# Anomalies, continued.

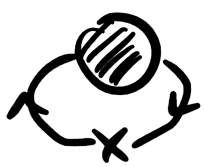
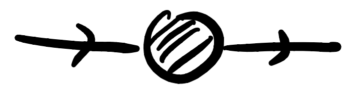
$$\partial^\mu j^5_\mu(x) = \mathcal{A}(x) \quad \text{where } Z \to Z\, e^{-i\int dx\, \alpha \mathcal{A}}$$

under chiral transf.

$\Rightarrow \mathcal{A}$ is a c-number

$$J^5_\mu(x) = \langle \bar\Psi(x)\gamma_\mu\gamma^5\Psi(x)\rangle = \langle j^5_\mu(x)\rangle$$

$$= Z^{-1}[A]\int [D\Psi D\bar\Psi]\, e^{-S[\Psi,\bar\Psi,A]}\, j^5_\mu(x)$$

$=$  w/  $= G^{[A]} = (i\not{D})^{-1}$

$$= -\mathrm{tr}\, \gamma_\mu\gamma^5 G^{[A]}(x,x) \qquad -\!\times\!- = \gamma_\mu\gamma^5$$

↑ trace over spinor indices.

if $H = \sum_n \epsilon_n |n\rangle\langle n|$ Then $H^{-1} = \sum_n \frac{1}{\epsilon_n}|n\rangle\langle n|$.

$$i\not{D}\,\xi_n(x) = \epsilon_n\,\xi_n(x) \qquad \bar\xi_n(x)\, i\gamma^\mu(-\overleftarrow{\partial}_\mu + iA_\mu) = \epsilon_n\bar\xi_n$$

$$\Rightarrow G^{[A]}(x,x') = \sum_n \frac{1}{\epsilon_n}\,\xi_n(x)\,\bar\xi_n(x') = (\ )(\ )$$

$$J^5_\mu(x) = \lim_{x' \to x} \sum_n \frac{1}{\epsilon_n} \bar{\xi}_n(x') \gamma^5 \gamma_\mu \xi_n(x).$$

The limit as $x \to x'$ is singular because of large $\epsilon_n$'s.

$$G^{(A)}(x,x') \rightsquigarrow G^{[A]}_s(x,x') \equiv \sum_n \underbrace{e^{-s\epsilon_n^2}}\frac{1}{\epsilon_n} \xi_n(x) \bar{\xi}_n(x')$$

[ 'heat kernel regulator' ]

$$J^5_\mu(x) \rightsquigarrow \sum_n e^{-s\epsilon_n^2} \frac{1}{\epsilon_n} \bar{\xi}_n(x) \gamma^5 \gamma_\mu \xi_n(x)$$

$$\partial^\mu J^5_\mu(x) = \partial^\mu \langle j^5_\mu \rangle$$

$$= -\sum_n \underbrace{i\partial^\mu (\bar{\xi}_n \gamma_\mu \gamma^5 \xi_n)}_{\text{def. } \not\partial \xi_n \; = -2\cancel{\epsilon_n}\, \bar{\xi}_n \gamma_5 \xi_n} \frac{e^{-s\epsilon_n^2}}{\cancel{\epsilon_n}}$$

$$= +2\, \underset{\text{spin trace}}{\mathrm{tr}} \langle x | \gamma_5 e^{-s(i\not D)^2} | x \rangle$$

$$\xi^\alpha_n(x) = \langle x, \alpha | n \rangle$$

$i\not{D}$ is a linear operator on $\mathcal{H}$

a basis for which is $|x,\alpha\rangle$

$= \int d^D p \, |p,\alpha\rangle\langle p\alpha|x\alpha\rangle$

$e^{ipx}$

$$(i\not{D})^2 = -(\gamma^\mu(\partial_\mu + iA_\mu))^2$$

$$= -(\partial_\mu + iA_\mu)^2 - \frac{i}{2}\Sigma_{\mu\nu}F^{\mu\nu}$$

$$\Sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad \left(\begin{array}{c}\text{generator} \\ \text{of} \\ \text{Lorentz}\end{array}\right)$$

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evaluate. small $s$ expansion = exp. in powers of $D, A, F$.

leading term: $A = 0$.

$$\langle x| \mathbb{1} e^{-s(i\not{\partial})^2}|x\rangle = \int d^D p \, e^{-sp^2}$$

$$\mathbb{1} = \int d^D p \, |p\rangle\langle p| \qquad = \frac{1}{s^{D/2}} \frac{\Omega_{D-1}}{(2\pi)^D} \overset{D=4}{=} \frac{1}{16\pi^2 s^2}$$

$$\rho(x) = \langle \psi^\dagger \psi(x)\rangle = \mathrm{tr}\, \gamma^0 G(x,x)$$

$$\mathrm{tr}_\gamma\, \gamma^5\, G(x,x)\Big|_{A=0} \propto \mathrm{tr}\,\gamma^5 = 0.$$

similarly $\mathrm{tr}\,\gamma^5\, e^{-s(\partial+A)^2} = 0.$

$$\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\mathrm{tr}\left(\gamma_5\, e^{-s(i\not{D})^2}\right)_{xx} =$$

$$= 4\,\epsilon^{\mu\nu\rho\lambda}$$

$$\underbrace{\langle x|\, e^{-s(iD)^2}\,|x\rangle}_{\frac{1}{16\pi^2 s^2}}\ \frac{s^2}{2}\left(\frac{-i}{2}\right)^2 \overbrace{\mathrm{tr}_\gamma\left(\gamma^5\,\Sigma^{\mu\nu}\Sigma^{\rho\lambda}\right)}$$

$$\mathrm{tr}_c\, F_{\mu\nu}F_{\rho\lambda} + O(s^1)$$

$$\Rightarrow \partial_\mu J_5^\mu = -2\,\frac{1}{16\pi^2 \cancel{s^2}}\,\frac{\cancel{s^2}}{8}\,4\,\epsilon^{\mu\nu\rho\lambda}\,\mathrm{tr}_c\, F_{\mu\nu}F_{\rho\lambda} + O(s)$$

$$= -\frac{1}{16\pi^2}\,\epsilon^{\mu\nu\rho\lambda}\,\mathrm{tr}_c\, F_{\mu\nu}F_{\rho\lambda}$$

$$= -\frac{1}{8\pi^2}\,\mathrm{tr}_c\, F_{\mu\nu}\left(*F^{\mu\nu}\right) = -\frac{1}{8\pi^2}\,\mathrm{tr}_c\, F\wedge F$$

<u>abelian</u>: $= -\frac{1}{32\pi^2}\,\vec{E}\cdot\vec{B}$

<u>Recall</u>: $F \wedge *F \propto F_{\mu\nu} F^{\mu\nu}$

$$F \wedge F \propto F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$$

$$= d(A \wedge F)$$

$\int_M \frac{F \wedge F}{8\pi^2}$ is a <u>topological invariant</u>

$\Rightarrow$ $\underline{\underline{\in \mathbb{Z}}}$.

<u>Argument for</u>:


[ everything $\to$ vacuum far away

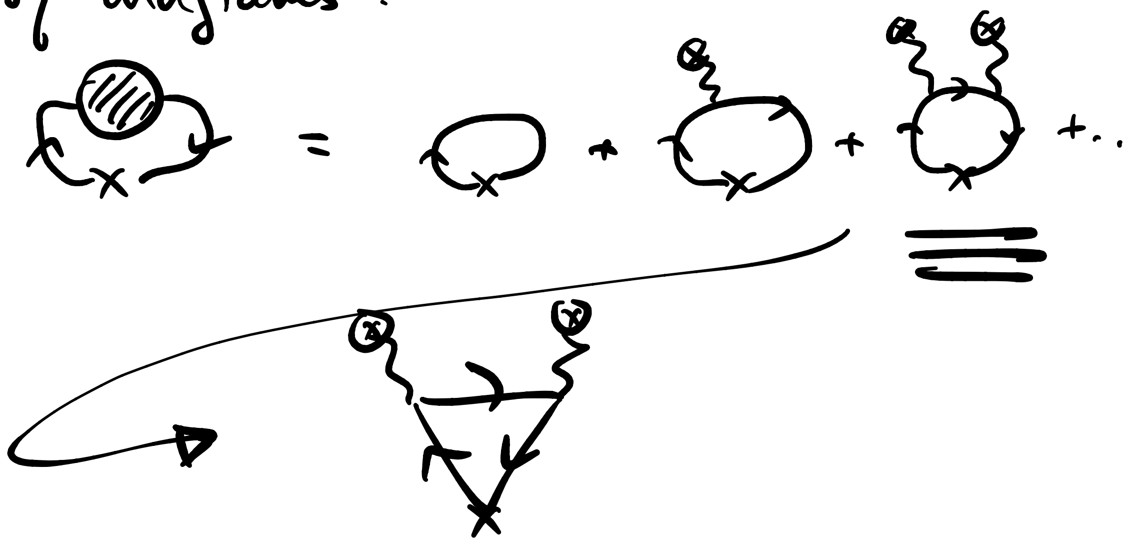
$$\mathbb{Z} \ni \Delta Q_A \equiv \Delta(N_L - N_R)$$

$$= (N_L' - N_L) - (N_R' - N_R)$$

$$= \int dt\, \partial_t \underbrace{\int d^d x\, J^5_0}_{Q_A(t)}$$

$$= \int_M d^D x\, \underbrace{\partial^\mu J^5_\mu(x)} \overset{\text{anomaly}}{=} \int_M \frac{F \wedge F}{8\pi^2}$$

- In terms of diagrams:

$J^5_\mu =$ 

- we chose heat kernel regulator.

  <u>claim</u>: same ans. for any chiral-symmetric regulator

- In even $D$ dims $\quad \mathrm{tr}\, \gamma^5 \underbrace{\gamma^0 \cdots \gamma^{D-1}} \neq 0.$

  needs $D/2$ powers of

  $$\left(\Sigma_{\mu\nu} F^{\mu\nu}\right)$$

- chiral <u>flavor</u> symms. $\quad \mathcal{L} = \bar{\Psi}_I \not{D} \Psi_I$

  $I = 1 \ldots N_f$

is preserved by: $\Psi_I \to e^{i\underline{\underline{(\gamma^5 \theta^a \tau^a)}}_{IJ}}\,\Psi_J$

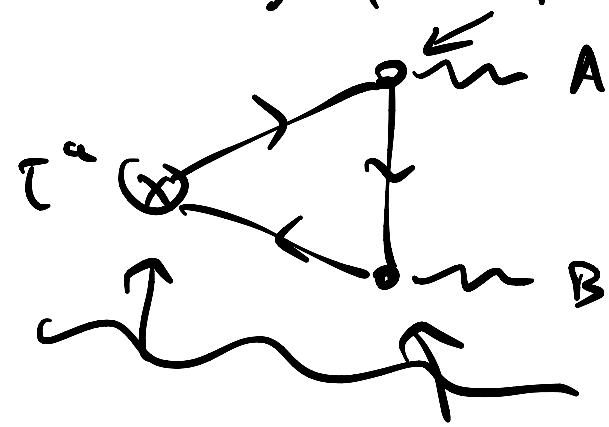
$\tau^a$ a generator of $SU(N_f)$.

$$\Longrightarrow \partial^\mu j^{5a}_\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}\underbrace{\mathrm{tr}_{c,f}\left(F_{\mu\nu}F_{\rho\lambda}\tau^a\right)}_{=\,\mathrm{tr}_c F_{\mu\nu}F_{\rho\lambda}\,\mathrm{tr}_f \tau^a}$$

$F = F^A T^A$

$\mathrm{tr}_c F_{\mu\nu}F_{\rho\lambda}$

$= \left(\mathrm{tr}_c T^A T^B\right) F^A F^B$




$\underline{\text{eg}}$: $\tau^a = \mathbb{1}$.

$$\Rightarrow \partial^\mu j^{5(1)}_\mu = N_f \frac{1}{8\pi^2}\mathrm{tr}\, F\wedge F.$$

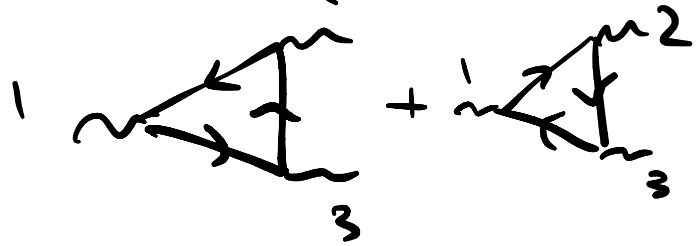
Most generally (in 4d): $\underline{G_1} \times G_2 \times G_3$ symmetry

couple to $A_1, A_2, A_3$

So far: 1 Dirac = R + L

$$\partial_\mu j^{A\mu 1} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F^{2B}_{\mu\nu} F^{3C}_{\rho\sigma} \sum_f \underline{(-1)^f} \times \underline{\mathrm{tr}_{R_f}\{T_1^A, T_2^B\} T_3^C}$$

$$\boxed{\begin{matrix} +1 \text{ for } L \\ -1 \text{ for } R \end{matrix}}$$



"$G_1 G_2 G_3$ anomaly"

($R_f$ is the rep of $G_1 \times G_2 \times G_3$ in which $f$ transforms.)

$\underline{\text{eg}}$: $G_1 = G_2 = G_3 = U(1)_Y$ in SM.

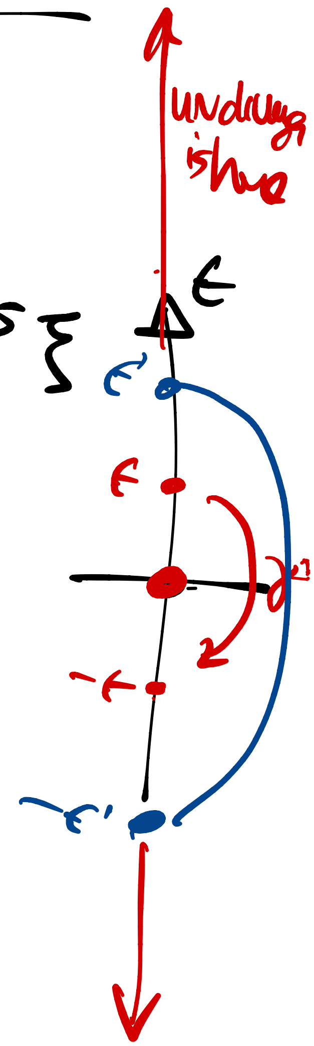
$G_1 = G_2 = G_3 = SU(2)_{EW}$ "

$G_1 = U(1)_Y \quad G_2 = G_3 = SU(2)_{EW}$.

# Zeromodes of the Dirac Operator

$$G(x,x') = \sum \underline{\frac{1}{\epsilon_n}}\, \xi_n(x)\bar{\xi}_n(x') \qquad ?$$

nonzero modes of $i\not{D}$ come in pairs




$[\,\{\gamma^5, i\not{D}\} = 0$

$\Longrightarrow$ if $\underline{i\not{D}\xi = \epsilon\xi}$ then $\gamma^5\xi$

has $\underline{i\not{D}(\gamma^5\xi) = -\epsilon(\gamma^5\xi)}$

let $\chi_n^{L/R} \equiv \frac{1}{2}(1\pm\gamma^5)\xi_n$

are simultaneous evecs of $(i\not{D})^2 = \epsilon_n^2$

and $\gamma^5$. $\gamma^5 = \pm 1$.

exception: if $\epsilon = 0$.

Atiyah-Singer Index Thm: only zeromodes contribute to $A$.

$$A \propto \mathrm{Tr}\,\gamma^5 e^{-s(i\not{D})^2} = \sum_{\substack{n \\ \epsilon_n\neq 0}} \underbrace{\sum_{L/R} \bar{\chi}_n^{L/R}\gamma^5\chi_n^{L/R}}_{0} e^{-s\epsilon_n^2}$$

$+$ zeromodes

$\Longrightarrow$ regulator independent. (ind. of $S$)

Choose $\int d^D x \, \bar{\xi}_n(x) \xi_m(x) = \delta_{nm}$ and $\gamma^5 \xi_n = \pm \xi_n$

$$\int_M d^D x \, \mathcal{A} = \int d^D x \sum_{\substack{\text{zeromodes} \\ i}} \bar{\chi}_{0i}(x) \gamma^5 \chi_{0i}(x)$$

$$= n_L - n_R = - \underbrace{\int_M \frac{\operatorname{tr} F \wedge F}{8\pi^2}}_{\in \mathbb{Z}} .$$

# of leftmoving zms
− # of rightmoving zms

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If $- \int \operatorname{tr} \frac{F \wedge F}{8\pi^2} \equiv q = n_L - n_R \neq 0$

$$Z[A^q] = \int [D\psi D\bar{\psi}] \, e^{i \int \bar{\psi} i \not{D} \psi} = \det i\not{D}$$

vac-to-vac amplitude

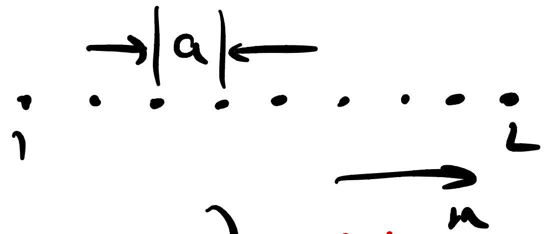
$$= \prod_n \epsilon_n$$

$$= 0 .$$

only amplitudes w/ $\Delta Q = q$ are nonzero.

A localized gauge field config w/ $\int \operatorname{tr} F \wedge F \neq 0$ is an INSTANTON.

# Physics of the anomaly




consider:

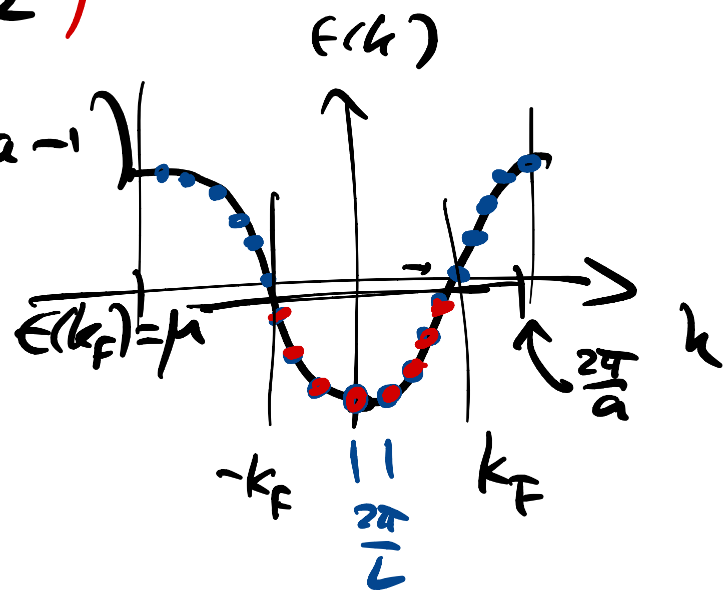
$$H = -t\sum \left(c_n^\dagger c_{n+1} + h.c.\right) - \mu N$$

$$c_n \equiv \oint dk\, e^{ikna}\, c_k$$

$$\boxed{\{c_n, c_m^\dagger\} = \delta_{nm}.}$$

$$= \oint dk\; c_k^\dagger c_k\,(\epsilon_k - \mu)$$

$$\epsilon_k = -2t(\cos ka - 1)$$




In a box of length L

$$k_\ell = \frac{2\pi \ell}{L} \quad \ell = 1, \dots L/a$$

$$|g.s.\rangle_{\text{of } N} = \prod_{\substack{\text{lowest}\\ N\\ \text{orbitals}\\ k_i}} c_{k_i}^\dagger |0\rangle$$

cheapest way to add an electron: is at $k = k_F + \dots$

has energy $\quad \epsilon(k) - \epsilon_F = \cancel{\epsilon(k_F)} + (k-k_F)\frac{\partial \epsilon}{\partial k}\Big|_{k_F} + O(k-k_F)^2 \; \cancel{-\mu}$

$$= \delta k\, v_F.$$

$$\epsilon(k \sim k_F) = \delta k\, v_F + \mathcal{O}(\delta k^2)$$

$$\delta k \equiv |k - k_F| \qquad v_F = \left.\frac{\partial \epsilon}{\partial k}\right|_{k_F} \quad \text{fermi velocity.}$$

eom for exc. near Fermi surface:

$$\left\{\begin{array}{l} (\omega - v_F \delta k)\,\underline{\psi_L} = 0 \\ (\omega + v_F \delta k)\,\underline{\psi_R} = 0 \end{array}\right] \text{ Dirac eqn}$$

are the eom for the action:

$$S[\Psi] = \int dx\, dt \quad \bar{\Psi}\, i \not{\partial}\, \Psi$$

w/ $\underline{\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}}$ $\qquad \left\{\begin{array}{l} \gamma^0 = \sigma^1 \\ \gamma^x = i\sigma^2 \end{array}\right. \Rightarrow \gamma^5 = -\sigma^3$

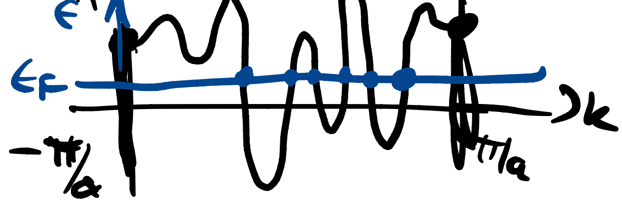
<u>we set $v_F = 1$.</u>

<u>Comments:</u> ① antiparticles = holes near FS

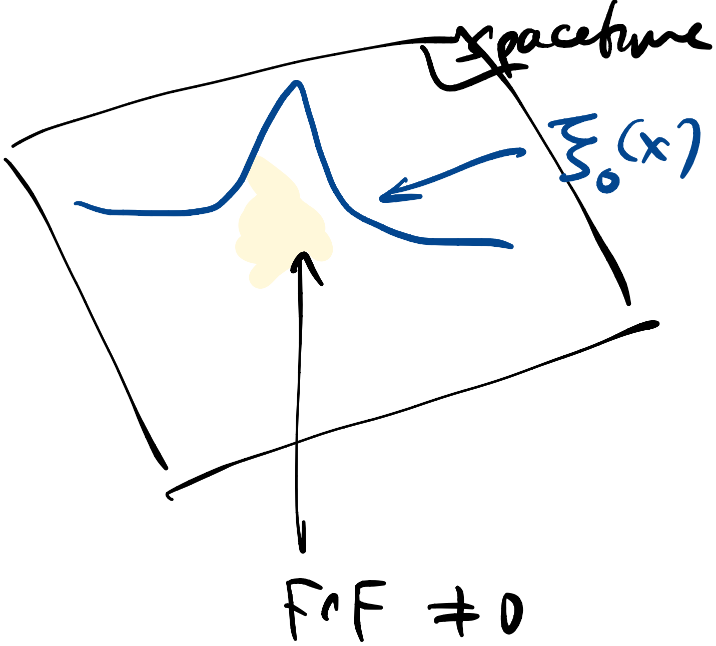
② Relativistic approx breaks down when $\delta k^2 \partial_k^2 \epsilon$ terms matter

$\epsilon(k) \overset{!}{=} \epsilon(k + \frac{2\pi}{a})$

③ Notice:




$\Rightarrow$ <u>$\# L = \# R$.</u>

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Symmetry Protected Topological phases (SPTs):

fermionic ← can be free fermions

bosonic ← requires interaction

anomalies on the bdy of space.

$$\{c_n, c_m\} = 0 \longleftarrow$$

$$\{c_n, c_m^\dagger\} = \delta_{nm}.$$

$$\{A, B\} = AB + BA.$$

$$c_n|0\rangle = 0 \quad \forall n.$$

$$c_n^\dagger|0\rangle \equiv \left\{ \begin{array}{c} \text{a state w} \\ \text{an electron at} \\ \text{site } n \end{array} \right\}$$

$$c_n^\dagger c_m^\dagger|0\rangle = \left\{ \text{" } n \,\&\, m \right\}$$

$$(c_n^\dagger)^2 = 0$$