvL \in 2\pi\mathbb{Z} \text{ is quantized}$$

$\mathbb{Z} \ni n \qquad \dfrac{mvL}{2\pi} \equiv$ vorticity of Φ



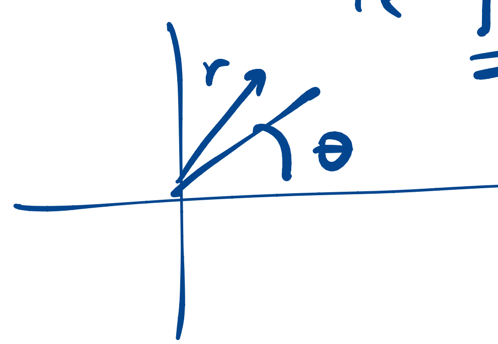
how can n change?

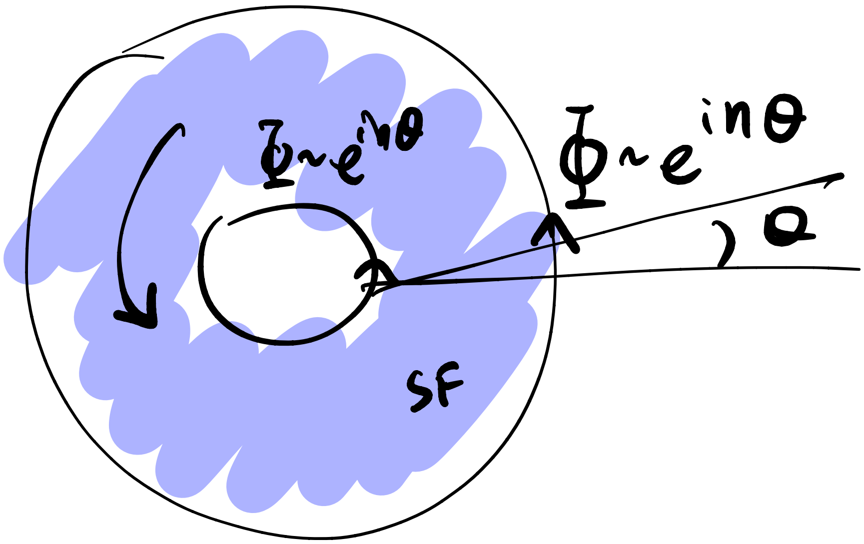
$$\Phi = \underset{\sim}{|\Phi_0|}\, e^{i\varphi}$$

Can only change by <u>vortices</u>

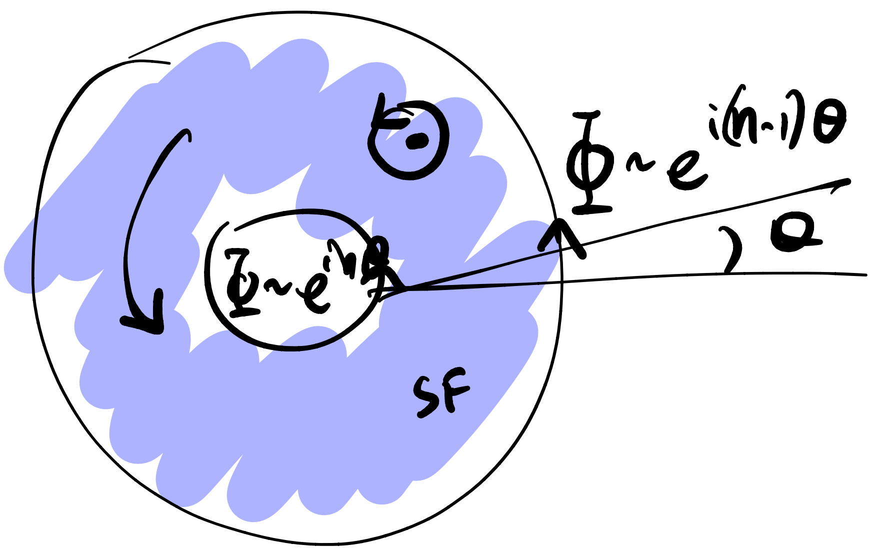
= locus where $\Phi(r,\theta) \sim r e^{i\theta}$

i.e. $\underline{\varphi \sim \theta}$





$\Phi \sim e^{in\theta}$
$\Phi \sim e^{in\theta}$
α
SF



$\Phi \sim e^{i(n-1)\theta}$
$\Phi \sim e^{in\theta}$
α
SF

$$E(\phi^2) \overset{\phi^2 \ll \ldots}{\sim} \phi^2 \ln \phi$$