Physics 215C: Quantum Field Theory, Part 3
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Contents

0.1 Introductory remarks

- The crux of many problems in physics is the correct choice of variables with which to label the degrees of freedom. Often the best choice is very different from the obvious choice; a name for this phenomenon is 'duality'. There are many examples of it and we will study some of them. This word is dangerous because it is *about* ambiguities in our (physics) language. I would like to reclaim it.

  An important bias in deciding what is meant by 'correct' or 'best' in the previous paragraph is: we will be interested in low-energy and long-wavelength physics, near the groundstate. For one thing, this is the aspect of the present subject which is like 'elementary particle physics'; the high-energy physics of these systems is of a very different nature and bears little resemblance to the field often called 'high-energy physics' (for example, there is volume-law entanglement).

- An important goal for the course is demonstrating that many fancy phenomena precious to particle physicists can emerge from humble origins in the kinds of (completely well-defined) local quantum lattice models we will study. Here I have in mind: fermions, gauge theory, photons, anyons, strings, topological solitons, CFT, and many other sources of wonder I'm forgetting right now.

Topics that I hope to discuss this quarter include:

- effects of topology in QFT (this includes anomalies, topological solitons and defects, topological terms in the action)
- some more illustrations of effective field theory (perhaps cleverly mixed in with the other subjects) in diverse areas of physics
- the uses and limitations of path integrals of various kinds
- large-N expansions
- more deep mysteries of gauge theory and its emergence in physical systems.
- duality.

Some other modern topics in QFT, which we could consider discussing, include: entanglement, generalized symmetries, various bootstrap methods, scattering amplitudes, QFT in curved spacetime. I welcome your suggestions regarding which subjects in QFT we should study.

0.2 Sources and acknowledgement

The material in these notes is collected from many places, among which I should mention in particular the following:

Peskin and Schroeder, *An introduction to quantum field theory*

Zee, *Quantum Field Theory* (2d Edition)

Banks, *Modern Quantum Field Theory: A Concise Introduction*

Schwartz, *Quantum field theory and the standard model*

David Tong's lectures on gauge theory

Many other bits of wisdom come from the Berkeley QFT courses of Prof. L. Hall and Prof. M. Halpern.

Some other books that might be useful to us are:

Xiao-Gang Wen, *Quantum Field Theory of Many-Body Systems*

Sidney Coleman, *Aspects of Symmetry*

Alexander Polyakov, *Gauge Fields and Strings*

Eduardo Fradkin, *Field Theories of Condensed Matter Systems*

Eduardo Fradkin, *Quantum Field Theory, an Integrated Approach*

R. Shankar, *Quantum Field Theory and Condensed Matter*

0.3 Conventions

Following most QFT books, I am going to use the $+---$ signature convention for the Minkowski metric. I am (somehow, still) used to the other convention, where time is the weird one, so I'll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x}\cdot d\vec{x} = \eta_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} .$$

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}_x\right)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. I'll use $\mu, \nu, ...$ for Lorentz indices, and $i, j, k, ...$ for spatial indices.

The convention that repeated indices are summed is always in effect unless otherwise indicated. d is the number of space dimensions, D is the number of spacetime dimensions (it's bigger!).

$\equiv$ means 'equals by definition'. $A \stackrel{!}{=} B$ means we are demanding that $A = B$. $A \stackrel{?}{=} B$ means A probably doesn't equal B.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is $đk \equiv \frac{dk}{2\pi}$. I will also write $\not{\delta}^d(q) \equiv (2\pi)^d\delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx}\tilde{f}(k) \equiv \int đ^d k\; e^{ikx}\tilde{f}(k) \equiv f(x).$$

IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
$+h.c. \equiv$ plus hermitian conjugate.
$\mathcal{L} \ni \mathcal{O}$ means the object $\mathcal{L}$ contains the term $\mathcal{O}$.

We work in units where $\hbar$ and the speed of light, c, are equal to one unless otherwise noted. When I say 'Peskin' I usually mean 'Peskin & Schroeder'.

Please tell me if you find typos or errors or violations of the rules above.

1 Anomalies

[Zee §IV.7; Polyakov, *Gauge Fields and Strings*, §6.3; K. Fujikawa, *Phys. Rev. Lett.* **42** (1979) 1195; Argyres, 1996 lectures on supersymmetry §14.3; Peskin, chapter 19; highly recommended: David Tong, Gauge Theory, chapter 3]

Topology means the study of quantities that can't vary smoothly, but can only vary by jumping. Good examples are quantities that must be integers. *Anomalies* provide an example of a topological phenomenon in QFT, which is therefore robust against any change in the QFT which can be made continuously (like varying masses or couplings, or the cutoff or the resolution of our description, *i.e.* a renormalization group transformation).

Suppose we have in our hands a classical field theory in the continuum which has some symmetry. Is there a well-defined QFT whose classical limit produces this classical field theory and preserves that symmetry? The path integral construction of QFT offers some insight here. The path integral involves two ingredients: (1) an action, which is shared with the classical field theory, and (2) a path integral measure. It is possible that the action is invariant but the measure is not. This is called an anomaly. It means that the symmetry is (explicitly) broken, and its current conservation is violated by a known amount, and this often has many other consequences that can be understood by humans. It means that the theory *cannot* be regulated in a way that preserves the symmetry.

Notice that here I am speaking about actual, global symmetries. I am not talking about gauge redundancies. If you think that two field configurations are equivalent but the path integral tells you that they would give different contributions, you are doing something wrong. Such a 'gauge anomaly' means that the system has more degrees of freedom than you thought. (It does not mean that the world is inconsistent. For a clear discussion of this, please see Preskill, 1990.)

You could say that we have already seen a dramatic example of an anomaly: the violation of classical scale invariance (*e.g.* in massless ϕ^4 theory, or in massless QED) by quantum effects. A regulator necessarily introduces a length scale into the problem and explicitly breaks scale invariance.

Notice that the name 'anomaly' betrays the bias that we imagine constructing a QFT by starting with a continuum action for a classical field theory; you would never imagine that *e.g.* scale invariance was an exact symmetry if you instead started from a well-defined quantum lattice model. Partly for this reason, the concept of 'anomaly' is not native to the condensed matter literature (but has recently been flourishing there).

The example we will focus on here is the *chiral anomaly*. This is encapsulated

by an equation for the violation of the chiral (aka axial) current for fermions coupled to a background gauge field. The chiral anomaly was first discovered in perturbation theory, by computing a certain Feynman diagram with a triangle; the calculation was motivated by the experimental observation of the process $\pi^0 \to \gamma\gamma$, which would not happen if the chiral current were conserved. (The relationship between the chiral current and the pion is explained in §3.5.)

I will outline a derivation of this effect (using the fermionic path integral) which is more illuminating than the triangle diagram. It shows that the one-loop result is exact – there are no other corrections. It shows that the quantity on the right hand side of the continuity equation for the would-be current integrates to an integer. It gives a (physics) proof of the *index theorem*, relating numbers of solutions of the Dirac equation in a background field configuration to a certain integral of field strengths. It butters your toast.

1.1 Chiral anomaly

Chiral symmetries. In even-dimensional spacetimes, the Dirac representation of $\mathrm{SO}(D-1,1)$ is reducible. This is because

$$\gamma^5 \equiv a \prod_{\mu=0}^{D-1} \gamma^\mu \neq 1, \quad \text{satisfies} \quad \{\gamma^5, \gamma^\mu\} = 0, \forall \mu$$

which means that γ^5 commutes with the Lorentz generators

$$[\gamma^5, \Sigma^{\mu\nu}] = 0, \qquad \Sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu].$$

We can choose a so that $(\gamma^5)^2 = 1$ so that $\frac{1}{2}(1 \pm \gamma^5)$ are projectors. A left- or right-handed Weyl spinor is an irreducible representation of $\mathrm{SO}(D-1,1)$, $\psi_{L/R} \equiv \frac{1}{2}(1 \pm \gamma^5)\psi$. This allows the possibility that the L and R spinors can transform differently under a symmetry; such a symmetry is called a chiral symmetry.

Note that in $D = 4k$ dimensions, if ψ_L is a left-handed spinor in representation $\mathbf{r}$ of some group G, then its image under CPT, $\psi_L^{CPT}(t,\vec{x}) \equiv \mathbf{i}\gamma^0 (\psi_L(-t,-\vec{x}))^\star$, is right-handed and transforms in representation $\bar{\mathbf{r}}$ of G. Therefore chiral symmetries arise when the Weyl fermions transform in *complex representations* of the symmetry group, where $\bar{\mathbf{r}} \neq \mathbf{r}$. (In $D = 4k+2$, CPT maps left-handed fields to left-handed fields. For more detail on discrete symmetries and Dirac fields, see Peskin §3.6.)

Some more explicit words (of review) about chiral fermions in $D = 3+1$, mostly notation. Recall Peskin's *Weyl* basis of gamma matrices in 3+1 dimensions, in which

γ^5 is diagonal:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\boldsymbol{\sigma}}^\mu \\ \boldsymbol{\sigma}^\mu & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^\mu \equiv (\mathbb{1}, \vec{\boldsymbol{\sigma}})^\mu, \quad \bar{\boldsymbol{\sigma}}^\mu \equiv (\mathbb{1}, -\vec{\boldsymbol{\sigma}})^\mu, \quad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

This makes the reducibility of the Dirac representation of SO(3, 1) manifest, since the Lorentz generators are $\propto [\gamma^\mu, \gamma^\nu]$ block diagonal in this basis. The gammas are a map from the $(1, \mathbf{2_R})$ representation to the $(\mathbf{2_L}, \mathbf{1})$ representation. It is sometimes useful to denote the $\mathbf{2_R}$ indices by $\alpha, \beta = 1, 2$ and the $\mathbf{2_L}$ indices by $\dot{\alpha}, \dot{\beta} = 1, 2$. Then we can define two-component Weyl spinors $\psi_{L/R} = P_{L/R}\psi \equiv \frac{1}{2}(1 \pm \gamma^5)\psi$ by simply forgetting about the other two components. The conjugate of a L spinor $\chi = \psi_L$ (L means $\gamma^5\chi = \chi$) is right-handed:

$$\bar{\chi} = \chi^\dagger\gamma^0, \quad \bar{\chi}\gamma^5 = \chi^\dagger\gamma^0\gamma^5 = -\chi^\dagger\gamma^5\gamma^0 = -\chi^\dagger\gamma^0 = -\bar{\chi}.$$

We can represent any system of Dirac fermions in terms of a collection of twice as many Weyl fermions.

For a continuous symmetry G, we can be more explicit about the meaning of a complex representation. The statement that ψ is in representation $\mathbf{r}$ means that its transformation law is

$$\delta\psi_a = \mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)_{ab} \psi_b$$

where $t^A_{\mathbf{r}}$, $A = 1..\dim G$ are generators of G in representation $\mathbf{r}$; for a compact Lie group G, we may take the t^A to be Hermitian. The conjugate representation, by definition, is one with which you can make a singlet of G – it's the way $\psi^{\star T}$ transforms:

$$\delta\psi^{\star T}_a = -\mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)^T_{ab} \psi^{\star T}_b \ .$$

So:

$$t^A_{\bar{\mathbf{r}}} = -\left(t^A_{\mathbf{r}}\right)^T .$$

The condition for a complex representation is that this is different from $t^A_{\mathbf{r}}$ (actually we have to allow for relabelling of the generators and the basis – two representations $\mathbf{r}_{1,2}$ are equivalent, $\mathbf{r}_1 \cong \mathbf{r}_2$ if there is a change of basis (the same for all A) that relates the generators: $t^A_{\mathbf{r}_1} = U^\dagger t^A_{\mathbf{r}_1} U$. $\mathbf{r}$ is complex if $\mathbf{r} \ncong \mathbf{r}_2$. The simplest case is $G = \mathrm{U}(1)$, where t is just a number indicating the charge. In that case, any nonzero charge gives a complex representation.

Consider the effective action produced by integrating out Dirac fermions coupled to a *background* gauge field (the gauge field is just going to sit there for this whole calculation):

$$e^{\mathbf{i}S_{\text{eff}}[A]} \equiv \int [D\psi D\bar{\psi}]\ e^{\mathbf{i}S[\psi,\bar{\psi},A]} \ .$$

We must specify how the fermions are coupled to the gauge field. The simplest example is if A is a $U(1)$ gauge field and ψ is minimally coupled:

$$S[\psi,\bar\psi,A]=\int \mathrm{d}^D x\bar\psi \mathbf{i}\not{D}\psi,\qquad \not{D}\psi\equiv\gamma^\mu\left(\partial_\mu+\mathbf{i}A_\mu\right)\psi.$$

We will focus on this abelian example, but you could imagine instead that A_μ is a non-Abelian gauge field for the group G, and ψ is in a representation R, with gauge generators $T^A(R)$ ($A=1...\mathrm{dim}G$), so the coupling would be

$$\bar\psi\not{D}\psi=\bar\psi_a\gamma^\mu\left(\partial_\mu\delta_{ab}+\mathbf{i}A^A_\mu T^A(R)_{ab}\right)\psi_b\ . \tag{1.1}$$

Much of the discussion below applies for any even D.

In the absence of a mass term, the action (in the Weyl basis) involves no coupling between L and R:

$$S[\psi,\bar\psi,A]=\int \mathrm{d}^D x\left(\psi_L^\dagger \mathbf{i}\sigma^\mu D_\mu\psi_L+\psi_R^\dagger\mathbf{i}\bar\sigma^\mu D_\mu\psi_R\right)$$

and therefore is invariant under the global chiral rotation

$$\psi\to e^{\mathbf{i}\alpha\gamma^5}\psi,\quad \psi^\dagger\to\psi^\dagger e^{-\mathbf{i}\alpha\gamma^5},\quad \bar\psi\to\bar\psi e^{+\mathbf{i}\alpha\gamma^5}.\quad \text{That is: }\ \psi_L\to e^{\mathbf{i}\alpha}\psi_L,\quad \psi_R\to e^{-\mathbf{i}\alpha}\psi_R.$$

(The mass term couples the two components

$$L_m=\bar\psi\left(\mathrm{Re}m+\mathrm{Im}m\gamma^5\right)\psi=m\psi_L^\dagger\psi_R+h.c.;$$

notice that the mass parameter is complex.) The associated Noether current is $j^5_\mu=\bar\psi\gamma^5\gamma_\mu\psi$, and it seems like we should have $\partial^\mu j^5_\mu\stackrel{?}{=}0$ if $m=0$. This follows from the massless (classical) Dirac equation $0=\gamma^\mu\partial_\mu\psi$. (With the mass term, we would have instead $\partial^\mu j^5_\mu\stackrel{?}{=}2\mathbf{i}\bar\psi\left(\mathrm{Re}m\gamma^5+\mathrm{Im}m\right)\psi$.)

Notice that there is another current $j^\mu=\bar\psi\gamma^\mu\psi$. j^μ is the current that is coupled to the gauge field, $L\ni A_\mu j^\mu$. The conservation of this current is required for gauge invariance of the effective action

$$S_{\mathrm{eff}}[A_\mu]\stackrel{!}{=}S_{\mathrm{eff}}[A_\mu+\partial_\mu\lambda]=-\mathbf{i}\log\left\langle e^{\mathbf{i}\int\lambda(x)\partial_\mu j^\mu}\right\rangle+S_{\mathrm{eff}}[A_\mu].$$

No matter what happens we can't find an anomaly in j^μ. The anomalous one is the other one, the *axial current*.

To derive the conservation law for the axial current we can use the Noether method. This amounts to substituting $\psi'(x)\equiv e^{\mathbf{i}\alpha(x)\gamma^5}\psi(x)$ into the action:

$$S_F[\psi']=\int \mathrm{d}^D x\bar\psi e^{+\mathbf{i}\alpha\gamma^5}\mathbf{i}\not{D}e^{\mathbf{i}\alpha\gamma^5}\psi=\int \mathrm{d}^D x\left(\bar\psi\mathbf{i}\not{D}\psi+\bar\psi\mathbf{i}\gamma^5\left(\not{\partial}\alpha\right)\psi\right)\stackrel{\mathrm{IBP}}{=}S_F[\psi]-\mathbf{i}\int\alpha(x)\partial^\mu\mathrm{tr}\bar\psi\gamma^5\gamma_\mu\psi$$

up to terms of $\mathcal{O}(\alpha^2)$. Then we can completely get rid of $\alpha(x)$ if the change of integration variables in the path integral, *i.e.* if $[D\psi'] \stackrel{?}{=} [D\psi]$. Usually this is true, but here we pick up an interesting Jacobian.

Claim:

$$e^{\mathbf{i}S_{\text{eff}}[A]} = \int [D\psi' D\bar{\psi}'] e^{\mathbf{i}S_F[\psi']} = \int [D\psi D\bar{\psi}] e^{\mathbf{i}S_F[\psi] + \mathbf{i}\int \mathrm{d}^D x \alpha(x)\left(\partial_\mu j_5^\mu - \mathcal{A}(x)\right)}$$

where $\mathcal{A}$ comes from the variation of the measure. That is,

$$[D\psi' D\bar{\psi}'] = [D\psi D\bar{\psi}] \det\left(e^{\mathbf{i}\alpha\gamma^5}\right) \equiv [D\psi D\bar{\psi}] e^{-\mathbf{i}\int \alpha\mathcal{A}}$$

so

$$e^{-\mathbf{i}\int \alpha\mathcal{A}} = e^{\mathrm{Tr}\log e^{\mathbf{i}\alpha\gamma^5}} = e^{\mathrm{Tr}\left(\mathbf{i}\alpha\gamma^5\right)}$$

or more explicitly but very formally we can write the anomaly as:

$$\mathcal{A}(x) = \sum_n \mathrm{tr}\bar{\xi}_n(x)\gamma^5\xi_n(x) \tag{1.2}$$

where ξ_n are a basis of eigenspinors of the Dirac operator. This big Tr is the trace over the space of functions on which $\not{D}$ acts, including both spinor indices and function labels. I'll use tr for the spinor trace.

The expression above for S_{eff} is actually independent of α, since the path integral is invariant under a change of variables. For a conserved current, α would multiply the divergence of the current and this demand would imply current conservation. Here this implies that instead of current conservation we have a specific violation of the current:

$$\partial^\mu j_\mu^5 = \mathcal{A}(x).$$

What is the anomaly $\mathcal{A}$? [Polyakov §6.3] An alternative useful (perhaps more efficient) perspective is that the anomaly arises from trying to define the axial current operator, which after all is a composite operator. Thus we should try to compute

$$\partial_\mu \langle j_5^\mu \rangle = \partial_\mu \left\langle \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \right\rangle$$

– the coincident operators on the RHS need to be regulated.

The classical (massless) Dirac equation immediately implies that the axial current is conserved (up to contact terms, meaning collisions with other operators in the expectation value)

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi\right) \stackrel{?}{=} 0.$$

Consider, on the other hand, the (Euclidean vacuum) expectation value

$$J^5_\mu \equiv \left\langle \bar\psi(x)\gamma_\mu\gamma^5\psi(x)\right\rangle \equiv Z^{-1}[A]\int [D\psi D\bar\psi] e^{-S_F[\psi]} j^5_\mu(x)$$

$$= \quad = \quad + \quad + \quad + \cdots$$

$$= -\mathrm{tr}_\gamma \gamma_\mu\gamma^5 G^{[A]}(x,x) \tag{1.3}$$

where the blob represents $G^{[A]}$, the Green's function of the Dirac operator in the gauge field background (and the figure is from Polyakov's book). The $\mathbf{x}$ is the insertion of the current $j^5_\mu = \bar\psi\gamma^5\gamma_\mu$. The minus sign in the last line is from the fermion loop.

We can construct it out of eigenfunctions of $\mathbf{i}\not{D}$:

$$\mathbf{i}\not{D}\xi_n(x) = \epsilon_n\xi_n(x), \qquad \bar\xi_n(x)\mathbf{i}\gamma^\mu\left(-\overleftarrow{\partial}_\mu + \mathbf{i}A_\mu\right) = \epsilon_n\bar\xi_n \tag{1.4}$$

in terms of which[1]

$$G^{[A]}(x,x') = \sum_n \frac{1}{\epsilon_n}\xi_n(x)\bar\xi_n(x'). \tag{1.5}$$

(I am suppressing spinor indices all over the place, note that here we are taking the outer product of the spinors to make a matrix.) [End of Lecture 1]

We want to define the coincidence limit, as $x' \to x$. The problem with this limit arises from the large $|\epsilon_n|$ eigenvalues; the contributions of such short-wavelength modes are local and most of them can be absorbed in renormalization of couplings. It should not (and does not) matter how we regulate them, but we must pick a regulator. A convenient choice here is heat-kernel regulator:

$$G^{[A]}_s(x,x') \equiv \sum_n e^{-s\epsilon_n^2}\frac{1}{\epsilon_n}\xi_n(x)\bar\xi_n(x')$$

and

$$J^5_\mu(x) = \sum_n e^{-s\epsilon_n^2}\frac{1}{\epsilon_n}\bar\xi_n(x)\gamma^5\gamma_\mu\xi_n(x)\ .$$

The anomaly is

$$\partial^\mu J^5_\mu = \partial^\mu\left\langle j^5_\mu\right\rangle = -\sum_n \mathbf{i}\partial^\mu\left(\bar\xi_n\gamma_\mu\gamma^5\xi_n\right)\frac{e^{-s\epsilon_n^2}}{\epsilon_n}.$$

The definition (1.4) says

$$\mathbf{i}\partial^\mu\left(\bar\xi_n\gamma_\mu\gamma^5\xi_n\right) = -2\epsilon_n\bar\xi_n\gamma_5\xi_n$$

[1] Actually, this step is full of danger, but I promise it works out. See §1.2 below for the full story.

using $\{\gamma^5, \gamma^\mu\} = 0$. (Notice that the story would deviate dramatically here if we were studying the vector current which lacks the γ^5.) This gives

$$\partial^\mu J^5_\mu(x) = 2\mathrm{tr}_\gamma \langle x| \gamma^5 e^{-s\left(\mathbf{i}\not{D}\right)^2} |x\rangle \tag{1.6}$$

with

$$(\mathbf{i}\not{D})^2 = -\left(\gamma_\mu \left(\partial_\mu + \mathbf{i}A_\mu\right)\right)^2 = -\left(\partial_\mu + \mathbf{i}A_\mu\right)^2 - \frac{\mathbf{i}}{2}\Sigma_{\mu\nu}F^{\mu\nu}$$

where $\Sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ is the spin Lorentz generator. We used $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \Sigma_{\mu\nu}$. (1.6) is the equation we got from the variation of the measure, (1.2), but now better defined by the heat kernel regulator.

We've shown that in any even dimension,

$$\partial^\mu \left\langle j^5_\mu(x)\right\rangle = 2\mathrm{tr}_\gamma \langle x| \gamma^5 e^{s\not{D}^2} |x\rangle \tag{1.7}$$

This can now be expanded in small s, which amounts to an expansion in powers of A, F. If there is no background field, $A = 0$, we get

$$\langle x| e^{-s\left(\mathbf{i}\not{\partial}\right)^2} |x\rangle = \int \mathrm{d}^D p\ e^{-sp^2} = \underbrace{K_D}_{=\frac{\Omega_{D-1}}{(2\pi)^D}} \frac{1}{s^{D/2}} \overset{D=4}{=} \frac{1}{16\pi^2 s^2}. \tag{1.8}$$

This term will renormalize the charge density

$$\rho(x) = \left\langle \psi^\dagger\psi(x)\right\rangle = \mathrm{tr}\gamma^0 G(x,x),$$

for which we must add a counterterm (in fact, it is accounted for by the counterterm for the gauge field kinetic term, *i.e.* the running of the gauge coupling). But it will not affect the axial current conservation which is proportional to

$$\mathrm{tr}\left(\gamma^5 G(x,x)\right)|_{A=0} \propto \mathrm{tr}\gamma^5 = 0.$$

Similarly, bringing down more powers of $(\partial + A)^2$ doesn't give something nonzero since the γ^5 remains.

In $D = 4$, the first term from expanding $\Sigma_{\mu\nu}F^{\mu\nu}$ is still zero from the spinor trace. (Not so in $D = 2$.) The first nonzero term comes from the next term:

$$\mathrm{tr}\left(\gamma_5 e^{-s\left(\mathbf{i}\not{D}\right)^2}\right)_{xx} = \underbrace{\left\langle x|e^{-s(\mathbf{i}D)^2}|x\right\rangle}_{\overset{(1.8)}{=}\frac{1}{16\pi^2 s^2}+\mathcal{O}(s^{-1})} \cdot \frac{s^2}{8} \cdot (\mathbf{i}^2) \underbrace{\mathrm{tr}\left(\gamma^5\Sigma^{\mu\nu}\Sigma^{\rho\lambda}\right)}_{=4\epsilon^{\mu\nu\rho\lambda}} \cdot \underbrace{\mathrm{tr}_c}_{\text{color}} (F_{\mu\nu}F_{\rho\lambda}) + \mathcal{O}(s^1)\ .$$

In the abelian case, just ignore the trace over color indices, tr_c. The terms that go like positive powers of s go away in the continuum limit. Therefore

$$\partial_\mu J^\mu_5 = -2 \cdot \frac{1}{16\pi^2 s^2} \cdot \frac{s^2}{8} \cdot 4\epsilon^{\mu\nu\rho\lambda}\mathrm{tr}_c F_{\mu\nu}F_{\rho\lambda} + \mathcal{O}(s^1) = -\frac{1}{8\pi^2}\mathrm{tr}F_{\mu\nu}\left(\star F\right)^{\mu\nu}. \tag{1.9}$$

(Here $(\star F)^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$.) This is the chiral anomaly formula. It can also be usefully written as:

$$\partial_\mu J_5^\mu = -\frac{1}{8\pi^2}\mathrm{tr}F\wedge F = -\frac{1}{32\pi^2}\vec{E}\cdot\vec{B}.$$

- This object on the RHS is a total derivative. In the abelian case it is

  $$F\wedge F = \mathrm{d}\left(A\wedge F\right)\ .$$

  Its integral over spacetime is a topological invariant (in fact $16\pi^2$ times an integer) characterizing the gauge field configuration. How do I know it is an integer? The anomaly formula! The change in the number of left-handed fermions minus the number of right-handed fermions during some time interval is:

  $$\Delta Q_A \equiv \Delta\left(N_L - N_R\right) = \int \mathrm{d}t\partial_t J_0^5 = \int_{M_4}\partial^\mu J_\mu^5 = \int_{M_4}\frac{F\wedge F}{8\pi^2}$$

  where M_4 is the spacetime region under consideration. If nothing is going on at the boundaries of this spacetime region (*i.e.* the fields go to the vacuum, or there is no boundary, so that no fermions are entering or leaving), we can conclude that the RHS is an integer.

  More generally, $\frac{\mathrm{tr}F\wedge F}{8\pi^2}$ integrates to an integer on any closed 4-manifold on which we can put fermions (*i.e.* which admits a *spin structure*).

- Look back at the diagrams in (1.3). Which term in that expansion gave the nonzero contribution to the axial current violation? In $D = 4$ it is the diagram with three current insertions, the ABJ triangle diagram. So in fact we did end up computing the triangle diagram. But this calculation also shows that nothing else contributes, even non-perturbatively.

- We chose a particular regulator above. The answer we got did not depend on the cutoff; in fact, whatever regulator we used (as long as it preserves the chiral symmetry!) we would get this answer. I am not proving this, but it must be true if the theory makes any sense. We will see strong evidence for it below.

- Consider what happens if we redo this calculation in other dimensions. We only consider even dimensions because in odd dimensions there is no analog of γ^5 – the Dirac spinor representation is irreducible, and there is no notion of chirality. In $2n$ dimensions, we need n powers of $\Sigma^{\mu\nu}F_{\mu\nu}$ to soak up the indices on the epsilon tensor.

  Actually there is an analogous phenomenon in odd dimensions (sometimes called parity anomaly) of an effect that is independent of the masses of the fields, where the spinor trace produces an $\epsilon_{\mu\nu\rho}$, which you already studied on the homework. Instead of F^n, the thing that appears is the Chern-Simons term.

- If we had kept the non-abelian structure in (1.1) through the whole calculation, the only difference is that the trace in (1.9) would have included a trace over representations of the gauge group. With multiple fermion flavors ($I = 1..N_f$), we could have considered also a non-abelian flavor transformation in the chiral symmetry

$$\psi_I \to \left(e^{\mathbf{i}\gamma^5 g^a \tau^a}\right)_{IJ} \psi_J$$

for some $\mathsf{su}(N_f)$ flavor rotation generator τ^a. This is a symmetry of $\bar{\psi}_I \mathbf{i}\not{D}\psi_I$ (with no mass terms). Then we would have found (recall that $F = F^A T^A$):

$$\partial^\mu j^{5a}_\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda} F^A_{\mu\nu} F^B_{\rho\lambda} \mathrm{tr}_{c,f}\left(T^A T^B \tau^a\right). \tag{1.10}$$

Note that in this expression, the matrix in the trace is more explicitly

$$(T^A \otimes \mathbb{1}_f)(T^B \otimes \mathbb{1}_f)(\mathbb{1}_c \otimes \tau^a) = T^A T^B \otimes \tau^a \ . \tag{1.11}$$

A special case of this is if we have multiple species of fermion fields but consider the diagonal chiral symmetry ($\tau^a = \mathbb{1}$): their contributions to the anomaly add. Sometimes they can cancel; the Electroweak gauge interactions are an example of this.

- Most generally, consider a collection of fermions transforming under symmetry group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$ and couple to background gauge fields $A^{1,2,3}$ for all three groups. We'll call a "$\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly" the diagram with insertions of currents for $\mathsf{G}_1, \mathsf{G}_2$ and G_3.

  Above we computed the contribution from whole Dirac fermions. We can compute separately the contributions of the L and R Weyl components: there is a factor of half and a relative sign. The result for the anomaly of the current for G^1 coming from the background gauge fields for $\mathsf{G}^{2,3}$ is

$$\partial_\mu j_1^{A\mu} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F^{2B}_{\mu\nu} F^{3C}_{\rho\sigma} \sum_f (-1)^f \mathrm{tr}_{R(f)}\{T_1^A, T_2^B\} T_3^C. \tag{1.12}$$

  The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$, and T_1^A are a basis of generators of the Lie algebra of G_1 etc. in the representation of the field f. By $(-1)^f$ I mean $\pm$ for left- and right-handed fermions respectively. Here $\{A, B\} = AB + BA$ means anticommutator; the second term comes from reversing the arrows of the fermion lines in the triangle diagram. Using this formula you can check that the Standard Model gauge group is anomaly-free.

1.2 Zeromodes of the Dirac operator

Do you see why I said that the step involving the fermion Green's function was full of danger? The danger arises because the Dirac operator (whose inverse is the Green's function) can have *zeromodes*, eigenspinors with eigenvalue $\epsilon_n = 0$. In that case, $\mathbf{i}\not{D}$ is *not* invertible, and the expression (1.5) for G is ambiguous. This factor of ϵ_n is about to be cancelled when we compute the divergence of the current and arrive at (1.2). Usually this kind of thing is not a problem because we can lift the zeromodes a little and put them back at the end. But here it is actually hiding something important. The zeromodes cannot just be lifted. This is true because nonzero modes of $\mathbf{i}\not{D}$ must come in left-right pairs: this is because $\{\gamma^5, \mathbf{i}\not{D}\} = 0$, so $\mathbf{i}\not{D}$ and γ^5 cannot be simultaneously diagonalized in general. That is: if $\mathbf{i}\not{D}\xi = \epsilon\xi$ then $(\gamma^5\xi)$ is also an eigenvector of $\mathbf{i}\not{D}$, with eigenvalue $-\epsilon$. By taking linear combinations

$$\chi_n^{L/R} = \frac{1}{2}\left(1 \pm \gamma^5\right)\xi_n$$

these two partners can be arranged into a pair of simultaneous eigenvectors of $(\mathbf{i}\not{D})^2$ (with eigenvalue ϵ_n^2) and of γ^5 with $\gamma^5 = \pm$ respectively.

Only for $\epsilon = 0$ does this fail, so zeromodes can come by themselves. So you can't just smoothly change the eigenvalue of some ξ_0 from zero unless it has a partner with whom to pair up.

This leads us to a deep fact, called the (Atiyah-Singer) *index theorem*: *only* zeromodes can contribute to the anomaly. Any mode ξ_n with nonzero eigenvalue has a partner (with the same eigenvalue of $(\mathbf{i}\not{D})^2$) with the opposite sign of γ^5; hence they cancel exactly in

$$\mathrm{tr}\gamma^5 e^{-s(\mathbf{i}\not{D})^2} = \sum_{n,\epsilon_n\neq 0} \underbrace{\sum_{L/R} \bar{\chi}_n^{L/R}\gamma^5\chi_n^{L/R} e^{-s\epsilon_n^2}}_{=0} + \text{ zeromodes } .$$

We can choose our eigenfunctions to be normalized $\int d^D x \bar{\chi}_i \chi_j = \delta_{ij}$ and of definite chirality $\gamma^5\chi = \pm\chi$. So the anomaly equation tells us that the number of zeromodes of the Dirac operator $\mathbf{i}\not{D}^{[A]}$ in some configuration of the background field, weighted by handedness (*i.e.* with a + for L and - for R) is equal to

$$n_L - n_R = \int d^D x \mathcal{A}(x) = -\int \frac{1}{8\pi^2} F \wedge F.$$

A practical consequence for us is that it makes manifest that the result is independent of the regulator s.

Another consequence is that in the background of a gauge field configuration with nonzero $n_L - n_R = q$, the vacuum to vacuum amplitude

$$Z[A^q] = \int [D\psi D\bar{\psi}] e^{\mathbf{i}\int d^D x \bar{\psi} \mathbf{i} \not{D} \psi} = \det \mathbf{i}\not{D} = 0$$

vanishes, since it is the determinant of an operator with a kernel. Rather, only amplitudes for transitions that change the chiral charge by q are allowed. A (localized) gauge field configuration with $\int F \wedge F \neq 0$ is called an *instanton*.

1.3 The physics of the anomaly

Emergence of the Dirac equation. Consider free fermions hopping on a chain of sites:

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c. \tag{1.13}$$

(The particular choice of nearest-neighbor hopping is not special for what I'm about to say.) Since this is translation-invariant, the single-particle Hamiltonian is diagonalized by Fourier modes: $c_n = \oint đk e^{\mathbf{i}kna} c_k$ (where a is the lattice spacing):

$$H = \oint đk c_k^\dagger c_k \epsilon(k) \tag{1.14}$$

and for the particular choice of nearest-neighbor hopping in one dimension we get

$$\epsilon(k) = -2t(\cos ka - 1) \ . \tag{1.15}$$

It is sometimes convenient to add a chemical potential term to account for the number of electrons:

$$H - \mu N = \oint đk c_k^\dagger c_k \left(\epsilon(k) - \mu\right) \ . \tag{1.16}$$

Introduce an infrared regulator so that the levels are discrete – put them in a box of length L, so that $k_\ell = \frac{2\pi\ell}{L}$, $\ell \in \mathbb{Z}$. The groundstate of N such fermions (N determines the chemical potential) is described by filling the N lowest-energy single particle levels, up to the Fermi momentum: $|k| \leq k_F$ are filled. The energy of the last filled level is $\epsilon(k_F) = \mu$, the Fermi energy. (In Figure 1, the red circles are possible 1-particle states, and the green ones are the occupied ones.)

Starting in the groundstate of N electrons, the lowest-energy state available in which to add a fermion is the one just above the Fermi level. Adding an electron in this level costs energy

$$\epsilon(k) - \mu = \epsilon(k_F) + (k - k_F)\partial_k \epsilon|_{k=k_F} + \mathcal{O}(k - k_F)^2 - \mu = v_F \delta k + \mathcal{O}(\delta k)^2. \tag{1.17}$$

The fields near these Fermi points $k = \pm k_F$ in k-space therefore satisfy the Dirac equation

$$(\omega - v_F \delta k)\, \psi_L = 0, \quad (\omega + v_F \delta k)\, \psi_R = 0 \tag{1.18}$$

where $\delta k \equiv k - k_F$ and $v_F \equiv |\partial_k \epsilon|_{k=k_F}$. I call this the Dirac equation because it is the equations of motion for the action

$$S[\psi, \bar{\psi}] = \int d^2x \bar{\psi} \mathbf{i} \partial\!\!\!/ \psi \tag{1.19}$$

where γ^μ are 2×2 and the upper/lower component of ψ creates fermions near the left/right Fermi point: $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ The basis of gammas that gives (1.18) is $\gamma^0 = \sigma^1, \gamma^1 = \mathbf{i}\sigma^2$. I chose units of length where $v_F = 1$ (rather than the actual speed of light).

Thus the Dirac equation emerges from a very generic and simple lattice model. The left-(right-)handed fermion is left-(right-)moving, very convenient. The Dirac antiparticle is a *hole*: the lowest energy state with one fewer particle is obtained by removing an electron just below the Fermi level. The relativistic approximation breaks down when the $\mathcal{O}(k - k_F)^2$ terms are appreciable, *i.e.* if we put in enough energy to see the curvature of the band.

[End of Lecture 2]

The chiral anomaly in the lattice model in finite volume. [Polyakov, page 102; Kaplan 0912.2560 §2.1; Alvarez-Gaumé]

This action is preserved by a chiral transformation and would therefore seem to imply a conserved axial current, whose conserved charge is the number of left moving fermions minus the number of right moving fermions. But the fields ψ_L and ψ_R are not independent; with high-enough energy excitations, you reach the bottom of the band (near $k = 0$ here) and you can't tell the difference. This means that the numbers are *not* separately conserved.

We can do better in this 1+1d example and show that the amount by which the axial current is violated is given by the anomaly formula. Consider subjecting our poor 1+1d free fermions to an electric field $E_x(t)$ which is constant in space and slowly varies in time[2] Suppose we gradually turn it on and then turn it off; here gradually means slowly enough that the process is adiabatic. Then each particle experiences a force $\partial_t p = eE_x$ and its net change in momentum is

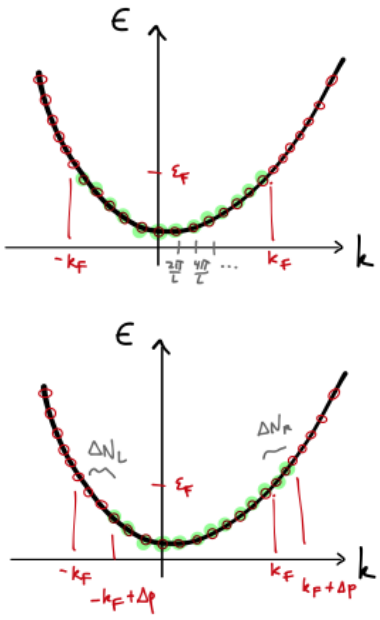


Figure 1: Green dots represent occupied 1-particle states. Top: In the groundstate. Bottom: After applying $E_x(t)$. Here I am just drawing the bottom of the band, where $\epsilon(k)$ can be approximated by $\frac{k^2}{2m}$.

$$\Delta p = e \int \mathrm{d}t E_x(t).$$

This means that the electric field puts the fermions in a state where the Fermi surface $k = k_F$ has shifted to the right by Δp, as in the figure. Notice that the total number of fermions is of course the same – charge is conserved.

Now consider the point of view of the low-energy theory at the Fermi points. This theory has the action

$$S[\psi] = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\mathbf{i}\gamma^\mu D_\mu\psi \ .$$

In the process above, we have added $N_R = \frac{\Delta p}{2\pi/L}$ right-moving particles and taken away $|N_L|$ left-moving particles (with $N_L = -N_R$), that is *added* N_L left-moving holes (aka

[2]To do this in the lattice model, modify the Hamiltonian by

$$H = -t\sum_n c_n^\dagger e^{\mathbf{i}eA_x(t)}c_{n+1} + h.c. \tag{1.20}$$

anti-particles). The axial charge of the state has changed by

$$\Delta Q_A = \Delta(N_R - N_L) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}\Delta p = \frac{L}{\pi}e\int \mathrm{d}t E_x(t) = \frac{e}{\pi}\int \mathrm{d}t\mathrm{d}x E_x = \frac{e}{2\pi}\int \epsilon_{\mu\nu}F^{\mu\nu}$$

On the other hand, the LHS is $\Delta Q_A = \int \partial^\mu J^A_\mu$. We can infer a local version of this equation by letting E vary slowly in space as well, and we conclude that

$$\partial_\mu J^\mu_A = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

This agrees exactly with the anomaly equation in $D = 1{+}1$ produced by the calculation above in (1.7) (see the homework).

1.4 't Hooft anomaly matching

The most important fact about anomalies is that they are RG invariants. The existence of the anomaly means that the partition function varies by some particular phase under the anomalous symmetry,

$$Z \to e^{\mathbf{i}\int \alpha \mathcal{A}} Z \ . \tag{1.21}$$

But an RG transformation (doing the integrals in a certain order and relabelling rulers) must preserve the partition function.

Why is this a big deal? Much of physics is about trying to match microscopic (UV) and long-wavelength (IR) descriptions. That is, we are often faced with questions of the form "what could be a microscopic Hamiltonian that produces these phenomena?" and "what does this microscopic Hamiltonian do at long wavelengths?". Anomalies are precious to us, because they are RG-invariant information: any anomaly in the UV description must be realized somehow in the IR description.

This tool has been used to great effect in the last few decades to study strongly-coupled and otherwise intractable theories. The first application (by 't Hooft) was to constrain the possibility that quarks themselves can be composite. He looked for gauge theories where there are massless gauge-invariant particles (baryons) with the quantum numbers of the quarks. Anomaly matching provides crucial evidence for the correctness of Seiberg duality.

It can also be used to help decide whether a gauge theory must spontaneously break chiral symmetry. The idea is: there may be no way for massless fermionic degrees of freedom in a candidate chiral-symmetry-preserving low-energy theory to saturate the anomaly, but there is another possibility. There is a way for bosons to contribute to the anomaly: if they transform non-linearly under the symmetry, *i.e.* if the symmetry is spontaneously broken, they can appear in Wess-Zumino-Witten terms, which can

produce the required anomalous variation of the action. More on this in the section about pions.

1.5 Further comments

- Another useful perspective on anomaly is as an obstruction to gauging the symmetry. Gauging a symmetry means creating a new system where the symmetry is a redundancy of the description, by coupling to gauge fields. If the symmetry is not conserved in the presence of background gauge fields, the resulting theory would be inconsistent.
- Above I've described an example of an anomaly of a continuous symmetry. Discrete symmetries can also be anomalous. One way to arrive at this is to start with a continuous symmetry with an anomaly and explicitly break it to a discrete subgroup.
- Anomaly is actually a more basic notion even than phase of matter: The anomaly is a property of the degrees of freedom (*i.e.* of the Hilbert space) and how the symmetry acts on them, independent of a choice of Hamiltonian or action. Multiple phases of matter can carry the same anomaly.
- There is a long story about anomalies and Symmetry Protected Topological (SPT) phases of matter. If you want to read about this, §3 here might be a place to start.

Other anomalies. There are some other examples of anomalies whose existence is worth mentioning.

One is that there can be anomalies where the role of the field strength F is played by the curvature of spacetime R. The chiral anomaly gets such a contribution. Since space is curved, this constrains the hypercharge assignments in the Standard Model.

Spinors in curved spacetime. To couple integer-spin fields to curved space is not such a big deal: just replace every $\eta_{\mu\nu}$ by $g_{\mu\nu}$, use covariant derivatives, and use the covariant volume form. For example:

$$\int d^D x \left(\partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} - V(\phi)\right) \rightsquigarrow \int d^D x \sqrt{g} \left(\partial_\mu \phi \partial_\nu \phi g^{\mu\nu}(x) - V(\phi)\right). \tag{1.22}$$

Coupling spinors to curved space is a little more involved, and requires the introduction of the spin connection. This is a useful device in other contexts, for example, for computing curvatures by hand (see *e.g.* §9 of these notes).

The first step is to introduce the vierbeins ('vier' means ''four' in German; in general dimension, they are called vielbeins instead, since 'viel' means 'many')

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab} \ . \tag{1.23}$$

Here $a, b = 0...D-1$ are tangent space indices, which are contracted with the ordinary Minkowski metric. The es are an orthonormal frame on the tangent space that varies from point to point. This description in terms of e introduces a redundancy under local Lorentz transformations $\mathsf{SO}(3,1)$ (that preserve $g_{\mu\nu}$). The required connection is the spin connection ω^{ab}_μ. It can be defined by demanding that the vielbeins are covariantly constant:

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{\mu b} e^b_\nu \stackrel{!}{=} 0 \tag{1.24}$$

where Γ is the usual Christoffel symbol. The field strength of the spin connection is

$$(R_{\mu\nu})^a{}_b = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} - [\omega_\mu, \omega_\nu]^a{}_b = R_{\mu\nu}{}^\rho{}_\sigma e^a_\rho e^\sigma_b \tag{1.25}$$

where the last object is the ordinary Riemann curvature.

In terms of these ingredients, the spinor covariant derivative is

$$\mathcal{D}_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \omega^{ab}_\mu \left(\Sigma_{ab}\right)^\beta{}_\gamma \psi_\beta \tag{1.26}$$

where $\Sigma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$ are the local Lorentz generators. The curved-space Dirac action is then

$$S[\Psi, g] = \int d^D x \sqrt{g} \bar{\Psi} \mathbf{i} \gamma^\mu \mathcal{D}_\mu \Psi \tag{1.27}$$

where $\gamma^\mu = \gamma^\mu(x) \equiv \gamma^a e^\mu_a(x)$. The γ^a are the ordinary flat-spacetime gammas.

Redoing our calculation above gives

$$\mathcal{D}_\mu j^\mu_A = -\frac{1}{384\pi^2} \sum_f Q_f (-1)^f \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}{}^{\lambda\tau}. \tag{1.28}$$

An extreme example of such an anomaly is an anomaly in diffeomorphism invariance – a purely gravitational anomaly. This only happens in $D = 8k+2$ dimensions.

Back in $D = 3+1$, $\mathsf{SU}(2)$ gauge theory with an odd number of Weyl fermions in a half-integer spin representation is anomalous. The case of spin-half is called the Witten anomaly or $\mathsf{SU}(2)$ anomaly.

The SU(2) anomaly. There are no perturbative anomalies (meaning, ones coming from triangle diagrams) for the case of SU(2), since the **2** is pseudo-real (isomorphic to its conjugate representation). But there can be a more subtle way for the fermion measure to vary under a gauge transformation – a non-perturbative anomaly.

The path integral over a Dirac fermion Ψ in some representation of a gauge group G is

$$\int D\Psi D\bar{\Psi} e^{-\int d^4x \bar{\Psi} \mathbf{i}\not{D}\Psi} = \det\left(\mathbf{i}\not{D}\right) = \prod_n \epsilon_n. \tag{1.29}$$

Here $\mathbf{i}\not{D}$ is the Dirac operator with background fields for G. We've discussed above how to regulate such things in a gauge-invariant way, and it is gauge invariant.

Now consider a single Weyl fermion $\psi_L = \frac{1}{2}(1+\gamma^5)\Psi$ in some representation of G:

$$\int D\psi_L D\bar{\psi}_L e^{-\int d^4x \bar{\psi}_L \mathbf{i}\sigma^\mu D_\mu \psi_L} = \pm \det\left(\mathbf{i}\not{D}\frac{1+\gamma^5}{2}\right) = \sqrt{\det \mathbf{i}\not{D}}. \tag{1.30}$$

Recall that the eigenvalues of $\mathbf{i}\not{D}$, when nonzero, come in left-right pairs. (Let's assume that no eigenvalues are zero, or else the whole thing is zero.) So the square root just means taking one of each pair. The problem is picking the sign.

To try to define the sign, pick a reference gauge field configuration $A^\star_\mu$, and define the square root for this configuration

$$\sqrt{\det \mathbf{i}\not{D}_{A^\star}} = \prod_{\epsilon>0} \epsilon_n \tag{1.31}$$

to be just the positive eigenvalues. To define the square root for any other configuration A, find a path from $A^\star$ to A, and follow the eigenvalues. If an odd number of the positive eigenvalues at $A^\star$ go through zero, then the sign of the square root flips.

Now we must ask: is this choice of sign gauge invariant? That is, do we get the same sign for A_μ and for its gauge image

$$A^\Omega_\mu = \Omega(x) A_\mu \Omega(x)^{-1} + \mathbf{i}\Omega(x)\partial_\mu \Omega(x)^{-1} \quad ? \tag{1.32}$$

If not, then clearly this gauge transformation $\Omega(x)$ cannot be regarded as an equivalence, since A and A^Ω would have different weight in

$$Z = \int DA e^{-S[A]} \int D\psi_L D\bar{\psi}_L e^{-\int d^4x \bar{\psi}_L \mathbf{i}\sigma^\mu D_\mu \psi_L}. \tag{1.33}$$

Now, there are many Ω we could consider. Only Ω that approach the identity map at $x \to \infty$ in $\mathbb{R}^4$ are equivalences. For such Ω, we can identify all the points at infinity

and $\mathbb{R}^4 \cup \infty \simeq S^4$. So such gauge transformations are maps $\Omega : S^4 \to \mathsf{G}$. In the case of $\mathsf{G} = \mathsf{SU}(2)$, it's a nontrivial fact that $\pi_4(\mathsf{SU}(2) \simeq S^3) = \mathbb{Z}_2$ – there are two classes of such gauge transformations.

The Witten anomaly happens because, with the definition of sign above,

$$\sqrt{\det \mathbf{i}\not{D}_{A^\Omega}} = -\sqrt{\det \mathbf{i}\not{D}_A} \tag{1.34}$$

if Ω is in the nontrivial homotopy class.

For more, see David Tong's notes on anomalies.

[End of Lecture 3]

2 Effective Field Theory, continued

2.1 General relativity as an EFT

Let's return to playing the effective field theory game. Recall that the game is played by filling out the following survey:

1. What are the dofs?
2. What are the symmetries and what are the redundancies of the description?
3. What is the cutoff?

Then the output is an action, which is a sum of all terms made from the dofs, respecting the symmetries and redundancies, organized as a derivative expansion with higher order terms suppressed by more powers of the cutoff.

In response to a question in lecture, I gave the following illustration of this procedure: Einstein sort of played this game in 1915 in building a theory of gravity. His answer for the dofs was: a metric on spacetime $g_{\mu\nu}(x)$. This is a coordinate-dependent description of a line element $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ that gives the distances between spacetime points. This description is redundant in that the same line element can be written in different coordinate systems (such as $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$). The cutoff is the mass scale appearing in Newtonian gravity: $G_N = \frac{\#}{M_P^2}$, the Planck mass. (Here I'm using units with $\hbar = c = 1$.)

The demand that physics is independent of the choice of coordinate system is highly constraining, and the only terms one can write down are

$$S[g_{\mu\nu}] = \#M_P^2 \int d^4x \left(\Lambda + \#R + \frac{\#}{M_P^2} R^2 + \frac{\#}{M_P^4} R^3 + \frac{\#}{M_P^4} DRDR \cdots \right) \tag{2.1}$$

where R is the Ricci scalar, and R^n represents various possible contractions of n powers of the Riemann tensor.

This the order in which Einstein should have written the terms, if he were following Wilson's rules. The first term is the cosmological constant, the constant operator of dimension zero. Here it matters quite a bit, because it changes the equations of motion of the metric. It is observed to be very small in units of M_P^2. We don't know why this is the case. It is a gross violation of the rules of EFT.

The next term is the Einstein-Hilbert term, which is the only one Einstein included. The higher-order terms are too small to have any effect on any observation so far. They have all been done in a regime where the curvature R is small compared to M_P^2.

The coupling to matter is also largely determined by demanding coordinate invariance (replace $\eta^{\mu\nu}$ by $g^{\mu\nu}$, replace ∂_μ by covariant derivative D_μ, and for spinors do the thing we described above). Included in the derivative expansion should also be terms involving matter fields and curvatures, like RH^2, where H is the higgs field.

What are the loopholes in this argument? Well, the statement that the dimensionless numerical coefficients (all written as '#' above) are order one may not be correct – the cosmological constant is already a violation of this rule, so maybe some of the higher derivative terms could be important. Another loophole is in the choice of dofs. There could well be other light dofs, like a scalar field, that couple

I emphasize that this is a perfectly good quantum field theory. It is nice enough to be non-renormalizable and to tell us its regime of validity. (Of course it could break down at a scale lower than M_P if we are missing some important other dofs.) It can be studied in perturbation theory about some vacuum geometry (such as flat space for $\Lambda = 0$, or anti-de Sitter space (AdS) for $\Lambda < 0$ or de Sitter space for $\Lambda > 0$). The tree-level approximation, *i.e.* classical physics, has been good enough for all observations so far.

The problem of Quantum Gravity arises in asking what is a more microscopic theory for which this is a low-energy EFT. The only candidate answer to that question that we have is string theory. The physics questions for we need to answer such a question involve large curvature or otherwise-strong fields (such as inside black holes, or in the very early universe) or if we care about which values of the coefficients # (or what choices of matter coupled to gravity) are possible.

2.2 The Standard Model as an EFT, continued.

The Standard Model. [Schwartz, §29]

Table 1 shows the matter content of one generation of fermions Standard Model (and the Higgs) once again.

Whence the values of the charges under the $\mathsf{U}(1)$ ("hypercharge")? In fact, they are completely determined by demanding that the gauge group is not anomalous, *i.e.* that the $\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly vanishes for all choices of $\mathsf{G}_i \in \{\mathsf{SU}(3), \mathsf{SU}(2), \mathsf{U}(1)_Y\}$ in the presence of gauge fields for all three gauge groups.

To check this, it is enough to ignore the Higgs field and the dynamics of the gauge fields. As we discussed, the coupling to the Higgs field produces masses for the fermions in a way that preserves all of the gauge invariance, despite the fact that $\mathsf{SU}(2) \times \mathsf{U}(1)_Y$ acts in a chiral manner. But the Higgs field is a scalar that transforms linearly, and so it doesn't contribute to the anomaly and we can just set it to zero and ignore it, and

	$L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	e_R	ν_R	$Q = \begin{pmatrix}u_L \\ d_L\end{pmatrix}$	u_R	d_R	H
$\mathsf{SU}(3)$	-	-	-	□	□	□	-
$\mathsf{SU}(2)$	□	-	-	□	-	-	□
$\mathsf{U}(1)_Y$	$-\frac{1}{2}$	-1	0	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$

Table 1: The Standard Model fields and their quantum numbers under the gauge group. □ indicates fundamental representation, - indicates singlet. Except for the Higgs, each column is copied three times; each copy is called a *generation*. Except for the Higgs all the matter fields are Weyl fermions of the indicated handedness. Gauge fields as implied by the gauge groups. (Some people might leave out the right-handed neutrino, ν_R, which is totally neutral and therefore would be hard to observe.)

the calculation reduces to the one in the previous section. The homework outlines all the choices of $\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$.

Beyond the Standard Model with EFT. At what energy does the Standard Model stop working? Because of the annoying feature of renormalizibility, it doesn't tell us. However, we have experimental evidence against a cutoff on the Standard Model (SM) at energies less than something like 10 TeV. The evidence I have in mind is the absence of interactions of the form

$$\delta L = \frac{1}{M^2}\left(\bar{\psi}A\psi\right)\cdot\left(\bar{\psi}B\psi\right)$$

(where ψ represent various SM fermion fields and A, B can be various gamma and flavor matrices) with $M \lesssim 10$ TeV. Notice that I am talking now about interactions *other* than the electroweak interactions, which as we've just discussed, for energies above $M_W \sim 80$GeV cannot be treated as contact interactions – you can see the Ws propagate!

If such operators were present, we would have found different answers for experiments at LEP. But such operators would be present if we consider new physics in addition to the Standard Model (in most ways of doing it) at energies less than 10 TeV. For example, many interesting ways of coupling in new particles with masses that make them accessible at the LHC would have generated such operators.

A little more explicitly: the Standard Model Lagrangian L_0 contains all the renormalizable (*i.e.* engineering dimension ≤ 4) operators that you can make from its fields (though the coefficients of the dimension 4 operators do vary through quite a large range, and the coefficients of the two relevant operators – namely the identity operator which has dimension zero, and the Higgs mass, which has engineering dimension two, are strangely small, and so is the QCD θ angle).

To understand what lies beyond the Standard Model, we can use our knowledge

that whatever it is, it is probably heavy (it could also just be very weakly coupled, which is a different story), with some intrinsic scale $\Lambda_{\rm new}$, so we can integrate it out and include its effects by corrections to the Standard Model:

$$L = L_0 + \frac{1}{\Lambda_{\rm new}}\mathcal{O}^{(5)} + \frac{1}{\Lambda_{\rm new}^2}\sum_i c_i \mathcal{O}_i^{(6)} + \cdots$$

where the $\mathcal{O}$s are made of SM fields, and have the indicated engineering dimensions, and preserve the necessary symmetries of the SM (Lorentz symmetry and gauge invariance).

In fact there is only one kind of operator of dimension 5 meeting these demands:

$$\mathcal{O}^{(5)} = c_5 \epsilon_{ij} \left(\bar{L}^c\right)^i H^j \epsilon_{kl} L^k H^l$$

where $H^i = (h^+, h^0)^i$ is the $\mathrm{SU}(2)_{EW}$ Higgs doublet and $L^i = (\nu_L, e_L)^i$ is an $\mathrm{SU}(2)_{EW}$ doublet of left-handed leptons, and $\bar{L}^c \equiv L^T C$ where C is the charge conjugation matrix. (I say 'kind of operator' because we can have various flavor matrices in here.) On the problem set you get to see from whence such an operator might arise, and what it does if you plug in the higgs vev $\langle H \rangle = (0, v)$. This term violates lepton number symmetry ($L \to e^{\mathbf{i}\alpha_L} L, Q \to Q, H \to H$).

At dimension 6, there are operators that directly violate baryon number, such as

$$\epsilon_{\alpha\beta\gamma} (\bar{u}_R)^c_\alpha (u_R)_\beta \, (\bar{u}_R)^c_\gamma \, e_R.$$

You should read the above tangle of symbols as '$qqq\ell$' – it turns three quarks into a lepton. The epsilon tensor makes a color SU(3) singlet; this thing ϵqqq has the quantum numbers of a baryon, such as the proton and neutron. The long lifetime of the proton (you can feel it in your bones – see Zee p. 413) then directly constrains the scale of new physics appearing in front of this operator.

Three more comments about this:

- The idea of Grand Unification means leptons and quarks are in the same representations of a larger gauge group – they can turn into each other by exchanging GUT gauge bosons. This predicts that the proton should not be perfectly stable, and integrating out the GUT gauge bosons should produce baryon-number violating operators like the ones above, suppressed by $M_{\rm GUT} \simeq 10^{16}$ GeV.

- If we didn't know about the Standard Model, (but after we knew about QM and GR and EFT (the last of which people didn't know before the SM for some reason)) we should have made the estimate that dimension-5 Planck-scale-suppressed operators like $\frac{1}{M_{\rm Planck}} p\mathcal{O}$ would cause proton decay (into whatever $\mathcal{O}$ makes). This

predicts $\Gamma_p \sim \frac{m_p^3}{M_{\rm Planck}^2} \sim 10^{-13} s^{-1}$ which is *not* consistent with our bodies not glowing. Actually it is a remarkable fact that there are no gauge-invariant operators made of SM fields of dimension less than 6 that violate baryon number symmetry ($L \to L, Q \to e^{i\alpha_B} Q, H \to H$). This is an *emergent* symmetry, expected to be violated by the UV completion.

- Surely nothing can prevent $\Delta L \sim \left(\frac{1}{M_{\rm Planck}}\right)^2 qqq\ell$. Happily, this is consistent with the observed proton lifetime.

There are $\sim 10^2$ dimension 6 operators that preserve baryon number, and therefore are not as tightly constrained[3]. (Those that induce flavor-changing processes in the SM are more highly constrained and must have $\Lambda_{\rm new} > 10^4$ TeV.) Two such operators are considered equivalent if they differ by something which vanishes by the tree-level SM equations of motion. This is the right thing to do, even for off-shell calculations (like green's functions and for fields running in loops). You know this from a previous problem set: the EOM are true as operator equations – Ward identities resulting from being free to change integration variables in the path integral[4].

A special pair of dimension 6 operators lead to mixing between the various electroweak gauge bosons:

$$\Delta L \ni S \frac{\alpha}{\sin\theta_W \cos\theta_W v^2} H^\dagger W_a^{\mu\nu} \tau^a H B_{\mu\nu} - T \frac{2\alpha}{v^2} |H^\dagger D_\mu H|^2 \tag{2.2}$$

where $B_{\mu\nu}$ is the hypercharge gauge field strength. These 'oblique parameters' S and T were introduced by Peskin and Takeuchi and are very strongly constrained by the large amount of data from e^+e^- collisions at the Z resonance. They are expected to be large in technicolor models. For a systematic discussion, see for example here.

[3] Recently, humans have gotten better at counting these operators. See this paper.

[4] There are a few meaningful subtleties here, as you might expect if you recall that the Ward identity is only true up to contact terms. The measure in the path integral can produce a Jacobian which renormalizes some of the couplings; the changes in source terms will drop out of S-matrix elements (recall our discussion of changing field variables in the Consequences of Unitarity section.) but can change the form of Green's functions. For more information on the use of eom to eliminate redundant operators in EFT, see Arzt, hep-ph/9304230 and Georgi, "On-Shell EFT".

2.3 Superconductors and superfluids

Who is Φ? Last quarter, we developed an effective (Landau-Ginzburg) description of superconductors which reproduces the Meissner effect (that magnetic flux is expelled or collimated into flux tubes); it is called the Abelian Higgs model:

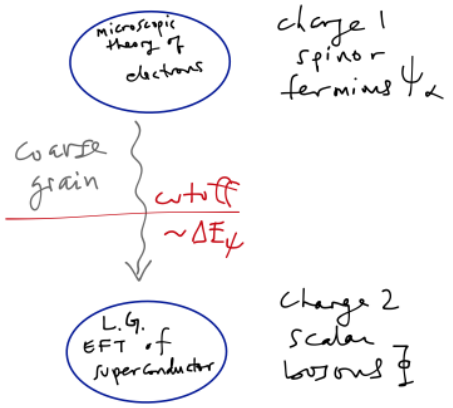
$$\mathcal{F} = \frac{1}{4}F_{ij}F_{ij} + |D_i\Phi|^2 + a|\Phi|^2 + \frac{1}{2}b|\Phi|^4 + ... \tag{2.3}$$

with $D_i\Phi \equiv (\partial_i - 2e\mathbf{i}A_i)\,\Phi$. Here A is the photon field. This is a slight modification of the previous expression to indicate that the Higgs field Φ has electric charge two. We could have guessed this description by playing the EFT game, knowing that the dofs involved are the photon and a charge-two scalar field. But who is this charge-two scalar field? (Relatedly: what is the cutoff on the validity of this description?)

New IR dofs. A feature of this example that I want you to notice: the microscopic description of real superconductor involves electrons – charge $1e$ spinor fermions, created by some fermionic operator ψ_α, $\alpha = \uparrow, \downarrow$.

We are describing the low-energy physics of a system of electrons in terms of a bosonic field, which (in simple 's-wave' superconductors) is roughly related to the electron field by

$$\Phi \sim \psi_\alpha \psi_\beta \epsilon^{\alpha\beta}\ ; \tag{2.4}$$



Φ is called a Cooper pair field. At least, the charges and the spins and the statistics work out. The details of this relationship are not the important point I wanted to emphasize. Rather I wanted to emphasize the dramatic difference in the correct choice of variables between the UV description (spinor fermions) and the IR description (scalar bosons). One reason that this is possible is that it costs a large energy to make a fermionic excitation of the superconductor. [End of Lecture 4]

This can be understood roughly as follows: The microscopic theory of the electrons looks something like (ignoring the coupling to electromagnetism for now, except for a screened (and therefore short-ranged) repulsion which ultimately is the Coulomb interaction)

$$S[\psi] = S_2[\psi] + \int \mathrm{d}t\mathrm{d}^dx\ u\psi^\dagger\psi\psi^\dagger\psi + h.c. \tag{2.5}$$

where

$$S_2 = \int \mathrm{d}t \int \mathrm{đ}^dk \psi_k^\dagger \left(\mathbf{i}\partial_t - \epsilon(k)\right)\psi_k.$$

Spin is important here so that $\psi^\dagger_\uparrow \psi_\uparrow \psi^\dagger_\downarrow \psi_\downarrow$ is nonzero. A mean field theory description of the condensation of Cooper pairs (2.4) is obtained by replacing the quartic term in (2.5) by expectation values:

$$\begin{aligned} S_{MFT}[\psi] &= S_2[\psi] - \int \mathrm{d}t \mathrm{d}^d x \; u \langle \psi\psi \rangle \, \psi^\dagger \psi^\dagger + h.c. \\ &= S_2[\psi] - \int \mathrm{d}t \mathrm{d}^d x \; u \Phi \psi^\dagger \psi^\dagger + h.c. \end{aligned} \tag{2.6}$$

So an expectation value for Φ is a mass for the fermions. It is a funny kind of symmetry-breaking mass, but if you diagonalize the quadratic operator in (2.6) (actually it is done below) you will find that it costs an energy of order $\Delta E_\psi = u \langle \Phi \rangle$ to excite a fermion. That's the cutoff on the LG EFT.

A general lesson from this example is: the useful degrees of freedom at low energies can be very different from the microscopic dofs.

2.3.1 Lightning discussion of BCS.

I am sure that some of you are nervous about the step from $S[\psi]$ to $S_{MFT}[\psi]$ above. To make ourselves feel better about it, I will say a few more words about the steps from the microscopic model of electrons (2.5) to the LG theory of Cooper pairs (these steps were taken by Bardeen, Cooper and Schreiffer (BCS)).

First recall the *Hubbard-Stratonovich transformation* aka *completing the square*. In 0+0 dimensional field theory:

$$e^{-\mathbf{i}u x^4} = \frac{1}{\sqrt{\mathbf{i}\pi u}} \int_{-\infty}^{\infty} \mathrm{d}\sigma \; e^{-\frac{1}{\mathbf{i}u}\sigma^2 - 2\mathbf{i}x^2\sigma} \; . \tag{2.7}$$

At the cost of introducing an extra field σ, we turn a quartic term in x into a quadratic term in x. The RHS of (2.7) is gaussian in x and we know how to integrate it over x. (The version with $\mathbf{i}$ is relevant for the real-time integral.) Notice the weird extra factor of $\mathbf{i}$ lurking in (2.7). This can be understood as arising because we are trying to use a scalar field, σ, to mediate a repulsive interaction (which it is, for positive u) (see Zee p. 193, 2nd Ed).

Actually, we'll need a complex H-S field:

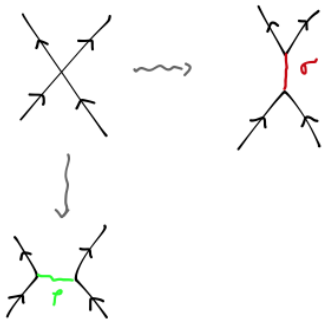
$$e^{-\mathbf{i}u x^2 \bar{x}^2} = \frac{1}{\mathbf{i}\pi u} \int_{\mathbb{C}} \mathrm{d}^2\sigma \; e^{-\frac{1}{\mathbf{i}u}|\sigma|^2 - \mathbf{i}x^2\bar{\sigma} + \mathbf{i}\bar{x}^2\sigma} \; , \tag{2.8}$$

where $\int_{\mathbb{C}} \mathrm{d}^2\sigma ... \equiv \int_{-\infty}^{\infty} d\mathrm{Re}\,\sigma \int_{-\infty}^{\infty} d\mathrm{Im}\,\sigma ...$ (The field-independent prefactor is, as usual, not important for path integrals.)

We can use a field theory generalization of (2.8) to 'decouple' the 4-fermion interaction in (2.5):

$$Z = \int [D\psi D\psi^\dagger] e^{\mathbf{i}S[\psi]} = \int [D\psi D\psi^\dagger D\sigma D\sigma^\dagger] e^{\mathbf{i}S_2[\psi] + \mathbf{i}\int d^D x \left(\bar\sigma \psi_\uparrow \psi_\downarrow + h.c.\right) - \int \mathrm{d}^D x \frac{|\sigma|^2(x)}{\mathbf{i}u}} \quad . \tag{2.9}$$

The point of this is that now the fermion integral is gaussian. At the saddle point of the σ integral (which is exact because it is gaussian), σ is the Cooper pair field, $\sigma_{\text{saddle}} = u\psi_\uparrow\psi_\downarrow$.



Notice that we made a choice here about in which 'channel' to make the decoupling – we could have instead introduces a different auxiliary field ρ and written $S[\rho,\psi] = \int \rho\psi^\dagger\psi + \int \frac{\rho^2}{2u}$, which would break up the 4-fermion interaction in the t-channel (as an interaction of the fermion density $\psi^\dagger\psi$) instead of the s (BCS) channel (as an interaction of Cooper pairs ψ^2). At this stage both are correct, but they lead to different mean-field approximations below. That the BCS mean field theory wins is a consequence of the RG, as I'll describe below.

How can you resist doing the fermion integral in (2.9)? Let's study the case where the single-fermion dispersion is $\epsilon(k) = \frac{\vec{k}^2}{2m} - \mu$.

$$I_\psi[\sigma] \equiv \int [D\psi D\psi^\dagger] e^{\mathbf{i}\int \mathrm{d}t\mathrm{d}^d x \left(\psi^\dagger \left(\mathbf{i}\partial_t - \frac{\nabla^2}{2m} - \mu\right)\psi + \bar\sigma\psi\psi + \bar\psi\bar\psi\sigma\right)}$$

The action here can be written as the integral of

$$L = \begin{pmatrix}\bar\psi & \psi\end{pmatrix} \begin{pmatrix} \mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla) & \sigma \\ \bar\sigma & -(\mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla))\end{pmatrix} \begin{pmatrix}\psi \\ \bar\psi\end{pmatrix} \equiv \begin{pmatrix}\bar\psi & \psi\end{pmatrix} M \begin{pmatrix}\psi \\ \bar\psi\end{pmatrix}$$

so the functional integral is

$$I_\psi[\sigma] = \det M = e^{\mathrm{tr}\log M(\sigma)}.$$

If σ is constant (which will lower the energy), the matrix M is diagonal in momentum space, and the integral remaining to be done is

$$\int [D\sigma D\sigma^\dagger] e^{-\int \mathrm{d}^D x \frac{|\sigma(x)|^2}{2\mathbf{i}u} + \int \mathrm{đ}^D k \log\left(\omega^2 - \epsilon_k^2 - |\sigma|^2\right)}.$$

It is often possible to do this integral by saddle point. This can be justified, for example, by the largeness of the volume of the Fermi surface, $\{k|\epsilon(k) = \mu\}$, or by a large number

N of species of fermions. The result is an equation that determines σ, which as we saw earlier determines the fermion gap.

$$0 = \frac{\delta \text{exponent}}{\delta \bar{\sigma}} = \mathbf{i}\frac{\sigma}{2u} + \int đ\omega đ^d k \frac{2\sigma}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} \ .$$

We can do the frequency integral by residues:

$$\int đ\omega \frac{1}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} = \frac{1}{2\pi} 2\pi \mathbf{i} \frac{1}{2\sqrt{\epsilon_k^2 + |\sigma|^2}}.$$

The resulting equation is naturally called the *gap equation*:

$$1 = -2u \int đ^d p' \frac{1}{\sqrt{\epsilon(p')^2 + |\sigma|^2}} \tag{2.10}$$

which you can imagine solving self-consistently for σ[5]. Plugging back into the action (2.9) says that σ determines the energy cost to have electrons around; more precisely, σ is the energy required to break a Cooper pair.

Comments:

- Notice that a solution of (2.10) requires $u < 0$, an *attractive* interaction. Superconductivity happens because the u that appears here is not the bare interaction between electrons, which is certainly repulsive (and long-ranged). This is where the phonons come in in the BCS discussion.
- If we hadn't restricted to a delta-function 4-fermion interaction $u(p, p') = u_0$ at the outset, we would have found a more general equation like

$$\sigma(\vec{p}) = -\frac{1}{2} \int đ^d p' \frac{u(p, p')\sigma(\vec{p}')}{\sqrt{\epsilon(p')^2 + |\sigma(p')|^2}} \ .$$

[5] I should have said: and in fact one can solve it. As we will learn in the next section, the integral is dominated by the behavior near the Fermi surface, near which $\epsilon(p') \simeq v_F \ell \equiv \varepsilon$; this approximation is valid for $|\epsilon| < E_D$, some UV cutoff on this description. The result is

$$1 = -2u \int \frac{đ^d p'}{\sqrt{\epsilon(p')^2 + |\sigma|^2}} \simeq -2u \int_{FS} \frac{đ^{d-1} k}{v_F} \int_{-E_D}^{E_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + |\sigma|^2}} = Nu \log\left(\frac{E_D + \sqrt{E_D^2 + |\sigma|^2}}{|\sigma|}\right)$$

where $N \equiv \int_{FS} \frac{đ^{d-1} k}{2\pi v_F}$ is the density of states at the Fermi surface. The largeness of this N justifies the saddle-point approximation. The solution for σ is

$$|\sigma| = \frac{2E_D e^{\frac{1}{2Nu}}}{e^{\frac{1}{Nu}} - 1} \overset{Nu \ll 1}{\simeq} 2E_D e^{-\frac{1}{Nu}}.$$

Notice that this is non-perturbative in the coupling strength u.

- A conservative perspective on the preceding calculation is that we have made a variational ansatz for the groundstate wavefunction, and the equation we solve for σ is minimizing the variational energy – finding the best wavefunction within the ansatz.

- I haven't included here effects of the fluctuations of the sigma field about its saddle point. In fact, they make the four-fermion interaction that leads to Cooper pairing marginally *relevant*. This breaks the degeneracy in deciding how to split up the $\psi\psi\psi^\dagger\psi^\dagger$ into *e.g.* $\psi\psi\sigma$ or $\psi^\dagger\psi\rho$. BCS wins. This is explained beautifully in Polchinski, lecture 2, and R. Shankar. I will summarize the EFT framework for understanding this in §2.4.

- I've tried to give the most efficient introduction I could here. I left out any possibility of k-dependence or spin dependence of the interactions or the pair field, and I've conflated the pair field with the gap. In particular, I've been sloppy about the dependence on k of σ above.

- You can study a very closely related manipulation on the problem set, in examples (the $\mathsf{O}(N)$ model and the Gross-Neveu model) where the saddle point is justified by large N.

2.3.2 Non-relativistic scalar fields

[Zee §III.5, V.1, Kaplan nucl-th/0510023 §1.2.1] In the previous discussion of the EFT for a superconductor, I just wrote the free energy, and so we didn't have to think about whether the complex scalar in question was relativistic or not.

It is not. In real superconductors, at least. How should we think about a non-relativistic field? A simple answer comes from realizing that a relativistic field which can make a boson of mass m can certainly make a boson of mass m which is moving slowly, with $v \ll c$. By taking a limit of the relativistic model, then, we can make a description which is useful for describing the interactions of an indefinite number of bosons moving slowly in some Lorentz frame. A situation that calls for such a description, for example, is a large collection of ^{4}He atoms.

Reminder: Non-relativistic limit of a relativistic scalar field. A non-relativistic particle in a relativistic theory (like the ϕ^4 theory that we've been spending time with) has energy

$$E = \sqrt{p^2 + m^2} \overset{\text{if } v \ll c}{=} m + \frac{p^2}{2m} + \ldots$$

This means that the field that creates and annihilates it looks like

$$\phi(\vec{x},t)=\sum_{\vec{k}}\frac{1}{\sqrt{2E_{\vec{k}}}}\left(a_{\vec{k}}e^{\mathbf{i}E_{\vec{k}}t-\mathbf{i}\vec{k}\cdot\vec{x}}+h.c.\right)$$

In particular, we have

$$\dot{\phi}^2\simeq m^2\phi^2$$

and the BHS of this equation is large. To remove this large number let's change variables:

$$\phi(x,t)\equiv\frac{1}{\sqrt{2m}}\left(e^{-\mathbf{i}mt}\underbrace{\Phi(x,t)}_{\text{complex},\dot{\Phi}\ll m\Phi}+h.c.\right)\ .$$

Notice that Φ is complex, even if ϕ is real.

Let's think about the action governing this NR sector of the theory. We can drop terms with unequal numbers of Φ and $\Phi^\star$ since such terms would come with a factor of $e^{\mathbf{i}mt}$ which gives zero when integrated over time. Starting from $(\partial\phi)^2-m^2\phi^2-\lambda\phi^4$ we get:

$$L_{\text{real time}}=\Phi^\star\left(\mathbf{i}\partial_t+\frac{\vec{\nabla}^2}{2m}\right)\Phi-g^2\left(\Phi^\star\Phi\right)^2+... \tag{2.11}$$

with $g^2=\frac{\lambda}{4m^2}$.

Notice that Φ is a complex field and its action has a U(1) symmetry, $\Phi\to e^{\mathbf{i}\alpha}\Phi$, even though the full theory did not. The associated conserved charge is the number of particles:

$$j_0=\Phi^\star\Phi,j_i=\frac{\mathbf{i}}{2m}\left(\Phi^\star\partial_i\Phi-\partial_i\Phi^\star\Phi\right),\quad\partial_t j_0-\nabla\cdot\vec{j}=0\ .$$

Notice that the 'mass term' $\Phi^\star\Phi$ is then actually the chemical potential term, which encourages a nonzero density of particles to be present.

This is another example of an *emergent* symmetry (like baryon number in the SM): a symmetry of an EFT that is not a symmetry of the microscopic theory. The ... in (2.11) include terms which break this symmetry, but they are irrelevant.

To see more precisely what we mean by irrelevant, let's think about scaling. To keep this kinetic term fixed we must scale time and space differently:

$$x\to\tilde{x}=sx,\ t\to\tilde{t}=s^2t,\ \Phi\to\tilde{\Phi}(\tilde{x},\tilde{t})=\zeta\Phi(sx,s^2t)\ .$$

A fixed point with this scaling rule has dynamical exponent $z=2$. The scaling of the

bare action (with no mode elimination step) is

$$S_E^{(0)} = \int \underbrace{dtd^d\vec{x}}_{=s^{d+z}d\tilde{t}d^d\tilde{x}} \left(\Phi^\star\left(sx, s^2t\right) \underbrace{\left(\partial_t - \frac{\vec{\nabla}^2}{2m}\right)}_{=s^{-2}\left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right)} \Phi(sx, s^2t) - g^2\left(\Phi^\star\Phi(sx, s^2t)\right)^2 + ... \right)$$
$$= \underbrace{s^{d+z-2}\zeta^{-2}}_{\stackrel{!}{=}1 \Longrightarrow \zeta = s^{-d/2}} \int \mathrm{d}\tilde{t}\mathrm{d}^d\tilde{x} \left(\tilde{\Phi}^\star\left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right)\tilde{\Phi} - \zeta^{-2}g^2\left(\tilde{\Phi}^\star\tilde{\Phi}(\tilde{x},\tilde{t})\right)^2 + ... \right) \quad (2.12)$$

From this we learn that $\tilde{g} = s^{2-d}g \to 0$ in the IR – the quartic term is irrelevant in $D = d+1 = 3+1$ with nonrelativistic scaling! Where does it become marginal? (Hint: look back at the first lecture of last quarter.)

Number and phase angle. In the NR theory, the canonical momentum for Φ is just $\frac{\partial L}{\partial \dot{\Phi}} \sim \Phi^\star$, with no derivatives. This statement becomes more shocking if we change variables to $\Phi = \sqrt{\rho}e^{\mathbf{i}\varphi}$ (which would be useful *e.g.* if we knew ρ didn't want to be zero); the action density is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\varphi - \frac{1}{2m}\left(\rho\left(\nabla\varphi\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - g^2\rho^2. \quad (2.13)$$

The first term is a total derivative. The second term says that the canonical momentum for the phase variable φ is $\rho = \Phi^\star\Phi = j_0$, the particle number density. Quantumly, then:

$$[\hat{\rho}(\vec{x},t), \hat{\varphi}(\vec{x}',t)] = \mathbf{i}\delta^d(\vec{x}-\vec{x}').$$

Number and phase are canonically conjugate variables. If we fix the phase, the amplitude is maximally uncertain.

If we integrate over space, $N \equiv \int d^dx\rho(\vec{x},t)$ gives the total number of particles, which is time independent, and satisfies $[N,\varphi] = \mathbf{i}$.

This relation explains why there's no Higgs boson in most non-relativistic superconductors and superfluids (in the absence of some extra assumption of particle-hole symmetry). In the NR theory with first order time derivative, the would-be amplitude mode that oscillates about the minimum of $V(\rho)$ is actually just the conjugate momentum for the goldstone boson!

2.3.3 Superfluids.

[Zee §V.1, Wen §3.3.3] Let me amplify the previous remark. A superconductor is just a superfluid coupled to an external U(1) gauge field, so we've already understood something about superfluids.

The effective field theory has the basic lagrangian (2.13), with $\langle\rho\rangle = \bar{\rho} \neq 0$. This nonzero density can be accomplished by adding an appropriate chemical potential to (2.13); up to an uninteresting constant, this is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\varphi - \frac{1}{2m}\left(\rho\left(\nabla\varphi\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - g^2\left(\rho-\bar{\rho}\right)^2.$$

Expand around such a condensed state in small fluctuations $\sqrt{\rho} = \sqrt{\bar{\rho}}+h$, $h \ll \sqrt{\bar{\rho}}$:

$$L = -2\sqrt{\bar{\rho}}h\partial_t\varphi - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\varphi\right)^2 - \frac{1}{2m}\left(\vec{\nabla}h\right)^2 - 4g^2\bar{\rho}h^2 + ...$$

Notice that h, the fluctuation of the amplitude mode, is playing the role of the canonical momentum of the goldstone mode φ. The effects of the fluctuations can be incorporated by doing the gaussian integral over h (What suppresses self-interactions of h?), and the result is

$$\begin{aligned} L &= \bar{\rho}\partial_t\varphi\frac{1}{4g^2\bar{\rho} - \frac{\nabla^2}{2m}}\bar{\rho}\partial_t\varphi - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\varphi\right)^2 \\ &= \frac{1}{4g^2}\left(\partial_t\varphi\right)^2 - \frac{\bar{\rho}}{2m}\left(\nabla\varphi\right)^2 + ... \end{aligned} \tag{2.14}$$

where in the second line we are expanding in the small wavenumber k of the modes, that is, we are constructing an action for Goldstone modes whose wavenumber is $k \ll \sqrt{8g^2\bar{\rho}m}$ so we can ignore higher gradient terms.

The linearly dispersing mode in this superfluid that we have found, sometimes called the phonon, has dispersion relation

$$\omega^2 = \frac{2g^2\bar{\rho}}{m}\vec{k}^2.$$

This mode has an emergent Lorentz symmetry with a lightcone with velocity $v_c = g\sqrt{2\bar{\rho}/m}$. The fact that the sound velocity involves g – which determined the steepness of the walls of the wine-bottle potential – is a consequence of the non-relativistic dispersion of the bosons. In the relativistic theory, in contrast, we have $L = \partial_\mu\Phi^\star\partial^\mu\Phi - \kappa\left(\Phi^\star\Phi - v^2\right)^2$ and we can take $\kappa \to \infty$ fixing v and still get a linearly dispersing mode by plugging in $\Phi = e^{\mathbf{i}\varphi}v$.

I've put the following paragraphs in an ugly color because they don't explain what I thought they explained.

What does this have to do with the phenomenology of superfluids, like dissipationless flow? The importance of the linearly dispersing phonon mode of the superfluid is that there is *no other* low energy excitation of the fluid. With a classical pile of (*e.g.* non interacting) bosons, a chunk of moving fluid can donate some small momentum $\vec{k}$ to a single boson at energy cost $\frac{(\hbar\vec{k})^2}{2m}$. A quadratic dispersion means more modes at small k than a linear one (the density of states is $N(E) \propto k^{D-1}\frac{dk}{dE}$). With only a linearly dispersing mode at low energies, there is a critical velocity below which a non-relativistic chunk of fluid cannot give up any momentum [Landau]: conserving momentum $M\vec{v} = M\vec{v}' + \hbar\vec{k}$ says the change in energy (which must be negative for this to happen on its own) is (eliminate $v' = v - \hbar k/M$):

$$\frac{1}{2}M(v')^2 + \hbar\omega(k) - \frac{1}{2}Mv^2 = -\hbar kv + \frac{(\hbar k)^2}{2M} + \hbar\omega(k) = (-v + v_c)k + \frac{(\hbar k)^2}{2M} \ .$$

For small k, this is only negative when $v > v_c$.

You can ask: an ordinary liquid also has a linearly dispersing sound mode; why doesn't Landau's argument mean that it has superfluid flow? The answer is that it has *other* modes with softer dispersion (so more contribution at low energies), in particular diffusion modes, with $\omega \propto k^2$ (there is an important factor of $\mathbf{i}$ in there).

The Goldstone boson has a compact target space, $\varphi(x) \equiv \varphi(x) + 2\pi$, since, after all, it is the phase of the boson field. This is significant because it means that as the phase wanders around in space, it can come back to its initial value after going around the circle – such a loop encloses a *vortex*. Somewhere inside, we must have $\Phi = 0$. And actually, our discussion of the vortices of the Abelian Higgs model did not depend on the form of the time-derivative terms. There is much more to say about this.

[Wen §3.7.3] The above argument about the Landau critical velocity does not really explain the phenomenon of superflow, where if we set up a current it keeps going for a very long time. One way to see this is that there are superfluids and superconductors where there are other light degrees of freedom besides the linearly-dispersing phonon. For example, sometimes the condensate fails to gap out the fermion excitations.

Here's the real reason for superflow. It happens entirely because the spatial components of the particle-number current have the form

$$\vec{j} = \frac{\rho}{m}\vec{\nabla}\varphi \tag{2.15}$$

where φ is a compact field $\varphi \simeq \varphi + 2\pi$. Consider the situation where the x direction is a circle $x \simeq x + L$ (for example if the superfluid lives in an annular region). Think about what is required to set up a flow of such a system in the x direction: we must

have

$$\varphi(x) = mvx. \tag{2.16}$$

But compactness of the boson and space requires that $mvL \in 2\pi\mathbb{Z}$ is quantized. This integer is the *vorticity* of the configuration. The reason is that the only way it can change is if a *vortex* (a point where $\Phi = 0$, so that φ is ill-defined) appears in the sample. (See the figure below.)

But as we've seen, vortices are costly. In a superfluid (where there is no dynamical gauge field), they are also confined, in the sense that a single vortex has infinite energy, and only a vortex-antivortex pair has finite energy. The difficulty of producing vortices is what makes the superflow configuration a long-lived metastable state.

[End of Lecture 5]

Notice that in a superconductor, only the combination $\vec{A} + \vec{\nabla}\varphi$ is gauge invariant, so (2.15) is the same as the London equation

$$\vec{j} = \frac{\rho}{m}(\vec{A} + \vec{\nabla}\varphi) \tag{2.17}$$

(φ can be set to zero by choosing unitary gauge). This equation implies the Meissner effect, as you can see by sticking it into the Maxwell equation.

In many accounts of the subject, the above explanation involves some discussion of Galilean invariance. This is not necessary, but it is useful to understand how the configuration (2.16) arises from a slightly more microscopic point of view. The key point is that in order to preserve the action $S = \int d^d x dt \mathcal{L}$,

$$\mathcal{L} = \Phi^\star \left(\mathbf{i}\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \Phi - V(\Phi^\star \Phi) \ , \tag{2.18}$$

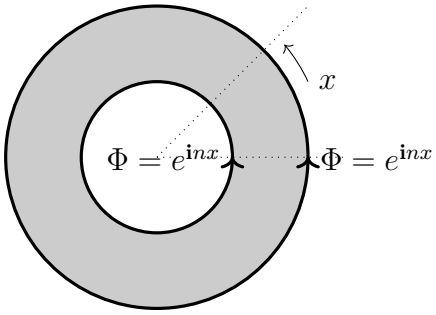
the Galilean boost

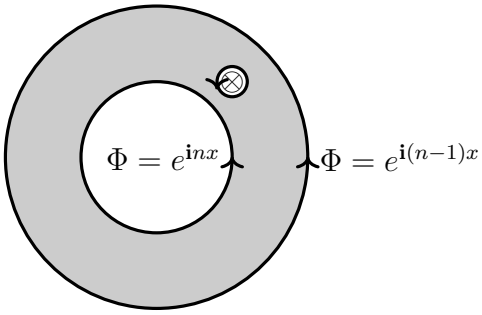
$$x_i' = x_i - v_i t, \quad t' = t \tag{2.19}$$

acts on the non-relativistic field Φ as

$$\Phi(x,t) \to \Phi'(x',t'), \quad \Phi(x,t) = e^{-\frac{1}{2}\mathbf{i}mv^2 + \mathbf{i}mv_i x^i} \Phi'(x',t'). \tag{2.20}$$

At fixed time, a boost therefore winds up the phase of Φ to $e^{\mathbf{i}mvx}\Phi$. If $|\Phi| \neq 0$, this winding cannot be removed continuously.





2.4 Effective field theory of metal

In previous subsections, we gave various descriptions of superconductors, appropriate at increasing energies. At the lowest energies, there was just a massive photon. At higher energies, there was a Cooper-pair field, (2.3). At even higher energies, where we can break apart Cooper pairs, there are electrons (2.5). In this subsection, we peel away one more layer of the onion: at even higher energies, those electrons are no longer paired up and constitute a metal.

[Polchinski, lecture 2 (I recommend these notes very strongly), and R. Shankar] Let us appreciate the remarkable phenomenon that is *metal*. An arbitrarily small electric field $\vec{E}$ leads to a nonzero current $\vec{j} = \sigma\vec{E}$. This means that there must be gapless modes with energies much less than the natural cutoff scale in the problem.

Scales involved: The Planck scale of solid state physics (made by the logic by which Planck made his quantum gravity energy scale, namely by making a quantity with dimensions of energy out of the available constants) is

$$E_0 = \frac{1}{2}\frac{e^4 m}{\hbar^2} = \frac{1}{2}\frac{e^2}{a_0} \sim 13\text{eV}$$

(where $m \equiv m_e$ is the electron mass and the factor of 2 is an abuse of outside information) which is the energy scale of *chemistry*. Chemistry is to solids as the melting of spacetime is to high-energy physics. As with high-energy physics, however, there are other scales involved. In particular a solid involves a lattice of nuclei, each with $M \gg m$ (of order the proton mass). So m/M is a useful small parameter which controls the coupling between the electrons and the lattice vibrations. Also, the actual speed of light $c \gg v_F$ can generally be treated as ∞ to first approximation. v_F/c suppresses spin-orbit couplings that break $\mathsf{SU}(2)_{\text{spin}} \times \mathsf{SO}(3)_{\text{spatial rotations}}$ down to the diagonal (though large atomic numbers enhance them: $\lambda_{\text{SO}} \propto Zv_F/c$).

Let us attempt to construct a Wilsonian-natural effective field theory of this phenomenon. The answer is called Landau Fermi Liquid Theory. What are the right low-energy degrees of freedom? Let's make a guess that they are like electrons – fermions with spin and electric charge. They will not have exactly the properties of free electrons, since they must incorporate the effects of interactions with all their colleagues. The 'dressed' electrons are called quasielectrons, or more generally quasiparticles.

Given the strong interactions between so many particles, why should the dofs have anything at all to do with electrons? Landau's motivation for this description (which is not always correct) is that we can imagine starting from the free theory and adiabatically turning up the interactions. If we don't encounter any phase transition along the way, we can follow each state of the free theory, and use the same labels in the interacting theory.

We will show that there is a nearly-RG-stable fixed point describing gapless quasielectrons. Notice that we are not trying to match this description directly to some microscopic lattice model of a solid; rather we will do bottom-up effective field theory.

Having guessed the necessary dofs, let's try to write an action for them consistent with the symmetries. A good starting point is the free theory:

$$S_{\text{free}}[\psi] = \int dt\, \mathrm{d}^d p \left(\mathbf{i} \psi^\dagger_\sigma(p) \partial_t \psi_\sigma(p) - (\epsilon(p) - \epsilon_F)\, \psi^\dagger_\sigma(p) \psi_\sigma(p) \right)$$

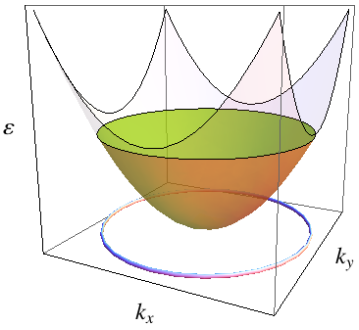
where σ is a spin index, ϵ_F is the Fermi energy (zero-temperature chemical potential), and $\epsilon(p)$ is the single-particle dispersion relation. For non-interacting non-relativistic electrons in free space, we have $\epsilon(p) = \frac{p^2}{2m}$. It will be useful to leave this as a general function of p. [6] [7]

[6] Notice that we are assuming translation invariance. I am not saying anything at the moment about whether translation invariance is discrete (the ions make a periodic potential) or continuous.

[7] We have chosen the normalization of ψ to fix the coefficient of the ∂_t term (this rescaling may depend on p).

The groundstate of the free theory is the filled Fermi sea:

$$|\text{gs}\rangle = \prod_{p|\epsilon(p)<\epsilon_F} \psi_p^\dagger |0\rangle\,, \quad \psi_p |0\rangle = 0, \quad \forall p.$$



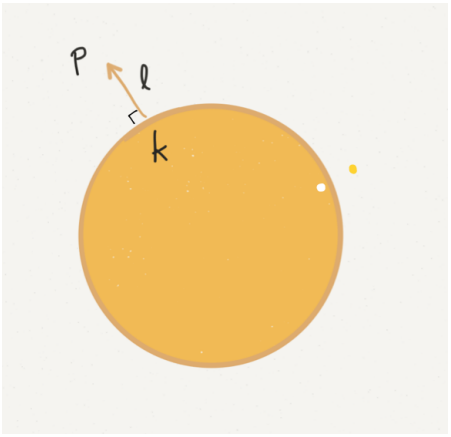
(If you don't like continuous products, put the system in a box so that p is a discrete label.) The Fermi surface is the set of points in momentum space at the boundary of the filled states:

$$\text{FS} \equiv \{p|\epsilon(p) = \epsilon_F\}.$$

The low-lying excitations are made by adding an electron just above the FS or removing an electron (creating a hole) just below.

In order to define the power-counting rules for our EFT, we would like to define a scaling transformation that focuses on the low-energy excitations. We scale energies by a factor $E \to bE, b < 1$. In relativistic QFT, $\vec{p}$ scales like E, toward zero, $\vec{p} \to b\vec{p}$, since all the low-energy stuff is near the single special point $\vec{p} = 0$. Here the situation is much more interesting because there is a whole surface of low-energy stuff on the FS. This will lead to what's called *hyperscaling violation* – we can't just count powers of momentum.

One way to implement this is to introduce a hierarchical labeling of points in momentum space, by breaking the momentum space into *patches* around the FS. (An analogous strategy of labeling is also used in heavy quark EFT and in SCET.)



We'll use a slightly different strategy, following Polchinski. To specify a point $\vec{p}$, we pick the nearest point $\vec{k}$ on the FS, $\epsilon(\vec{k}) = \epsilon_F$ (draw a line perpendicular to the FS from $\vec{p}$), and let

$$\vec{p} = \vec{k} + \vec{\ell}.$$

So $d-1$ of the components are determined by $\vec{k}$ and one is determined by ℓ. (There are some exceptional cases if the FS gets too wiggly. Ignore these for now.)

$$\epsilon(p) - \epsilon_F = \ell v_F(\vec{k}) + \mathcal{O}(\ell^2), \qquad v_F \equiv \partial_p \epsilon|_{p=k}.$$

So a scaling rule that accomplishes our goal of focusing on the FS is

$$E \to bE, \quad \vec{k} \to \vec{k}, \quad \vec{\ell} \to b\vec{\ell}.$$

This implies

$$dt \to b^{-1}dt, \quad d^{d-1}\vec{k} \to d^{d-1}\vec{k}, \quad d\vec{\ell} \to bd\vec{\ell}, \quad \partial_t \to b\partial_t$$

$$S_{\text{free}} = \int \underbrace{dt\ d^{d-1}\vec{k}\ d\vec{\ell}}_{\sim b^0} \left(\mathbf{i}\psi^\dagger(p) \underbrace{\partial_t}_{\sim b^1} \psi(p) - \underbrace{\ell v_F(k)}_{\sim b^1} \psi^\dagger(p)\psi(p) \right)$$

In order to make this go like b^0 we require $\psi \to b^{-\frac{1}{2}}\psi$ near the free fixed point.

Next we will play the EFT game. To do so we must enumerate the symmetries we demand of our EFT:

1. Particle number, $\psi \to e^{\mathbf{i}\theta}\psi$
2. Spatial symmetries: time-translation invariance, and either (a) continuous translation invariance and rotation invariance (as for *e.g.* liquid ^{3}He) or (b) lattice symmetries. This means that momentum space is periodically identified, roughly $p \simeq p + 2\pi/a$ where a is the lattice spacing (the set of independent momenta is called the Brillouin zone (BZ)) and p is only conserved modulo an inverse lattice vector $2\pi/a$. There can also be some remnant of rotation invariance preserved by the lattice. Case (b) reduces to case (a) if the Fermi surface does not go near the edges of the BZ.
3. Spin rotation symmetry, $\mathsf{SU}(n)$ if $\sigma = 1..n$. In the limit with $c \to \infty$, this is an internal symmetry, independent of rotations.
4. Let's assume that $\epsilon(p) = \epsilon(-p)$, which is a consequence of *e.g.* parity invariance (or, on the lattice, an inversion symmetry).

Now we enumerate all terms analytic in ψ and its momenta (since we are assuming that there are no other low-energy dofs integrating out which is the only way to get non-analytic terms in ψ) and consistent with the symmetries; we can order them by the number of fermion operators involved. Particle number symmetry means every ψ comes with a $\psi^\dagger$. The possible **quadratic terms** are:

$$\int \underbrace{dt\ d^{d-1}\vec{k}\ d\vec{\ell}}_{\sim b^0}\ \mu(k) \underbrace{\psi^\dagger_\sigma(p)\psi_\sigma(p)}_{\sim b^{-1}} \sim b^{-1}$$

is relevant. This is like a mass term. But don't panic: it just shifts the FS around. The *existence* of a Fermi surface is Wilson-natural (*i.e.* a stable assumption given generic coefficients of all possible terms in the action); any precise location or shape (modulo something enforced by symmetries, like roundness) is not.

Adding one extra ∂_t or factor of ℓ costs a b^1 and makes the operator marginal; those terms are already present in S_{free}. Adding more than one makes it irrelevant.

Quartic terms:

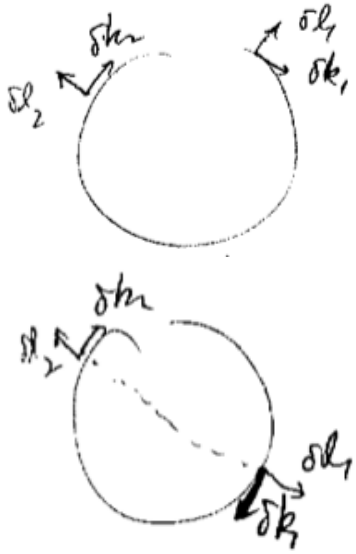
$$S_4 = \int \underbrace{dt \prod_{i=1}^{4} d^{d-1}\vec{k}_i d\vec{\ell}_i}_{\sim b^{-1+4}} u(4\cdots 1) \underbrace{\psi^\dagger_\sigma(p_1)\psi_\sigma(p_3)\psi^\dagger_{\sigma'}(p_2)\psi_{\sigma'}(p_4)}_{\sim b^{-\frac{1}{2}\cdot 4}} \delta^d(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$

The minus signs on $p_{3,4}$ is because $\psi(p)$ *removes* a particle with momentum p. We assume u depends only on k, σ, so does not scale – this will give the most relevant piece. How does the delta function scale?

$$\delta^d\left(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4\right) = \delta^d\left(k_1+k_2-k_3-k_4+\ell_1+\ell_2-\ell_3-\ell_4\right) \stackrel{?}{\simeq} \delta^d\left(k_1+k_2-k_3-k_4\right)$$

In the last (questioned) step, we used the fact that $\ell \ll k$ to ignore the contributions of the ℓs. If this is correct then the delta function does not scale (since ks do not), and $S_4 \sim b^1$ is irrelevant (and quartic interactions with derivatives are moreso). If this were correct, the free-fixed point would be exactly stable.

There are two important subtleties: (1) the questioned equality above is questionable because of kinematics of the Fermi surface, and (2) there exist phonons. We will address these two issues in order.



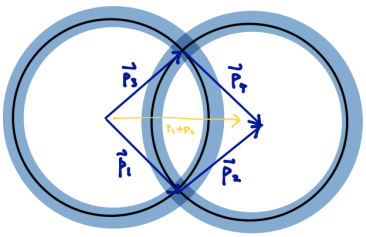
The **kinematic subtlety** in the treatment of the scaling of $\delta(p_1+p_2-p_3-p_4)$ arises because of the geometry of the Fermi surface. Consider scattering between two points on the FS, where (in the labeling convention above)

$$p_3 = p_1 + \delta k_1 + \delta\ell_1, \quad p_4 = p_2 + \delta k_2 + \delta\ell_2,$$

in which case the momentum delta function is

$$\delta^d(p_1+p_2-p_3-p_4) = \delta^d(\delta k_1 + \delta\ell_1 + \delta k_2 + \delta\ell_2).$$

For generic choices of the two points $p_{1,2}$ (top figure at left), δk_1 and δk_2 are linearly independent and the $\delta\ell$s can indeed be ignored as we did above. However, for two points with $p_1 = -p_2$ (they are called *nested*, as depicted in the bottom figure at left), then one component of $\delta k_1 + \delta k_2$ is automatically zero, revealing the tiny $\delta\ell$s to the force of (one component of) the delta function. In this case, $\delta(\ell)$ scales like b^{-1}, and for this particular kinematic configuration the four-fermion interaction is (classically) marginal. Classically marginal means quantum mechanics has a chance to make a big difference.



A useful visualization is at right ($d = 2$ with a round FS is shown; this is what's depicted on the cover of the famous book by Abrikosov-Gorkov-Dzyaloshinski): the blue circles have radius k_F; the yellow vector is the sum of the two initial momenta $p_1 + p_2$, both of which are on the FS; the condition that $p_3 + p_4$, each also on the FS, add up to the same vector means that p_3 must lie on the intersection of the two circles (spheres in $d > 2$). But when $p_1 + p_2 = 0$, the two circles are on top of each other so they intersect everywhere! Comments:

1. We assumed that both p_1 and $-p_2$ were actually *on* the FS. This is automatic if $\epsilon(p) = \epsilon(-p)$, *i.e.* if ϵ is only a function of p^2.

2. This discussion works for any $d > 1$.

3. **Forward scattering.** There is a similar phenomenon for the case where $p_1 = p_3$ (and hence $p_2 = p_4$). This is called *forward scattering* because the final momenta are the same as the initial momenta. (We could just as well take $p_1 = p_4$ (and hence $p_2 = p_3$).) In this case too the delta function will constrain the ℓs and will therefore scale.

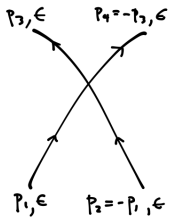
The tree-level-marginal 4-Fermi interactions at special kinematics leads to a *family* of fixed points labelled by 'Landau parameters'. In fact there is whole *function*'s worth of fixed points. In 2d, the points on the FS are parametrized by an angle θ, and the fixed point manifold is parametrized by the forward-scattering function

$$F(\theta_1, \theta_2) \equiv u(\theta_4 = \theta_2, \theta_3 = \theta_1, \theta_2, \theta_1)$$

(Fermi statistics implies that $u(\theta_4 = \theta_1, \theta_3 = \theta_2, \theta_2, \theta_1) = -F(\theta_1, \theta_2)$) and the BCS-channel (nesting) interaction:

$$V(\theta_1, \theta_3) = u(\theta_4 = -\theta_3, \theta_3, \theta_2 = -\theta_1, \theta_1).$$

Now let's think about what decision the fluctuations make about the fate of the nested interactions. The most interesting bit is the renormalization of the BCS interaction:



$= -\mathbf{i}V$

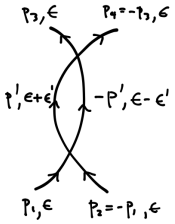
The electron propagator, obtained by inverting the kinetic operator in S_{free}, is

$$G(\epsilon, p = k + l) = \frac{\mathbf{i}}{\epsilon(1 + \mathbf{i}\eta) - v_F(k)\ell + \mathcal{O}(\ell)^2}$$

where I used $\eta \equiv 0^+$ for the infinitesimal specifying the contour prescription. [8]

Let's assume rotation invariance. Then $V(\theta_3, \theta_1) = V(\theta_3 - \theta_1)$, $V_l = \int đ\theta e^{\mathbf{i}l\theta} V(\theta)$. Different angular momentum sectors decouple from each other at one loop.

We will focus for simplicity on the s-wave bit of the interaction, so V is independent of momentum. We will integrate out just a shell in energy (depicted by the blue shaded shell in the Fermi surface figures). The interesting contribution comes from the following diagram: [End of Lecture 6]

$-\mathbf{i}\delta^{(1)}V =$ 

$$
\begin{aligned}
&= -(-\mathbf{i}V)^2 \int_{b\epsilon_0}^{\epsilon_0} đ\epsilon' đ^{d-1}k' đ\ell' \frac{\mathbf{i}^2}{(\epsilon + \epsilon' - v_F(k')\ell')\,(\epsilon - \epsilon' - v_F(k')\ell')} \\
&\quad \text{do } \textstyle\int d\ell' \text{ by residues} \\
&= -V^2 \int \frac{đ\epsilon' đ^{d-1}k'}{(2\pi)} \frac{2\pi\mathbf{i}}{v_F(k')} \left(\underbrace{\epsilon - \epsilon' - (\epsilon + \epsilon')}_{=-2\epsilon'} \right)^{-1} \\
&= +\mathbf{i}V^2 \underbrace{\int_{b\epsilon_0}^{\epsilon_0} \frac{d\epsilon'}{\epsilon'}}_{=\log(1/b)} \underbrace{\frac{1}{2} \int \frac{d^{d-1}k'}{(2\pi)^d v_F(k')}}_{\text{dos at FS}} .
\end{aligned}
\tag{2.22}
$$

Don't forget the fermion loop minus sign (in red, because I forgot it at first). Between the first and second lines, we did the ℓ' integral by residues. The crucial point is that we are interested in external energies $\epsilon \sim 0$, but we are integrating out a shell near the cutoff, so $\epsilon' > \epsilon$ and the sign of $\epsilon + \epsilon'$ is opposite that of $\epsilon - \epsilon'$; therefore there is a pole on either side of the real ℓ axis and we get the same answer by closing the contour either way. On one side the pole is at $\ell' = \frac{1}{v_F(k')}(\epsilon + \epsilon')$. (In the t-channel diagram (what Shankar calls ZS), the poles are on the *same* side and it therefore does not renormalize the four-fermion interaction.)

The result to one-loop is then

$$V(b) = V - V^2 N \log(1/b) + \mathcal{O}(V^3)$$

with $N \equiv \frac{1}{4\pi} \int đ^{d-1}k' v_F(k')$ is the density of states at the Fermi surface. From this we

[8] It's in a unfamiliar place. But this is the $\mathbf{i}\epsilon$ (rather, $\mathbf{i}\eta$) prescription that we get by analytic continuation from Euclidean time. Think about the integral

$$\int đ\omega \frac{e^{i\omega t}}{\omega(1 + \mathbf{i}\eta) - \omega_0} = -\mathbf{i}\theta(t \text{sgn}\,(\omega_0)) e^{i\omega_0 t}. \tag{2.21}$$

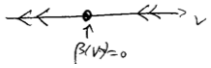
So it's the retarded green's function for particles and the advanced green's function for holes.

derive the beta function (recall that $b \to 0$ in the IR in this section)

$$-b\frac{d}{db}V(b) = \beta_V = -NV^2(b) + \mathcal{O}(V^3)$$

and the solution of the flow equation at $E = bE_1$ is

$$V(E) = \frac{V_1}{1 + NV_1 \log(E_1/E)} \begin{cases} \to 0 & \text{in IR for } V_1 > 0 \text{ (repulsive)} \\ \to -\infty & \text{in IR for } V_1 < 0 \text{ (attractive)} \end{cases} \tag{2.23}$$

There is therefore a very significant dichotomy depending on the sign of the coupling at the microscopic scale E_1, as in this phase diagram: 

The conclusion is that if the interaction starts attractive at some scale it flows to large attractive values. The thing that is decided by our perturbative analysis is that (if $V(E_1) < 0$) the decoupling we did with σ ('the BCS channel') wins over the decoupling with ρ ('the particle-hole channel').

What happens at $V \to -\infty$? Here we need non-perturbative physics. The non-perturbative physics is in general hard, but we've already done what we can in §2.3.

The remaining question is: Who is V_1 and why would it be attractive (given that Coulomb interactions between electrons, while screened and therefore short-ranged, are repulsive)? The answer is:

Phonons. The lattice of positions taken by the ions making up a crystalline solid spontaneously break many spacetime symmetries of their governing Hamiltonian. This implies a collection of gapless Goldstone modes in any low-energy effective theory of such a solid[9]. The Goldstone theorem is satisfied by including a field

$\vec{D}(\vec{r}) \propto$ (local) displacement $\delta\vec{r}$ of ions near $\vec{r}$ from their equilibrium positions

Most microscopically we have a bunch of coupled springs:

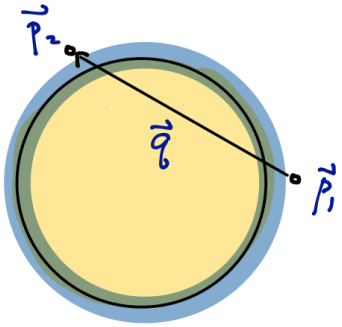
$$L_{\text{ions}} \sim \frac{1}{2}M\sum_I \left(\delta\dot{\vec{r}}_I\right)^2 - k_{ij}^{IJ}\delta r_I^i \delta r_J^j + ...$$

where $\delta\vec{r}_J$ is the displacement from equilibrium of ion J. We don't want to ask about the spring constants k, except to say that they are independent of the nuclear mass M. It is useful to introduce a canonically normalized field in terms of which the action is

$$S[\vec{D} = (M)^{1/2}\,\delta\vec{r}] = \frac{1}{2}\int dt d^d q \left(\partial_t D_i(q)\partial_t D_i(-q) - \omega_{ij}^2(q)D_i(q)D_j(-q)\right).$$

[9]Note that there is a subtlety in counting Goldstone modes from spontaneously broken spacetime symmetries: there are more symmetry generators than Goldstones. Basically it's because the associated currents differ only by functions of spacetime; but a localized Goldstone particle is anyway made by a current times a function of spacetime, so you can't sharply distinguish the resulting particles. Some useful references on this subject are Low-Manohar and more recently Watanabe-Murayama.

Here $\omega^2 \propto M^{-1}$. Their status as Goldstones means that the eigenvalues of $\omega_{ij}^2(q) \sim |q|^2$ at small q: moving everyone by the same amount does not change the energy. This also constrains the coupling of these modes to the electrons: they can only couple through derivative interactions.



For purposes of their interactions with the electrons, a nonzero q that keeps the e^- on the FS must scale like $q \sim b^0$. Therefore

$$dtd^dq\,(\partial_t D)^2 \sim b^{+1+2[D]} \quad \Longrightarrow \quad D \sim b^{-\frac{1}{2}}$$

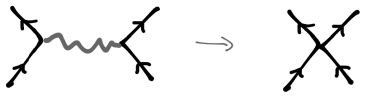
and the restoring force term $dtd^dqD^2\omega^2(q) \sim b^{-2}$ is relevant, and dominates over the ∂_t^2 term for

$$E < E_D = \sqrt{\frac{m}{M}}E_0 \qquad \text{the Debye energy.}$$

(For the more traditional derivation of the relation between E_D and E_0, see *e.g.* De-Gennes' *Superconductivity of Metals and Alloys*, pages 99-102.) This means that phonons mediate *static* interactions below E_D – we can ignore retardation effects, and their effects on the electrons can be fully incorporated by the four-fermion interaction we used above (with some $\vec{k}$ dependence). How do they couple to the electrons?

$$S_{\rm int}[D,\psi] = \int dtd^3qd^2k_1d\ell_1d^2k_2d\ell_2\; M^{-\frac{1}{2}}g_i(q,k_1,k_2)D_i(q)\psi_\sigma^\dagger(p_1)\psi_\sigma(p_2)\delta^3(p_1-p_2-q)$$
$$\sim b^{-1+1+1-3/2} = b^{-1/2} \tag{2.24}$$

– here we took the delta function to scale like b^0 as above. This is relevant when we use the $\dot{D}^2$ scaling for the phonons; when the restoring force dominates we should scale D differently and this is irrelevant for generic kinematics. This is consistent with our previous analysis of the four-fermion interaction.

The summary of this discussion is: phonons do not destroy the Fermi surface, but they *do* produce an attractive contribution to the 4-fermion interaction, which is relevant in some range of scales (above the Debye energy). Below the Debye energy, it amounts to an addition to V that goes like $-g^2$: 

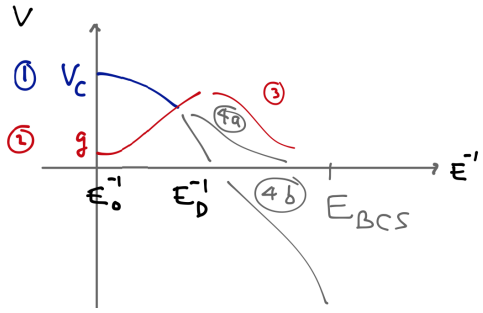
Notice that the scale at which the coupling V becomes strong ($V(E_{\rm BCS}) \equiv 1$ in (2.23)) is

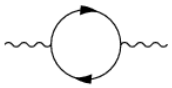
$$E_{\rm BCS} \sim E_D e^{-\frac{1}{NV_D}}.$$

Two comments about this: First, it is non-perturbative in the interaction V_D. Second, it provides some verification of the role of phonons, since $E_D \sim M^{-1/2}$ can be varied

by studying the same material with different isotopes and studying how the critical superconducting temperature ($\sim E_{\rm BCS}$) scales with the nuclear mass.

Actually, we can make some headway towards understanding the result of this interaction going strong. Because the diagrams with the special kinematics are marginal and hence unsuppressed, while all other interactions flow to zero at low energy, certain diagrams dominate. In particular, bubble-chains dominate.



Here's the narrative, proceeding as a function of decreasing energy scale, beginning at E_0, the Planck scale of solids: (1) Electrons repel each other by the Coulomb interaction. However, in a metal, this interaction is *screened* by processes like this:  (the intermediate state is an electron-hole pair) and is short-ranged. It is still repulsive, however. As we coarse-grain more and more, we see more and more electron-hole pairs and the force weakens. (2) While this is happening, the electron-phonon interaction is relevant and growing. This adds an attractive bit to V. This lasts until E_D. (3) At E_D the restoring force term in the phonon lagrangian dominates (for the purposes of their interactions with the electrons) and we can integrate them out. (4) What happens next depends on the sign of $V(E_D)$. If it's positive, V flows harmlessly to zero. If it's negative, it becomes moreso until we exit the perturbative analysis around $E_{\rm BCS}$, and vindicate our choice of Hubbard-Stratonovich channel above.

Further brief comments, for which I refer you to Shankar:

1. Putting back the possible angular dependence of the BCS interaction, the result at one loop is

$$\frac{dV(\theta_1-\theta_3)}{d\ell} = -\frac{1}{8\pi^2}\int_0^{2\pi} đ\theta V(\theta_1-\theta)V(\theta-\theta_3)$$

   or in terms of angular momentum components,

$$\frac{dV_l}{d\ell} = -\frac{V_l^2}{4\pi}.$$

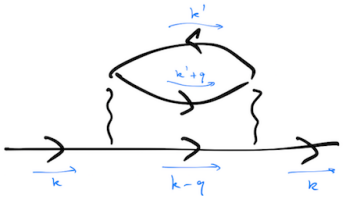
2. This example is interesting and novel in that it is a (family of) fixed point(s) characterized by a dimensionful quantity, namely k_F. This leads to a phenomenon called *hyperscaling violation* where thermodynamic quantities need not have their naive scaling with temperature.

3. The one loop analysis gives the right answer to all loops in the limit that $N \sim (k_F/\Lambda)^{d-1} \gg 1$, where Λ is the UV cutoff on the momentum.

4. The forward scattering interaction (for any choice of function $F(\theta_{13})$) is not renormalized at one loop. This means it is exactly marginal at leading order in N.

5. Like in ϕ^4 theory, the sunrise diagram at two loops is the first appearance of wavefunction renormalization. In the context of the Fermi liquid theory, this leads to the renormalization of the effective mass which is called $m^\star$.

Another consequence of the FS kinematics which I should emphasize more: it allows the quasiparticle to be stable. The leading contribution to the decay rate of a one-quasiparticle state with momentum k can be obtained applying the optical theorem to the following process.



In the figure, the object  is the four-fermion vertex (the wiggly line is just for clarity). The intermediate state is two electrons with momenta $k'+q$ and $k-q$, and one *hole* with momentum k'. (To understand the contour prescription for the propagator, it is useful to begin with

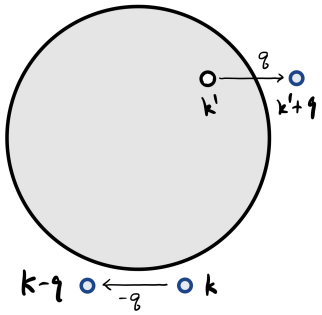
$$G(t,p) = \langle \mathrm{gs}| \mathcal{T} c_p^\dagger(t) c_p(0) |\mathrm{gs}\rangle \,, \quad c_p^\dagger(t) \equiv e^{-\mathbf{i}\mathbf{H}t} c_p^\dagger e^{\mathbf{i}\mathbf{H}t}$$

and use the free-fermion fact $[\mathbf{H}, c_p^\dagger] = \epsilon_p c_p^\dagger$. For more details, see the steps leading up to equation (7.7) of AGD (Abrikosov, Gorkov, Dzyaloshinski, *Methods of QFT in Statistical Physics.*)) Notice that this is the eyeball diagram which gives the lowest-order contribution to the wavefunction renormalization of a field with quartic interactions.

After doing the frequency integrals by residues, we get something of the form

$$\Sigma(k,\epsilon) = \int đq\, đk' \frac{|u_q|^2}{D} \theta(\epsilon_{k'}\epsilon_{k'+q})\theta(\epsilon_{k'}\epsilon_{k-q})$$

$$D \equiv \epsilon_k(1+\mathbf{i}\eta) + \epsilon_{k'}(1-\mathbf{i}\eta) - \epsilon_{k'+q}(1+\mathbf{i}\eta) - \epsilon_{k-q}(1+\mathbf{i}\eta)$$



By the optical theorem, its imaginary part is the (leading contribution to the) inverse-lifetime of the quasiparticle state with fixed k:

$$\tau^{-1}(k) = \mathrm{Im}\,\Sigma(k,\epsilon) = \pi \int đq\, đk' \delta(D) |u_q|^2 f(-\epsilon_{k'}) f(\epsilon_{k'+q}) f(\epsilon_{k-q})$$

where

$$f(\epsilon) = \lim_{T\to 0} \frac{1}{e^{\frac{\epsilon-\epsilon_F}{T}}+1} = \theta(\epsilon < \epsilon_F)$$

is the Fermi function. This is just the demand that a particle can only scatter into an empty state and a hole can only scatter into a filled state. These constraints imply that all the energies are near the Fermi energy: both $\epsilon_{k'+q}$ and $\epsilon_{k'}$ lie in a shell of radius ϵ about the FS; the answer is proportional to the density of possible final states, which is thus

$$\tau^{-1} \propto \left(\frac{\epsilon}{\epsilon_F}\right)^2 .$$

So the width of the quasiparticle resonance is

$$\tau^{-1} \propto \epsilon^2 \ll \epsilon$$

much smaller than its frequency – it is a sharp resonance, a well-defined particle.

The fact that the single-particle lifetime goes like ω^2 implies (as long as the decay of quasiparticles is the main source of current dissipation) that the electrical resistivity goes like $\rho(T) \sim T^2$. Rather, this is the contribution from electron-electron scattering. Disorder, in the form of static impurities, contributes an additive constant. In $d = 3$, phonons contribute $\rho \sim T^5$ (T^3 from the density of states of bosons with linear dispersion and an extra factor of T^2 for the derivative coupling), for $T < T_D$. For $T > T_D$, the phonons can be treated as classical springs and contribute $\rho \sim T$.

In high-temperature superconductors (at optimal doping), in sharp contrast, the resistivity goes like $\rho \sim T$ in a large range of temperatures, including temperatures well below the Debye temperature. The above analysis shows that some other EFT must describe them. A metal that is not described by the Landau Fermi liquid theory is called a non-Fermi liquid.

One significant loophole is that there could be other light degrees of freedom besides the electronic quasiparticles and the phonons. One way in which extra bosonic degrees of freedom can arise is at a quantum critical point.

3 Geometric and topological terms in field theory actions

The following is an unpaid advertisement: When studying a quantum mechanical system, isn't it annoying to have to worry about the order in which you write the symbols? What if they don't commute?! If you have this problem, too, the path integral is for you. In the path integral, the symbols are just integration variables – just ordinary numbers, and you can write them in whatever order you want. You can write them upside down if you want. You can even change variables in the integral (Jacobian not included).

(In what order do the operators end up? As we showed last quarter, in the kinds of path integrals we're thinking about, they end up in *time-order*. If you want a different order, you will need the *Schwinger-Keldysh* extension package, sold separately.)

Resolving the identity. Much of the following is about how to go back and forth from Hilbert space to path integral representations, aka Hamiltonian and Lagrangian descriptions of QFT. Starting with an operator expression for some physical quantity, we can make a path integral representation by sticking lots of 1ls in there, and then resolving each of the identity operators in some basis that we like. Different bases, different integrals. Some are useful, mostly because we have intuition for the behavior of integrals.

[End of Lecture 7]

3.1 Coherent state path integrals for bosons

[Wen §3.3] Let's develop a path integral for a collection of bosons, using as our basis of the local Hilbert space ordinary SHO (simple harmonic oscillator) coherent states. What I mean by 'bosons' is a many-body system whose Hilbert space can be written as $\mathcal{H} = \otimes_k \mathcal{H}_k$ where k is a label (could be real space, could be momentum space) and

$$\mathcal{H}_k = \text{span}\{|0\rangle_k, \mathbf{a}_k^\dagger |0\rangle_k, \frac{1}{\sqrt{2!}}\left(\mathbf{a}_k^\dagger\right)^2 |0\rangle_k, ...\} = \text{span}\{|n\rangle_{\vec{k}}, n = 0, 1, 2...\}$$

is the SHO Hilbert space. Assume the modes satisfy

$$[\mathbf{a}_{\vec{k}}, \mathbf{a}_{\vec{k}'}^\dagger] = \delta^d(\vec{k} - \vec{k}').$$

A good example hamiltonian to keep in mind is the free one,

$$\mathbf{H}_0 = \sum_{\vec{k}} \left(\epsilon_{\vec{k}} - \mu\right) \mathbf{a}_{\vec{k}}^\dagger \mathbf{a}_{\vec{k}} \ .$$

The object $\epsilon_{\vec{k}} - \mu$ is the energy of the state with one boson of momentum $\vec{k}$: $\mathbf{a}^\dagger_{\vec{k}} |0\rangle$. The chemical potential μ shifts the energy of any state by an amount proportional to

$$\left\langle \sum_{\vec{k}} \mathbf{a}^\dagger_{\vec{k}} \mathbf{a}_{\vec{k}} \right\rangle = N$$

the number of bosons. To this we could consider adding an interaction term $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$, such as

$$V = \sum_{x,y} V_{xy} a^\dagger_x a_x a^\dagger_y a_y.$$

For each normal mode $\mathbf{a}$, coherent states are the eigenstates of the annihilation operator[10]

$$\mathbf{a} |\phi\rangle = \phi |\phi\rangle ; \qquad |\phi\rangle = \mathcal{N} e^{\phi \mathbf{a}^\dagger} |0\rangle .$$

The eigenbra of $\mathbf{a}^\dagger$ is $\langle\phi|$, with

$$\langle\phi| \mathbf{a}^\dagger = \langle\phi| \phi^\star, \quad \langle\phi| = \langle 0| e^{+\phi^\star \mathbf{a}} \mathcal{N}.$$

(In this case, this equation is the adjoint of the previous one.) Their overlap is[11]:

$$\langle\phi_1|\phi_2\rangle = e^{\phi_1^\star \phi_2}.$$

If we choose $\mathcal{N} = e^{-|\phi|^2/2}$, they are normalized, but it is more convenient to set $\mathcal{N} = 1$. The overcompleteness relation on $\mathcal{H}_k$ is[12]

$$\mathbb{1}_k = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} |\phi\rangle\langle\phi|. \tag{3.1}$$

It will be convenient to arrange all our operators into sums of *normal-ordered* operators:

$$: \mathbf{a}_k \mathbf{a}^\dagger_l :=: \mathbf{a}^\dagger_l \mathbf{a}_k := \mathbf{a}^\dagger_l \mathbf{a}_k$$

[10]The right equation is true because

$$\mathbf{a} e^{\phi \mathbf{a}^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \underbrace{\mathbf{a} \left(\mathbf{a}^\dagger\right)^n |0\rangle}_{n \left(\mathbf{a}^\dagger\right)^{n-1} |0\rangle} = \sum_{m=n-1} \frac{\phi^{m+1}}{m!} \left(\mathbf{a}^\dagger\right)^m |0\rangle .$$

[11]Since $\mathbf{a}$ is not hermitian, eigenvectors with different eigenvalues need not be orthogonal. You can check this by expanding the coherent states in the number basis and using $\langle 0|a^n a^{\dagger m}|0\rangle = \delta_{mn} n!$.

[12]Again we can go to the number basis and do the integrals:

$$\int \frac{d\phi d\phi^\star}{\pi} e^{-\phi\phi^\star} \phi^n \left(\phi^\star\right)^{n'} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\mathbf{i}(n-n')\theta} \int_0^\infty du e^{-u} u^{\frac{n+n'}{2}}$$

to get $\mathbb{1} = \sum_n |n\rangle\langle n|$.

with all annihilation operators to the right of all creation operators. Coherent state expectation values of such operators can be built from the monomials

$$\langle\phi|\prod_k \left(\mathbf{a}_k^\dagger\right)^{M_k}\left(\mathbf{a}_k\right)^{N_k}|\phi\rangle = \prod_k (\phi_k^\star)^{M_k}\left(\phi_k\right)^{N_k}.$$

Also useful will be the representation of the trace in this basis:

$$\mathrm{tr}\cdot = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2}\,\langle\phi|\cdot|\phi\rangle\,.$$

Let the Hamiltonian be $\mathbf{H} = H(\{\mathbf{a}_k^\dagger\},\{\mathbf{a}_k\}) =: \mathbf{H}:$, normal ordered. First let's study a single mode. To derive a path integral (for *e.g.* the thermal partition function) using this resolution of the identity $\mathbb{1} = \prod_{\vec{k}} \mathbb{1}_{\vec{k}}$, we write

$$Z = \mathrm{tr} e^{-\mathbf{H}/T} = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2}\,\langle\phi|\underbrace{e^{-\mathbf{H}/T}}_{=e^{-\Delta\tau\mathbf{H}}e^{-\Delta\tau\mathbf{H}}\cdots e^{-\Delta\tau\mathbf{H}}}|\phi\rangle$$

and repeatedly insert $\mathbb{1}$ in the form (3.1):

$$\begin{aligned} Z &= \mathrm{tr} e^{-\mathbf{H}/T} \\ &= \int \prod_{l=0}^{M-1} \frac{d\phi_l d\phi_l^\star}{\pi} e^{-|\phi_l|^2}\,\langle\phi_{l+1}|\,e^{-\Delta\tau H}\,|\phi_l\rangle \\ &= \int_{\phi_M=\phi_0} \prod_{l=0}^{M-1} \frac{d^2\phi_l}{\pi}\; e^{-\sum_{l=0}^{M-1}\left(\phi_{l+1}^\star(\phi_{l+1}-\phi_l)-\Delta\tau H(\phi_{l+1}^\star\phi_l)\right)} \\ &\simeq \int_{\phi(0)=\phi(1/T)} [D\phi]\; e^{-\int_0^{1/T} d\tau(\phi^\star\partial_\tau\phi + H(\phi^\star,\phi))}. \end{aligned} \tag{3.2}$$

Here we used

$$\langle\phi_{\ell+1}|\,e^{-\Delta\tau\mathbf{H}}\,|\phi_\ell\rangle \simeq \langle\phi_{\ell+1}|\,(1-\Delta\tau\mathbf{H})\,|\phi_\ell\rangle = \langle\phi_{\ell+1}|\,\left(1-\Delta\tau H(\phi_{\ell+1}^\star,\phi_\ell)\right)|\phi_\ell\rangle \simeq e^{-\Delta\tau H(\phi_{\ell+1}^\star,\phi_\ell)}e^{-\phi_{\ell+1}^\star\phi_\ell}$$

for small enough $\Delta\tau$.

Now let's put back the mode labels. Doing the above for each momentum mode, we get

$$Z = \int [Da] e^{\int d\tau \sum_{\vec{k}}\left(\frac{1}{2}\left(a_{\vec{k}}^\star\dot{a}_{\vec{k}} - a_{\vec{k}}\dot{a}_{\vec{k}}^\star\right) - \left(\epsilon_{\vec{k}}-\mu\right)a_{\vec{k}}^\star a_{\vec{k}}\right) + \int d\tau V}.$$

In real space $a_{\vec{k}} \equiv \int \mathrm{d}^{D-1}x e^{i\vec{k}\cdot\vec{x}}\Phi(\vec{x})$, Taylor expanding $\epsilon_{\vec{k}} - \mu = -\mu + \frac{\vec{k}^2}{2m} + \mathcal{O}(k^4)$, this is

$$Z = \int [D\Phi] e^{\int \mathrm{d}^d\vec{x}\mathrm{d}\tau\left(\frac{1}{2}(\Phi^\star\partial_\tau\Phi - \Phi\partial_\tau\Phi^\star) - \frac{1}{2m}\vec{\nabla}\Phi^\star\cdot\vec{\nabla}\Phi - \mu\Phi^\star\Phi\right) + \int d\tau V}.$$

The first term in the exponent is sometimes called a Berry phase term.

Real time. If you are interested in real-time propagation, rather than euclidean time, just replace the euclidean propagator $e^{-\tau \mathbf{H}} \mapsto e^{-\mathbf{i}t\mathbf{H}}$. This amounts to a replacement $\tau \to \mathbf{i}t$; as usual, for convergence purposes, it's better to rotate not quite all the way: $\tau \to e^{\mathbf{i}\left(\frac{\pi}{2}-\eta\right)}t$. The result, for example, for the amplitude to propagate from one bose coherent state to another is

$$\langle \Phi_f, t_f |\, e^{-\mathbf{i}t\mathbf{H}} \,| \Phi_0, t_0 \rangle = \int_{\Phi(t_0)=\Phi_0}^{\Phi(t_f)=\Phi_f} D\Phi \; e^{\frac{\mathbf{i}}{\hbar}\int_{t_0}^{t_f} dt(\mathbf{i}\hbar\Phi^\star\partial_t\Phi - H(\Phi,\Phi^\star))}.$$

(Actually there is some funny business with the boundary conditions in this expression. In particular, notice that with a first-order kinetic term, we only get to impose one (complex) initial condition. A more careful treatment can be found here, §1.9.)

Note that a distinguishing feature of the Berry phase term (because it has a single time derivative) is that it produces a complex term in the real-time action. Another distinguishing feature of the Berry phase term is that it is *geometric*: for a history of field configurations, $\Phi(t)$, the Berry phase term

$$\int_{t_0}^{t_f} dt \Phi^\star(t)\dot{\Phi}(t) = \int_{\Phi_0}^{\Phi_f} \Phi^\star d\Phi$$

doesn't depend on how fast we go, only on the path in field space.

Question to ponder: how would you get a second-order-in-time kinetic term?

This is the same non-relativistic field theory that we found earlier by taking the $E \ll m$ limit of a relativistic scalar field, and that described a non-relativistic superfluid (or superconductor if we gauge the $\mathsf{U}(1)$ symmetry). Notice that the field Φ is actually the coherent state eigenvalue!

If instead we had an interaction term in H, say $\Delta H = \int d^dx \int d^dy \frac{1}{2}\Phi^\star(x,t)\Phi(x,t)V(x-y)\Phi^\star(y,t)\Phi(y,t)$, it would lead to a term in the path integral action

$$S_i = -\int \mathrm{d}t \int d^dx \int d^dy \frac{1}{2}\Phi^\star(x,t)\Phi(x,t)V(x-y)\Phi^\star(y,t)\Phi(y,t) \ .$$

In the special case $V(x-y) = V(x)\delta^d(x-y)$, this is the local quartic interaction we considered briefly earlier.

3.2 Coherent state path integral for fermions

[Shankar, *Principles of QM*, path integrals revisited. In this chapter of his great QM textbook, Shankar sneaks in lots of insights useful for modern condensed matter physics. For much more see Shankar, *Quantum field theory and condensed matter*.]

Consider the algebra of a single fermion mode operator[13]:

$$\{\mathbf{c}, \mathbf{c}\} = 0, \quad \{\mathbf{c}^\dagger, \mathbf{c}^\dagger\} = 0, \quad \{\mathbf{c}, \mathbf{c}^\dagger\} = 1 \ .$$

With a single mode, the most general Hamiltonian is

$$\mathbf{H} = \mathbf{c}^\dagger \mathbf{c} \, (\omega_0 - \mu)$$

(ω_0 and μ are (redundant when there is only one mode) constants, and I've omitted an additive constant). This algebra is represented on a two-state system $|1\rangle = \mathbf{c}^\dagger |0\rangle$. We might be interested in its thermal partition function

$$Z = \text{tr } e^{-\frac{\mathbf{H}}{T}} \ .$$

(In this example, it happens to equal $Z = 1 + e^{-\frac{\omega_0 - \mu}{T}}$, as you can see by computing the trace in the eigenbasis of $\mathbf{n} = \mathbf{c}^\dagger \mathbf{c}$. But never mind that; the one mode is a proxy for many, where it's not quite so easy to sum. The kind of model we really want to think about looks like

$$\mathbf{H} = \sum_k \mathbf{c}_k^\dagger \mathbf{c}_k \, (\omega_k - \mu) + \sum_{x,y} \mathbf{c}_x^\dagger \mathbf{c}_x V_{xy} \mathbf{c}_y^\dagger \mathbf{c}_y .)$$

How do we trotterize this? That is, what is 'the' corresponding classical system? We can do the Trotterizing using any resolution of the identity on $\mathcal{H}$, so there can be many very-different-looking answers to this question. (One answer is to use the (0d) Jordan-Wigner map which relates spins and fermions. More about that later. Here's another, different, answer.)

Let's define coherent states for fermionic operators:

$$\mathbf{c} \, |\psi\rangle = \psi \, |\psi\rangle \, . \tag{3.3}$$

Here ψ is a c-number (not an operator), but acting twice with $\mathbf{c}$ we see that we must have $\psi^2 = 0$. So ψ is a grassmann number. These satisfy

$$\psi_1 \psi_2 = -\psi_2 \psi_1, \quad \psi \mathbf{c} = -\mathbf{c} \psi \tag{3.4}$$

[13] For many modes,

$$\{\mathbf{c}_i, \mathbf{c}_j\} = 0, \quad \{\mathbf{c}_j^\dagger, \mathbf{c}_j^\dagger\} = 0, \quad \{\mathbf{c}_j, \mathbf{c}_j^\dagger\} = \mathbb{1} \delta_{ij} \ .$$

– they anticommute with each other and with fermionic operators, and commute with ordinary numbers and bosons. They seem weird but they are easy. We'll need to consider multiple grassmann numbers when we have more than one fermion mode, where $\{\mathbf{c}_1, \mathbf{c}_2\} = 0$ will require that they anticommute $\{\psi_1, \psi_2\} = 0$ (as in the definition (3.4)); note that we will be simultaneously diagonalizing operators that *anti*commute. This is only possible because the eigenvalues themselves anticommute!

The solution to equation (3.3) is very simple:

$$|\psi\rangle = |0\rangle - \psi\,|1\rangle = |0\rangle - \psi\mathbf{c}^\dagger\,|0\rangle = e^{-\psi\mathbf{c}^\dagger}\,|0\rangle$$

where as above $|0\rangle$ is the empty state ($\mathbf{c}\,|0\rangle = 0$) and $|1\rangle = \mathbf{c}^\dagger\,|0\rangle$ is the filled state. (Check: $\mathbf{c}\,|\psi\rangle = \mathbf{c}\,|0\rangle - \mathbf{c}\psi\,|1\rangle = +\psi\mathbf{c}\,|1\rangle = \psi\,|0\rangle = \psi\,|\psi\rangle$.)

Similarly, the left-eigenvector of the creation operator is

$$\langle\bar\psi|\,\mathbf{c}^\dagger = \langle\bar\psi|\,\bar\psi, \qquad \langle\bar\psi| = \langle 0| - \langle 1|\,\bar\psi = \langle 0| + \bar\psi\,\langle 1|\,.$$

Notice that these states are weird in that they are elements of an enlarged hilbert space with grassmann coefficients (usually we just allow complex numbers). Also, $\bar\psi$ is *not* the complex conjugate of ψ and $\langle\bar\psi|$ is not the adjoint of $|\psi\rangle$. Rather, their overlap is

$$\langle\bar\psi|\psi\rangle = 1 + \bar\psi\psi = e^{\bar\psi\psi}.$$

Grassmann calculus summary. In the last expression we have seen an example of the amazing simplicity of Taylor's theorem for grassmann functions:

$$f(\psi) = f_0 + f_1\psi\ .$$

Integration is just as easy and it's the same as taking derivatives:

$$\int \psi d\psi = 1, \qquad \int 1 d\psi = 0.$$

With more than one grassmann we have to worry about the order:

$$1 = \int \bar\psi\psi d\psi d\bar\psi = -\int \bar\psi\psi d\bar\psi d\psi.$$

The only integral, really, is the gaussian integral:

$$\int e^{-a\bar\psi\psi} d\bar\psi d\psi = a.$$

Many of these give

$$\int e^{-\bar\psi\cdot A\cdot\psi} d\bar\psi d\psi = \det A.$$

Here $\bar\psi\cdot A\cdot\psi\equiv\left(\bar\psi_1,\cdots,\bar\psi_M\right)\begin{pmatrix}A_{11}&A_{12}&\cdots\\A_{21}&\ddots&\cdots\\\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}\psi_1\\\vdots\\\psi_M\end{pmatrix}$. One way to get this expression is to change variables to diagonalize the matrix A.

$$\left\langle\bar\psi\psi\right\rangle\equiv\frac{\int\bar\psi\psi e^{-a\bar\psi\psi}d\bar\psi d\psi}{\int e^{-a\bar\psi\psi}d\bar\psi d\psi}=-\frac{1}{a}=-\left\langle\psi\bar\psi\right\rangle.$$

If for many grassman variables we use the action $S=\sum_i a_i\bar\psi_i\psi_i$ (diagonalize A above) then

$$\left\langle\bar\psi_i\psi_j\right\rangle=\frac{\delta_{ij}}{a_i}\equiv\langle\bar ij\rangle \tag{3.5}$$

and Wick's theorem here is

$$\left\langle\bar\psi_i\bar\psi_j\psi_k\psi_l\right\rangle=\langle\bar il\rangle\langle\bar jk\rangle-\langle\bar ik\rangle\langle\bar jl\rangle.$$

Back to quantum mechanics: The resolution of $\mathbb{1}$ in this basis is

$$\mathbb{1}=\int d\bar\psi d\psi\; e^{-\bar\psi\psi}\left|\psi\right\rangle\left\langle\bar\psi\right| \tag{3.6}$$

And if $\mathbf{A}$ is a bosonic operator (made of an even number of grassmann operators),

$$\mathrm{tr}\mathbf{A}=\int d\bar\psi d\psi\; e^{-\bar\psi\psi}\left\langle-\bar\psi\right|\mathbf{A}\left|\psi\right\rangle\ .$$

(Note the minus sign; it will lead to a deep statement.) So the partition function is:

$$Z\mathrm{tr}e^{-\mathbf{H}/T}=\int d\bar\psi_0 d\psi_0\; e^{-\bar\psi_0\psi_0}\left\langle-\bar\psi_0\right|\underbrace{e^{-\frac{\mathbf{H}}{T}}}_{=\underbrace{(1-\Delta\tau\mathbf{H})\cdots(1-\Delta\tau\mathbf{H})}_{M\ \text{times}}}\left|\psi_0\right\rangle$$

Now insert (3.6) in between each pair of Trotter factors to get

$$Z=\int\prod_{l=0}^{M-1}d\bar\psi_l d\psi_l e^{-\bar\psi_l\psi_l}\left\langle\bar\psi_{l+1}\right|(1-\Delta\tau\mathbf{H})\left|\psi_l\right\rangle\ .$$

Because of the $-\bar\psi$ in (3.6), to get this nice expression we had to define an extra letter

$$\bar\psi_M=-\bar\psi_0,\quad\psi_M=-\psi_0 \tag{3.7}$$

so we could replace $\left\langle-\bar\psi_0\right|=\left\langle\bar\psi_M\right|$.

Now we use the coherent state property to turn the matrix elements into grassmann-valued functions:

$$\left\langle \bar{\psi}_{l+1} \right| \left(1 - \Delta\tau H(\mathbf{c}^\dagger, \mathbf{c})\right) \left|\psi_l\right\rangle = \left\langle \bar{\psi}_{l+1} \right| \left(1 - \Delta\tau H(\bar{\psi}_{l+1}, \psi_l)\right) \left|\psi_l\right\rangle \overset{\Delta\tau\to 0}{=} e^{\bar{\psi}_{l+1}\psi_l} e^{-\Delta\tau H(\bar{\psi}_{l+1},\psi_l)}.$$

It was important that in $\mathbf{H}$ all $\mathbf{c}$s were to the right of all $\mathbf{c}^\dagger$s, *i.e.* that $\mathbf{H}$ was normal ordered.

So we have

$$\begin{aligned} Z &= \int \prod_{l=0}^{M-1} d\bar{\psi}_l d\psi_l e^{-\bar{\psi}_l\psi_l} e^{\bar{\psi}_{l+1}\psi_l} e^{-\Delta\tau H(\bar{\psi}_{l+1},\psi_l)} \\ &= \int \prod_{l=0}^{M-1} d\bar{\psi}_l d\psi_l \exp\left(\Delta\tau \left(\underbrace{\frac{\bar{\psi}_{l+1} - \bar{\psi}_l}{\Delta\tau}}_{=\partial_\tau\bar{\psi}} \psi_l - H(\bar{\psi}_{l+1}, \psi_l)\right)\right) \\ &\simeq \int [D\bar{\psi}D\psi] \exp\left(\int_0^{1/T} d\tau\ \bar{\psi}(\tau)\left(-\partial_\tau - \omega_0 + \mu\right)\psi(\tau)\right) = \int [D\bar{\psi}D\psi] e^{-S[\bar{\psi},\psi]}. \end{aligned} \tag{3.8}$$

Points to note:

- In the penultimate step we defined, as usual, continuum fields

$$\psi(\tau_l = \Delta\tau l) \equiv \psi_l, \qquad \bar{\psi}(\tau_l = \Delta\tau l) \equiv \bar{\psi}_l.$$

- We elided the difference $H(\bar{\psi}_{l+1}, \psi_l) = H(\bar{\psi}_l, \psi_l) + \mathcal{O}(\Delta\tau)$ in the last expression. This difference is usually negligible and sometimes helpful (an example where it's helpful is the discussion of the number density below).

- The APBCs (3.7) on $\psi(\tau + \frac{1}{T}) = -\psi(\tau)$ mean that in its fourier representation[14]

$$\psi(\tau) = T\sum_n \psi(\omega) e^{-\mathbf{i}\omega_n\tau}, \qquad \bar{\psi}(\tau) = T\sum_n \bar{\psi}(\omega) e^{\mathbf{i}\omega_n\tau} \tag{3.9}$$

  the *Matsubara frequencies*

$$\omega_n = (2n+1)\pi T, \quad n \in \mathbb{Z}$$

  are half-integer multiples of πT. This has the important physical consequence that at finite temperature, there is no zero-mode of a fermion field – all the modes have a restoring force. This means that the quantum Fermi gas at any $T \neq 0$ is adiabatically connected to the classical gas at $T = \infty$, unlike the case of bosons.

[14] $\bar{\psi}$ is still not the complex conjugate of ψ but the relative sign is convenient.

- The measure $[D\bar{\psi}D\psi]$ is defined by this equation, just as in the bosonic path integral.
- The derivative of a grassmann function is also defined by this equation; note that $\psi_{l+1} - \psi_l$ is not 'small' in any sense.
- In the last step we integrated by parts, *i.e.* relabeled terms in the sum, so

$$\sum_l \left(\bar{\psi}_{l+1} - \bar{\psi}_l\right)\psi_l = \sum_l \bar{\psi}_{l+1}\psi_l - \sum_l \bar{\psi}_l\psi_l = \sum_{l'=l-1} \bar{\psi}_{l'}\psi_{l'-1} - \sum_l \bar{\psi}_l\psi_l = -\sum_l \bar{\psi}_l\left(\psi_l - \psi_{l-1}\right).$$

  Note that no grassmanns were moved through each other in this process.

The punchline of this discussion for now is that the euclidean action is

$$\boxed{S[\bar{\psi},\psi] = \int d\tau \left(\bar{\psi}\partial_\tau\psi + H(\bar{\psi},\psi)\right)}\,.$$

The first-order kinetic term we've found $\bar{\psi}\partial_\tau\psi$ is sometimes called a 'Berry phase term'. Note the funny-looking sign.

Continuum limit warning (about the red $\simeq$ in (3.8)). The Berry phase term is *actually*

$$\sum_{l=0}^{M-1} \bar{\psi}_{l+1}\left(\psi_{l+1} - \psi_l\right) = T\sum_{\omega_n} \bar{\psi}(\omega_n)\left(1 - e^{\mathbf{i}\omega_n\tau}\right)\psi(\omega_n)$$

and in (3.8) we have kept only the leading nonzero term:

$$\left(1 - e^{\mathbf{i}\omega_n\tau}\right) \to \mathbf{i}\omega_n\tau.$$

Clearly this replacement is just fine if

$$\omega_n\tau \ll 1$$

for all ω_n which matter. Which ω_n contribute? I claim that if we use a reasonable $\mathbf{H} = \mathbf{H}_{\text{quadratic}} + \mathbf{H}_{\text{int}}$, reasonable quantities like Z, $\left\langle \mathcal{O}^\dagger\mathcal{O}\right\rangle$, are dominated by $\omega_n \ll \tau^{-1}$.

There's more we can learn from what we've done here that I don't want to pass up. Let's use this formalism to compute the fermion density at $T = 0$:

$$\langle \mathbf{N}\rangle = \frac{1}{Z}\mathrm{tr}e^{-\mathbf{H}/T}\mathbf{c}^\dagger\mathbf{c}.$$

This is an example where the annoying $\Delta\tau$s in the path integral not only matter, but are extremely friendly to us.

Frequency space, $T \to 0$.

Let's change variables to frequency-space fields, which diagonalize S. The Jacobian is 1 (since fourier transform is unitary):

$$D\bar{\psi}(\tau)D\psi(\tau) = \prod_n d\bar{\psi}(\omega_n)d\psi(\omega_n) \overset{T\to 0}{\to} D\bar{\psi}(\omega)D\psi(\omega).$$

The partition function is

$$Z = \int D\bar{\psi}(\omega)D\psi(\omega)\ \exp\left(T\sum_{\omega_n}\bar{\psi}(\omega_n)\left(\mathbf{i}\omega_n - \omega_0 + \mu\right)\psi(\omega_n)\right).$$

Notice that in the zero-temperature limit

$$T\sum_{\omega_n} \mapsto \int \frac{d\omega}{2\pi} \equiv \int \text{đ}\omega.$$

(This is the same fact as $V\sum_k \mapsto \int \text{đ}^d k$ in the thermodynamic limit.) So the zero-temperature partition function is

$$Z \overset{T\to 0}{=} \int D\bar{\psi}(\omega)D\psi(\omega)\ \exp\left(\int_{-\infty}^{\infty}\text{đ}\omega\bar{\psi}(\omega)\left(\mathbf{i}\omega - \omega_0 + \mu\right)\psi(\omega)\right).$$

Using the gaussian-integral formula (3.5) you can see that the *propagator* for ψ is

$$\left\langle\bar{\psi}(\omega_1)\psi(\omega_2)\right\rangle = \underbrace{\frac{\delta_{\omega_1,\omega_2}}{T}}_{\overset{T\to 0}{\to}\delta(\omega_1-\omega_2)}\ \frac{2\pi}{\mathbf{i}\omega_1 - \omega_0 + \mu}. \tag{3.10}$$

In particular $\left\langle\bar{\psi}(\omega)\psi(\omega)\right\rangle = \frac{2\pi/T}{\mathbf{i}\omega-\omega_0+\mu}$. $\delta(\omega = 0) = 1/T$ is the 'volume' of the time direction.

Back to the number density. Using the same strategy as above, we have

$$\langle\mathbf{N}\rangle = \frac{1}{Z}\int \prod_{l=0}^{M-1+1}\left(d\bar{\psi}_l d\psi_l e^{-\bar{\psi}_l\psi_l}\right)\prod_{l=1}^{M-1}\langle\bar{\psi}_{l+1}|(1-\Delta\tau\mathbf{H}(\mathbf{c}^\dagger\mathbf{c}))|\psi_l\rangle\ \underbrace{\left\langle\bar{\psi}_{N+1}\right|\mathbf{c}^\dagger\mathbf{c}\left|\psi_N\right\rangle}_{=\bar{\psi}_{N+1}\psi_N=\bar{\psi}(\tau_N+\Delta\tau)\psi(\tau_N)},$$

where τ_N is any of the time steps. This formula has a built-in point-splitting of the operators!

$$\langle\mathbf{N}\rangle = \frac{1}{Z}\int D\bar{\psi}D\psi\ e^{-S[\bar{\psi},\psi]}\bar{\psi}(\tau_N+\Delta\tau)\psi(\tau_N)$$

$$= \int_{-\infty}^{\infty} đ\omega \frac{e^{\mathbf{i}\omega\Delta\tau}}{\mathbf{i}\omega - \omega_0 + \mu} = \theta(\mu - \omega_0). \tag{3.11}$$

Which is the right answer: the mode is occupied in the groundstate only if $\omega_0 < \mu$. In the last step we used the fact that $\Delta\tau > 0$ to close the contour in the UHP; so we only pick up the pole if it is in the UHP. Notice that this quantity is very *UV sensitive*: if we put a frequency cutoff on the integral, $\int^{\Lambda} \frac{d\omega}{\omega} \sim \log\Lambda$, the integral diverges logarithmically. For most calculations the $\Delta\tau$ can be ignored, but here it told us the right way to treat the divergence. [15]

[End of Lecture 8]

[15] The calculation between the first and second lines of (3.11) is familiar to us – it is a single Wick contraction, and can be described as a feynman diagram with one line between the two insertions. More prosaically, it is

$$\left\langle \bar{\psi}(\tau_N + \Delta\tau)\psi(\tau_N) \right\rangle \overset{(3.9)}{=} T^2 \sum_{nm} e^{\mathbf{i}(\omega_n - \omega_m)\tau + \mathbf{i}\omega_n\Delta\tau} \left\langle \bar{\psi}(\omega_n)\psi(\omega_m) \right\rangle \overset{(3.10)}{=} T \sum_m \frac{e^{\mathbf{i}\omega_n\Delta\tau}}{\mathbf{i}\omega_n - \omega_0 + \mu} \overset{T\to 0}{\to} \int đ\omega \frac{e^{\mathbf{i}\omega\Delta\tau}}{\mathbf{i}\omega - \omega_0 + \mu}.$$

3.3 Path integrals for spin systems

In this subsection we develop path integral descriptions of spin systems.

Quantum spin systems. To be clear, let me say a few introductory words about quantum spin systems, the flagship family of examples of well-regulated QFTs. These include useful models for magnetic insulators, and for possible hardware platforms for quantum computing.

Such a thing is a collection of two-state systems (aka qbits) $\mathcal{H}_j = \text{span}\{|\uparrow_j\rangle, |\downarrow_j\rangle\}$ distributed over space and coupled somehow:

$$\mathcal{H} = \bigotimes_j \mathcal{H}_j \ , \quad \dim(\mathcal{H}) = 2^N$$

where N is the number of sites.

One qbit: To begin, consider just one two-state system. There are four independent hermitian operators acting on this Hilbert space. Besides the identity, there are the three Pauli operators, which I will denote by $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ instead of $\boldsymbol{\sigma}^x, \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^z$:

$$\mathbf{X} \equiv \boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} \equiv \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \mathbf{Z} \equiv \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This notation (which comes to us from the quantum information community) makes the important information larger and is therefore better, especially for those of us with limited eyesight.

They satisfy

$$\mathbf{XY} = \mathbf{iZ}, \quad \mathbf{XZ} = -\mathbf{ZX}, \quad \mathbf{X}^2 = \mathbb{1},$$

and all cyclic permutations $\mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to \mathbf{X}$ of these statements.

Multiple qbits: If we have more than one site, denote, *e.g.*

$$\mathbf{X}_j \equiv \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underbrace{\mathbf{X}}_{j\text{th site}} \otimes \cdots . \tag{3.12}$$

Since they act nontrivially only on different Hilbert spaces, the paulis on different sites *commute*:

$$[\boldsymbol{\sigma}_j^\alpha, \boldsymbol{\sigma}_l^\beta] = 0, \quad j \neq l \quad \textit{i.e.} \quad \mathbf{X}_j \mathbf{Z}_l = (-1)^{\delta_{jl}} \mathbf{Z}_l \mathbf{X}_j,$$

where $\boldsymbol{\sigma}_j^\alpha$ is any of the three Pauli operators acting on $\mathcal{H}_j$.

Quantum-classical correspondence. We've seen by now that *any* resolution of the identity on the local Hilbert space will give us some form of 'sum over histories'. In a spin system, perhaps the most obvious 'path integral' is the one associated with the $\mathbf{Z}$-basis resolution, $\mathbb{1} = |+1\rangle\langle+1| + |-1\rangle\langle-1|$, with $\mathbf{Z}|s\rangle = s|s\rangle$. In this case, the

labels on the states are classical spins ± 1 (or equivalently, classical bits). I put 'path integral' in quotes because it is instead a 'path sum', since the integration variables are discrete. Using this basis would allow us to further harness our knowledge of stat mech for QFT purposes. An important conclusion from that analysis is the (inverse) relationship between the correlation length and the energy gap above the groundstate.

To see the idea, consider the quantum system consisting of a single spin with $\mathbf{H} = E_0 - \frac{\Delta}{2}\mathbf{X} + \bar{h}\mathbf{Z}$. Set $\bar{h} = 0$ for a moment. Then Δ is the energy gap between the groundstate and the first excited state (hence the name). The thermal partition function is

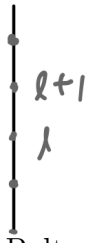
$$Z_Q(T) = \mathrm{tr} e^{-\mathbf{H}/T} = \sum_{s=\pm} \langle s| e^{-\beta \mathbf{H}} |s\rangle \,, \tag{3.13}$$

where we've evaluated the trace in the $\mathbf{Z}$ basis, $\mathbf{Z}|s\rangle = s|s\rangle$. I emphasize that T here is the temperature to which we are subjecting our quantum spin; $\beta = \frac{1}{T}$ is the length of the euclidean time circle. Break up the euclidean time circle into M_τ intervals of size $\Delta\tau = \beta/M_\tau$. Insert many resolutions of unity (this is called 'Trotter decomposition')

$$Z_Q = \sum_{s_1 \ldots s_{M_\tau}} \langle s_{M_\tau}| e^{-\Delta\tau \mathbf{H}} |s_{M_\tau - 1}\rangle \langle s_{M_\tau - 1}| e^{-\Delta\tau \mathbf{H}} |s_{M_\tau - 2}\rangle \cdots \langle s_1| e^{-\Delta\tau \mathbf{H}} |s_{M_\tau}\rangle \ .$$

The RHS is the partition function of a classical Ising chain,

$$Z_1 = \sum_{\{s_l = \pm 1\}} e^{-S}, \qquad S = -K \sum_{l=1}^{M_\tau} s_l s_{l+1} - h \sum_{l=1}^{M_\tau} s_l \tag{3.14}$$



These ss are now just M_τ numbers, each ± 1 – there are 2^{M_τ} terms in this sum[16]. The parameter $K > 0$ is the 'inverse temperature' in the Boltzmann distribution; I put these words in quotes because I want you to think of it as merely a parameter in the classical hamiltonian.

Even if we didn't care about quantum spins, this way of organizing the partition sum of the Ising chain does the sum for us (since the trace is basis-independent, and so we might as well evaluate it in the basis where the transfer matrix $\mathbf{T} = e^{-\Delta\tau \mathbf{H}}$ is diagonal):

$$Z_1 = \mathrm{tr}\mathbf{T}^{M_\tau} = \lambda_+^{M_\tau} + \lambda_-^{M_\tau}$$

[16]Here I used

$$\langle s_1| e^{a\mathbf{X}} |s_2\rangle = e^{K s_1 s_2} \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix}_{s_1 s_2} \qquad \langle s_1| e^{b\mathbf{Z}} |s_2\rangle = e^{b s_1} \delta_{s_1 s_2} \tag{3.15}$$

and the identity $e^{a\mathbf{X}} = \cosh a \mathbb{1} + \sinh a \mathbf{X}$. For $h \neq 0$, we choose $\Delta\tau$ small enough that we can approximate

$$e^{a\Delta\tau \mathbf{X} + b\Delta\tau \mathbf{Z}} = e^{a\Delta\tau \mathbf{X}} e^{b\Delta\tau \mathbf{X}} + \mathcal{O}(\Delta\tau^2). \tag{3.16}$$

where $\lambda_\pm$ are the two eigenvalues of the transfer matrix, $\lambda_+ \geq \lambda_-$:

$$\lambda_\pm = e^K \cosh h \pm \sqrt{e^{2K}\sinh^2 h + e^{-2K}} \overset{h\to 0}{\to} \begin{cases} 2\cosh K \\ 2\sinh K. \end{cases} \tag{3.17}$$

In the thermodynamic limit, $M_\tau \gg 1$, the bigger one dominates the free energy is

$$e^{-F} = Z_1 = \lambda_+^{M_\tau}\left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^{M_\tau}\right) \sim \lambda_+^{M_\tau}.$$

In the Z_1 obtained from Trotterizing the single qubit, we have $h = 0$ and K determined by the relation

$$e^{-2K} = \tanh\left(\frac{\beta\Delta}{2M_\tau}\right) \ . \tag{3.18}$$

Notice that if our interest is in the quantum model with couplings E_0, Δ, we can use any M_τ we want – there are many classical models we could use[17]. For given M_τ, the couplings we should choose are related by (3.18).

A quantum system with just a single spin (for any $\mathbf{H}$ not proportional to $\mathbb{1}$) clearly has a unique groundstate; this statement means the absence of a phase transition in the 1d Ising chain.

Correlation functions. [Sachdev, 2d ed p. 69] For now, let's construct correlation functions of spins in the classical Ising chain, (3.14), using the transfer matrix. (We'll study correlation functions in the TFIM later, I think.) Let

$$C(l,l') \equiv \langle s_l s_{l'}\rangle = \frac{1}{Z_1}\sum_{\{s_l\}_l} e^{-H_c} s_l s_{l'}$$

By translation invariance, this is only a function of the difference $C(l,l') = C(l-l')$. For simplicity, set the external field $h = 0$. Also, assume that $l' > l$ (as we'll see, this is *time-ordering* of the correlation function). In terms of the transfer matrix, $\mathbf{T} = e^{-\Delta\tau \mathbf{H}}$, it is:

$$C(l-l') = \frac{1}{Z}\mathrm{tr}\left(\mathbf{T}^{M_\tau - l'}\mathbf{Z}\mathbf{T}^{l'-l}\mathbf{Z}\mathbf{T}^{l}\right). \tag{3.19}$$

Notice that there is only *one* operator $\mathbf{Z} = \boldsymbol{\sigma}^z$ here; it is the matrix

$$\mathbf{Z}_{ss'} = \delta_{ss'} s \ .$$

[17] If we include the $\mathbf{Z}$ term, we'll want to take $\Delta\tau$ small enough so that we can write

$$e^{-\Delta\tau\mathbf{H}} = e^{\Delta\tau\frac{\Delta}{2}\mathbf{X}} e^{-\Delta\tau\left(E_0 - \bar{h}\mathbf{Z}\right)} + \mathcal{O}(\Delta\tau^2)$$

All the information about the index l, l' is encoded in the location in the trace.

Let's evaluate this trace in the basis of $\mathbf{T}$ eigenstates. When $h = 0$, we have $\mathbf{T} = e^K \mathbb{1} + e^{-K}\mathbf{X}$, so these are $\mathbf{X}$ eigenstates:

$$\mathbf{T}\,|{\rightarrow}\rangle = \lambda_+\,|{\rightarrow}\rangle\,, \qquad \mathbf{T}\,|{\leftarrow}\rangle = \lambda_-\,|{\rightarrow}\rangle\ .$$

Here $|{\rightarrow}\rangle \equiv \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + |{\downarrow}\rangle\right)$.

In this basis

$$\langle\alpha|\,\mathbf{Z}\,|\beta\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta}, \quad \alpha, \beta = {\rightarrow}\ \text{ or }\ {\leftarrow}\ .$$

So the trace (aka path integral) has two terms: one where the system spends $l' - l$ steps in the state $|{\rightarrow}\rangle$ (and the rest in $|{\leftarrow}\rangle$), and one where it spends $l' - l$ steps in the state $|{\rightarrow}\rangle$. The result (if we take $M_\tau \to \infty$ holding fixed $l' - l$) is

$$C(l' - l) = \frac{\lambda_+^{M_\tau - l' + l}\lambda_-^{l' - l} + \lambda_-^{M_\tau - l' + l}\lambda_+^{l' - l}}{\lambda_+^{M_\tau} + \lambda_-^{M_\tau}} \overset{M_\tau \to \infty}{\to} \tanh^{l' - l} K\ . \tag{3.20}$$

You should think of the insertions as

$$s_l = \mathbf{Z}(\tau),\ \tau = \Delta\tau l.$$

So what we've just computed is

$$C(\tau) = \langle \mathcal{T}\mathbf{Z}(\tau)\mathbf{Z}(0)\rangle = \tanh^l K = e^{-|\tau|/\xi} \tag{3.21}$$

where the *correlation time* ξ satisfies

$$\frac{1}{\xi} = \frac{1}{\Delta\tau} \ln\coth K\ . \tag{3.22}$$

Notice that this is the same as our formula for the gap, Δ, in (3.18).[18] This connection between the correlation length in euclidean time and the energy gap is general and important, and we'll understand it more clearly in (3.25).

---

[18] Seeing this requires the following cool hyperbolic trig fact:

$$\text{If}\ \ e^{-2K} = \tanh X\ \ \text{ then }\ \ e^{-2X} = \tanh K \tag{3.23}$$

(*i.e.* this equation is 'self-dual') which follows from algebra. Here (3.18) says $X = \frac{\Delta}{TM_\tau} = \Delta\tau\Delta$ while (3.22) says $X = \Delta\tau/\xi$. Actually this relation (3.23) can be made manifestly symmetric by writing it as

$$1 = \sinh 2X \sinh 2K\ .$$

(You may notice that this is the same combination that appears in the Kramers-Wannier self-duality condition.) I don't know a slick way to show this, but if you just solve this quadratic equation for e^{-2K} and boil it enough, you'll find $\tanh X$.

For large K, ξ is much bigger than the lattice spacing:

$$\frac{\xi}{\Delta\tau} \overset{K\gg 1}{\simeq} \frac{1}{2}e^{2K} \gg 1.$$

This is the limit we had to take to make the euclidean time continuous.

Notice that if we had taken $l < l'$ instead, we would have found the same answer with $l' - l$ replaced by $l - l'$.

Dictionary. More generally, this set of steps establishes a mapping between classical systems in $d+1$ dimensions and quantum systems in d space dimensions. Some of the dictionary is shown in Table 2.

statistical mechanics in $d+1$ dimensions	quantum system in d space dimensions
transfer matrix	euclidean-time propagator, $e^{-\Delta\tau \mathbf{H}}$
statistical 'temperature'	(lattice-scale) coupling K
free energy in infinite volume	groundstate energy: $e^{-F} = Z = \mathrm{tr} e^{-\beta \mathbf{H}} \overset{\beta\to\infty}{\to} e^{-\beta E_0}$
periodicity of euclidean time L_τ	temperature: $\beta = \frac{1}{T} = \Delta\tau M_\tau$
statistical averages	groundstate expectation values of time-ordered operators

Table 2: Quantum-classical dictionary

Note that this correspondence between classical and quantum systems is not an isomorphism. For one thing, we've seen that *many* classical systems are related to the same quantum system, which does not care about the lattice spacing in time. There is a set of physical quantities which agree between these different classical systems, called *universal*, which is the information in the quantum system.

Continuum scaling limit and universality. [Sachdev, 2d ed §5.5.1, 5.5.2] Now we are going to grapple with the term 'universal'. Let's think about the Ising chain some more. We'll regard $M_\tau \Delta\tau$ as a physical quantity, the proper length of the chain. We'd like to take a continuum limit, where $M_\tau \to \infty$ or $\Delta\tau \to 0$ or maybe both. Such a limit is useful if $\xi \gg \Delta\tau$. This determines how we should scale K, h in the limit. More explicitly, here is the prescription: Hold fixed physical quantities (*i.e.* eliminate

the quantities on the RHS of these expressions in favor of those on the LHS):

$$\begin{aligned} \text{the correlation length,} \quad \xi &\simeq \Delta\tau \frac{1}{2} e^{2K}, \\ \text{the length of the chain,} \quad L_\tau &= \Delta\tau M_\tau, \\ \text{physical separations between operators,} \quad \tau &= (l - l')\Delta\tau, \\ \text{the applied field in the quantum system,} \quad \bar{h} &= h/\Delta\tau. \end{aligned} \tag{3.24}$$

while taking $\Delta\tau \to 0, K \to \infty, M_\tau \to \infty$.

What physics of the various chains will agree? Certainly only quantities that don't depend explicitly on the lattice spacing; such quantities are called *universal*.

Consider the thermal free energy of the single quantum spin (3.13)[19]: The energy spectrum of our spin is $E_\pm = E_0 \pm \sqrt{(\Delta/2)^2 + \bar{h}^2}$, which means

$$F = -T \log Z_Q = E_0 - T \ln \left(2 \cosh \left(\beta \sqrt{(\Delta/2)^2 + \bar{h}^2} \right) \right)$$

(just evaluate the trace in the energy eigenbasis).

Now compare this with the classical Ising chain: In the scaling limit (3.24), (3.17) becomes

$$\lambda_\pm \simeq \sqrt{\frac{2\xi}{\Delta\tau}} \left(1 \pm \frac{\Delta\tau}{2\xi} \sqrt{1 + 4\bar{h}^2 \xi^2} \right)$$

and so in the scaling limit (3.24)

$$F \simeq L_\tau \left(\underbrace{-\frac{K}{\Delta\tau}}_{\text{cutoff-dependent vac. energy}} \quad -\frac{1}{L_\tau} \ln \left(2 \cosh \frac{L_\tau}{2} \sqrt{\xi^{-2} + 4\bar{h}^2} \right) \right) ,$$

which is the same (up to an additive constant) as the quantum formula under the previously-made identifications $T = \frac{1}{L_\tau}$, $\xi^{-1} = \Delta$.

We can also use the quantum system to compute the correlation functions of the classical chain in the scaling limit (3.20). They are *time-ordered* correlation functions:

$$C(\tau_1 - \tau_2) = Z_Q^{-1} \mathrm{tr} e^{-\beta \mathbf{H}} \left(\theta(\tau_1 - \tau_2) \mathbf{Z}(\tau_1) \mathbf{Z}(\tau_2) + \theta(\tau_2 - \tau_1) \mathbf{Z}(\tau_2) \mathbf{Z}(\tau_1) \right)$$

where

$$\mathbf{Z}(\tau) \equiv e^{\mathbf{H}\tau} \mathbf{Z} e^{-\mathbf{H}\tau}.$$

This time-ordering is just the fact that we had to decide whether l' or l was bigger in (3.19).

[19] [Sachdev, 1st ed p. 19, 2d ed p. 73]

For example, consider what happens to this when $T \to 0$. Then (inserting $\mathbb{1} = \sum_n |n\rangle \langle n|$, in an energy eigenbasis $\mathbf{H}|n\rangle = E_n |n\rangle$),

$$C(\tau)|_{T=0} = \sum_n |\langle 0| \mathbf{Z} |n\rangle|^2 e^{-(E_n - E_0)|\tau|}$$

where the $|\tau|$ is taking care of the time-ordering. This is a *spectral representation* of the correlator. For large τ, the contribution of $|n\rangle$ is exponentially suppressed by its energy, so the sum is approximated well by the lowest energy state for which the matrix element is nonzero. Assuming this is the first excited state (which in our two-state system it has no choice!), we have

$$C(\tau)|_{T=0} \overset{\tau\to\infty}{\simeq} e^{-\tau/\xi}, \quad \xi = 1/\Delta, \tag{3.25}$$

where Δ is the energy gap.

In these senses, the quantum theory of a single qbit is *the* universal theory of the Ising chain. For example, if we began with a chain that had in addition next-nearest-neighbor interactions, $\Delta H_c = K' \sum_j s(j)s(j+2)$, we could redo the procedure above. The scaling limit would not be exactly the same; we would have to scale K' somehow (it would also have to grow in the limit). But we would find the same 2-state quantum system, and when expressed in terms of physical variables, the $\Delta\tau$-independent terms in F would be identical, as would the form of the correlation functions, which is

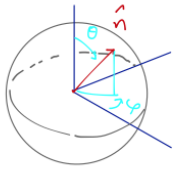
$$C(\tau) = \langle \mathbf{Z}(\tau)\mathbf{Z}(0)\rangle = \frac{e^{-|\tau|/\xi} + e^{-(L_\tau - |\tau|)/\xi}}{1 + e^{-L_\tau/\xi}}.$$

(Note that in this expression we did not assume $|\tau| \ll L_\tau$ as we did before in (3.21), to which this reduces in that limit.)

There is a lot more to say about this relationship between QM in d dimensions and stat mech in $d+1$ dimensions.

3.3.1 Geometric quantization and coherent state path integral for spin systems

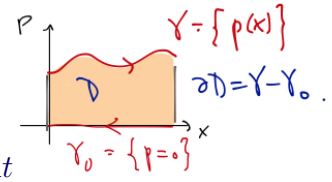
[Zinn-Justin, Appendix A3; XGW §2.3] We're going to spend some time talking about QFT in $D = 0{+}1$, then we'll work our way up to $D = 1{+}1$, and beyond. Consider the nice, round two-sphere. It has an area element that can be written



$$\omega = s\mathrm{d}\cos\theta \wedge \mathrm{d}\varphi \qquad \text{and satisfies} \qquad \int_{S^2} \omega = 4\pi s.$$

s is a number. Suppose we think of this sphere as the *phase space* of some dynamical system. We can use ω as the symplectic form. What is the associated quantum mechanics system?

Let me remind you what I mean by 'the symplectic form'. Recall the phase space formulation of classical dynamics. The action associated to a trajectory is



$$\mathcal{A}[x(t), p(t)] = \int_{t_1}^{t_2} \mathrm{d}t\,(p\dot{x} - H(x,p)) = \int_\gamma p(x)\mathrm{d}x - \int H\mathrm{d}t$$

where γ is the trajectory through the phase space. The first term is the area 'under the graph' in the classical phase space – the area between (p, x) and $(p = 0, x)$. We can rewrite it as

$$\int p(t)\dot{x}(t)\mathrm{d}t = \int_{\partial D} p\mathrm{d}x = \int_D \mathrm{d}p \wedge \mathrm{d}x$$

using Stokes' theorem; here ∂D is the closed curve made by the classical trajectory and some reference trajectory ($p = 0$) and it bounds some region D. Here $\omega = \mathrm{d}p \wedge \mathrm{d}x$ is the symplectic form. More generally, we can consider an $2n$-dimensional phase space with coordinates $u_\alpha, \alpha = 1..2n$ and symplectic form

$$\omega = \omega_{\alpha\beta}\mathrm{d}u^\alpha \wedge \mathrm{d}u^\beta$$

and action

$$\mathcal{A}[u] = \int_D \omega - \int_{\partial D} \mathrm{d}t H(u,t).$$

The symplectic form says who is canonically conjugate to whom (and therefore, quantumly, determines the canonical commutators). It's important that $\mathrm{d}\omega = 0$ so that the equations of motion resulting from $\mathcal{A}$ depend only on the trajectory $\gamma = \partial D$ and not on the choice of D. The equations of motion from varying u are

$$\omega_{\alpha\beta}\dot{u}^\beta = \frac{\partial H}{\partial u^\alpha}.$$

Locally, we can find coordinates p, x so that $\omega = \mathrm{d}(p\mathrm{d}x)$. Globally on the phase space this is not guaranteed – the symplectic form needs to be closed, but need not be exact.

So the example above of the two-sphere is one where the symplectic form is closed (there are no three-forms on the two sphere, so $\mathrm{d}\omega = 0$ automatically), but is not exact. One way to see that it isn't exact is that if we integrate it over the whole two-sphere, we get the area:

$$\int_{S^2} \omega = 4\pi s \ .$$

On the other hand, the integral of an exact form over a closed manifold (meaning a manifold without boundary, like our sphere) is zero:

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha = 0.$$

So there can't be a *globally defined* one-form α such that $\mathrm{d}\alpha = \omega$. Locally, we can find one; for example:

$$\alpha = s\cos\theta \mathrm{d}\varphi \ ,$$

but this is singular at the poles, where φ is not a good coordinate.

So: what I mean by "what is the associated quantum system..." is the following: let's construct a system whose path integral is

$$Z = \int [\mathrm{d}\theta \mathrm{d}\varphi] e^{\frac{\mathrm{i}}{\hbar}\mathcal{A}[\theta,\varphi]} \tag{3.26}$$

with the action above, and where $[\mathrm{d}x]$ denotes the path integral measure:

$$[\mathrm{d}x] \equiv \aleph \prod_{i=1}^{N} \mathrm{d}x(t_i)$$

where $\aleph$ involves lots of awful constants that drop out of ratios. It is important that the measure does not depend on our choice of coordinates on the sphere.

- Hint 1: the model has an action of $\mathsf{O}(3)$, by rotations of the sphere.
- Hint 2: We actually didn't specify the model yet, since we didn't choose the Hamiltonian. For definiteness, let's pick the hamiltonian to be

$$H = -s\vec{h}\cdot\vec{n}$$

where $\vec{n} \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$[20]. WLOG, we can take the polar axis to be along the 'magnetic field': $\vec{h} = \hat{z}h$. The equations of motion are then

$$0 = \frac{\delta\mathcal{A}}{\delta\theta(t)} = -s\sin\theta\,(\dot{\varphi} - h)\,, \quad 0 = \frac{\delta\mathcal{A}}{\delta\varphi(t)} = -\partial_t\,(s\cos\theta)$$

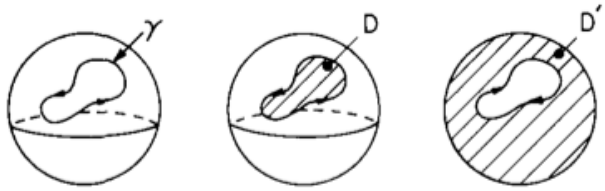
which by rotation invariance can be written better as

$$\partial_t\vec{n} = \vec{h}\times\vec{n}. \tag{3.27}$$

This is a big hint about the answer to the question.

- Hint 3: Semiclassical expectations. Semiclassically, each patch of phase space of area $2\pi\hbar$ contributes one quantum state. Therefore we expect that if our whole phase space has area $4\pi s$, we should get approximately $\frac{4\pi s}{2\pi\hbar} = \frac{2s}{\hbar}$ states, at least at large $s/\hbar$. (Notice that s appears out front of the action.) This will turn out to be very close – the right answer is $2s+1$ (in units with $\hbar = 1$)!

In QM we care that the action produces a well-defined phase – the action must be defined modulo additions of 2π times an integer. We should get the same answer whether we fill in one side D of the trajectory γ or the other D'. The difference between them is



[from Witten]

$$s\left(\int_D - \int_{D'}\right)\text{area} = s\int_{S^2}\text{area}\,.$$

So in this difference s multiplies $\int_{S^2}\text{area} = 4\pi$ (actually, this can be multiplied by an integer which is the number of times the area is covered). Our path integral will be well-defined (*i.e.* independent of our arbitrary choice of 'inside' and 'outside') only if $4\pi s \in 2\pi\mathbb{Z}$, that is if $\boxed{2s \in \mathbb{Z} \text{ is an integer}}$.

The conclusion of this discussion is that the coefficient of the area term must be an integer. We will interpret this integer below.

WZW term. We have a nice geometric interpretation of the 'area' term in our action $\mathcal{A}$ – it's the solid angle swept out by the particle's trajectory. But how do we write it in a manifestly SU(2) invariant way? We'd like to be able to write it, not in terms of the annoying coordinates θ, ϕ, but directly in terms of

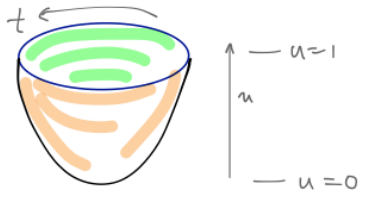
$$n^a \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^a.$$

[20]Here and below I sometimes write $\vec{n}$ and sometimes write $\check{n}$ to emphasize that this is a unit vector. It's always a unit vector.

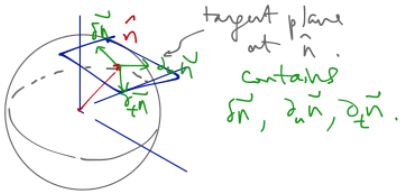
One way to do this is to add an extra dimension (!):

$$\frac{1}{4\pi}\int \mathrm{d}t\cos\theta\partial_t\phi = \frac{1}{8\pi}\int_0^1 \mathrm{d}u\int \mathrm{d}t\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c\epsilon^{abc} \equiv W_0[\vec{n}]$$

where $x^\mu = (t,u)$, and the ϵ tensors are completely antisymmetric in their indices with all nonzero entries 1 and -1.



In order to write this formula we have to extend the $\vec{n}$-field into the extra dimension whose coordinate is u. We do this in such a way that the real spin lives at $u=1$: $\vec{n}(t,u=1)=\vec{n}(t)$, and $\vec{n}(t,u=0)=(0,0,1)$ – it goes to the north pole at the other end of the extra dimension for all t. If we consider periodic boundary conditions in time $n(\beta)=n(0)$, then this means that the space is really a disk with the origin at $u=0$, and the boundary at $u=1$. Call this disk B, its boundary ∂B is the real spacetime ('B' is for 'ball').



This WZW term has the property that its variation with respect to $\vec{n}$ depends only on the values at the boundary (that is: δW_0 is a total derivative). The crucial reason is that allowed variations $\delta\vec{n}$ lie on the 2-sphere, as do derivatives $\partial_\mu\vec{n}$; this means $\epsilon^{abc}\delta n^a\partial_\mu n^b\partial_\nu n^c = 0$, since they all lie in a two-dimensional tangent plane to the 2-sphere at $\vec{n}(t)$. Therefore:

$$\begin{aligned}
\delta W_0 &= \int_0^1 \mathrm{d}u\int \mathrm{d}t\frac{1}{4\pi}\epsilon^{\mu\nu}n^a\partial_\mu\delta n^b\partial_\nu n^c\epsilon^{abc} = \int_B \frac{1}{4\pi}n^a\mathrm{d}\delta n^b\wedge\mathrm{d}n^c\epsilon^{abc}\\
&= \int_0^1 \mathrm{d}u\int \mathrm{d}t\;\partial_\mu\left(\frac{1}{4\pi}\epsilon^{\mu\nu}n^a\delta n^b\partial_\nu n^c\epsilon^{abc}\right) = \int_B \mathrm{d}\left(\frac{1}{4\pi}n^a\delta n^b\mathrm{d}n^c\epsilon^{abc}\right)\\
&\overset{\text{Stokes}}{=} \frac{1}{4\pi}\int \mathrm{d}t\delta\vec{n}\cdot\left(\dot{\vec{n}}\times\vec{n}\right).
\end{aligned} \tag{3.28}$$

(Note that $\epsilon^{abc}n^am^b\ell^c = \vec{n}\cdot\left(\vec{m}\times\vec{\ell}\right)$. The right expressions in red in each line are a rewriting in terms of differential forms; notice how much prettier they are.) So the equations of motion coming from this term do not depend on how we extend it into the auxiliary dimension.

And in fact they are the same as the ones we found earlier:

$$0 = \frac{\delta}{\delta\vec{n}(t)}\left(4\pi sW_0[n] + s\vec{h}\cdot\vec{n} + \lambda\left(\vec{n}^2-1\right)\right) = s\partial_t\vec{n}\times\vec{n} + s\vec{h} + 2\lambda\vec{n}$$

(λ is a Lagrange multiplier to enforce unit length.) The cross product of this equation with $\vec{n}$ is $\partial_t \vec{n} = \vec{h} \times \vec{n}$.

In QM we also care that the action produces a well-defined phase – the action must be defined modulo additions of 2π times an integer. There may be many ways to extend $\hat{n}$ into an extra dimension; another obvious way is shown in the figure above. The demand that the action is the same modulo $2\pi\mathbb{Z}$ gives the same quantization law as above for the coefficient of the WZW term. So the WZW term is topological in the sense that because of topology its coefficient must be quantized.

(This set of ideas generalizes to many other examples, with other fields in other dimensions. WZW stands for Wess-Zumino-Witten.) [End of Lecture 9]

Coherent-state path integral for spin systems. [Wen §2.3.1, Fradkin, Sachdev, *QPT*, chapter 13 and §2.2 of cond-mat/0109419] To understand more about the path integral we've just constructed, we now go in the opposite direction. Start with a spin one-half system, with

$$\mathcal{H}_{\frac{1}{2}} \equiv \mathrm{span}\{|\uparrow\rangle, |\downarrow\rangle\}.$$

Define spin coherent states $|\vec{n}\rangle$ by[21]:

$$\vec{\boldsymbol{\sigma}} \cdot \vec{n} \, |\vec{n}\rangle = |\vec{n}\rangle \quad .$$

These states form another basis for $\mathcal{H}_{\frac{1}{2}}$; they are related to the basis where $\boldsymbol{\sigma}^z$ is diagonal by:

$$|\vec{n}\rangle = z_1 |\uparrow\rangle + z_2 |\downarrow\rangle, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{-\mathbf{i}\varphi/2} \cos\frac{\theta}{2} e^{\mathbf{i}\psi/2} \\ e^{+\mathbf{i}\varphi/2} \sin\frac{\theta}{2} e^{\mathbf{i}\psi/2} \end{pmatrix} \tag{3.29}$$

as you can see by diagonalizing $\vec{n} \cdot \vec{\boldsymbol{\sigma}}$ in the $\boldsymbol{\sigma}^z$ basis. Notice that

$$\vec{n} = z^\dagger \vec{\boldsymbol{\sigma}} z, \quad |z_1|^2 + |z_2|^2 = 1$$

and the phase of z_α does not affect $\vec{n}$ (this is the Hopf fibration $S^3 \to S^2$). In (3.29) I chose a representative of the phase. The space of independent states is a two-sphere:

$$S^2 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1\} / (z_\alpha \simeq e^{\mathbf{i}\chi} z_\alpha).$$

It is just the ordinary Bloch sphere of pure states of a qbit.

[21] For more general spin representation with spin $s > \frac{1}{2}$, and spin operator $\vec{\mathbf{S}}$, we would generalize this equation to

$$\vec{\mathbf{S}} \cdot \vec{n} \, |\vec{n}\rangle = s \, |\vec{n}\rangle .$$

These states are not orthogonal (there are infinitely many of them and the Hilbert space is only 2-dimensional!): the overlap between two of them is

$$\langle \check{n}|\check{n}'\rangle = z^\dagger z' = (z_1^\star, z_2^\star)\begin{pmatrix} z_1' \\ z_2' \end{pmatrix},$$

as you can see using the σ^z-basis representation (3.29). The (over-)completeness relation in this basis is:

$$\int \frac{\mathrm{d}^2\vec{n}}{2\pi}|\vec{n}\rangle\langle\vec{n}| = \mathbb{1}_{2\times 2}. \tag{3.30}$$

As always, we can construct a path integral representation of any amplitude by inserting many copies of $\mathbb{1}$ in between successive time steps. For example, using (3.30) many times, we can construct such a representation for the propagator:

$$\begin{aligned}\mathbf{i}G(\vec{n}_f,\vec{n}_0,t) &\equiv \langle\vec{n}_f|\, e^{-\mathbf{i}\mathbf{H}t}\,|\vec{n}_1\rangle \\ &= \int \prod_{i=1}^{M\equiv\frac{t}{\mathrm{d}t}} \frac{\mathrm{d}^2\vec{n}(t_i)}{2\pi} \lim_{\mathrm{d}t\to 0} \langle\vec{n}(t)|\vec{n}(t_M)\rangle \ldots \langle\vec{n}(t_2)|\vec{n}(t_1)\rangle\,\langle\vec{n}(t_1)|\vec{n}(0)\rangle \end{aligned} \tag{3.31}$$

with $\vec{n}_0 = \vec{n}(0), \vec{n}_f = \vec{n}(t)$. (Notice that $\mathbf{H} = 0$ here, so $\mathbf{U} \equiv e^{-\mathbf{i}\mathbf{H}t}$ is actually the identity.) The crucial ingredient is

$$\langle\vec{n}(t+\epsilon)|\vec{n}(t)\rangle = z^\dagger(\mathrm{d}t)z(0) \overset{0=1-1}{=} 1 - z^\dagger(\mathrm{d}t)\,(z(\mathrm{d}t) - z(0)) \approx e^{-z^\dagger\partial_t z\mathrm{d}t}.$$

$$\mathbf{i}G(\vec{n}_f,\vec{n}_0,t) = \int_{\vec{n}(0)=\vec{n}_0}^{\vec{n}(t)=\vec{n}_f} [D\vec{n}]\, e^{\mathbf{i}S_B[\vec{n}(t)]}, \qquad S_B[\vec{n}(t)] = \int_0^t \mathrm{d}t\mathbf{i}z^\dagger\dot{z}\ . \tag{3.32}$$

Even though the Hamiltonian of the spins was zero – whatever their state, they have no potential energy and no kinetic energy – the action in the path integral is not zero. This phase $e^{\mathbf{i}S_B}$ is a quantum phenomenon (again) called a Berry phase.

Starting from the action S_B and doing the Legendre transform to find the Hamiltonian you will get zero. The first-derivative action says that $z^\dagger$ is the canonical momentum conjugate to z: the space with coordinates $(z, z^\dagger)$ becomes the phase space (just like position and momentum)! But this phase space is curved. In fact it is the two-sphere

$$S^2 = \{(z_1,z_2)||z_1|^2 + |z_2|^2 = 1\}/(z_\alpha \simeq e^{\mathbf{i}\psi}z_\alpha).$$

In terms of the coordinates θ, φ above, we have

$$S_B[z] = S_B[\theta,\varphi] = \int dt \frac{1}{2}\left(\cos\theta\dot{\phi} + \dot{\psi}\right)|_{\psi=0} = 4\pi s W_0[\hat{n}]|_{s=\frac{1}{2}}. \tag{3.33}$$

At the last step we chose a gauge $\psi = 0$. BIG CONCLUSION: This is the 'area' term that we studied above, with $s = \frac{1}{2}$! So the expression in terms of z in (3.32)

gives another way to write the area term which is manifestly SU(2) invariant; this time the price is introducing these auxiliary z variables, with their gauge redundancy $z(t) \to e^{\mathbf{i}\chi(t)} z(t)$.

Making different choices of the phase ψ at different times can shift the constant in front of the second ($\dot{\psi}$) term in (3.33); as we observed earlier, this term is a total derivative. Different choices of ψ change the overall phase of the wavefunction, which doesn't change physics (recall that this is why the space of normalized states of a qbit is a two-sphere and not a three-sphere). Notice that $\mathcal{A}_t = z^\dagger \partial_t z$ is like the time component of a gauge field. Adding a total derivative to the action (by changing $\psi(t)$) imparts a gauge transformation.

The Berry phase $S_B[n]$ is *geometric*, in the sense that it depends on the trajectory of the spin through time, but not on its parametrization, or speed or duration. It is called the Berry phase of the spin history because it is the phase acquired by a spin that follows the instantaneous groundstate (*i.e.* adiabatic evolution) $|\Psi_0(t)\rangle$ of $H(\check{n}(t), t) \equiv -h\check{n}(t) \cdot \mathbf{S}$, with $h > 0$. This is Berry's adiabatic phase, $S_B[\check{n}] = -\lim_{\partial_t h \to 0} \int dt \mathrm{Im} \, \langle \Psi_0(t)| \, \partial_t \, |\Psi_0(t)\rangle$.

Since S_B is geometric, like integrals of differential forms, let's take advantage of this to make it pretty and relate it to familiar objects. Introduce a vector potential (the Berry connection) on the sphere $A^a, a = x, y, z$ so that

$$S_B = \oint d\tau \dot{n}_a A^a = \oint_\gamma A \overset{\text{Stokes}}{=} \int_D F$$

where $\gamma = \partial D$ is the trajectory. ($F = \mathrm{d}A$ is the Berry curvature.) What is the correct form? We must have $(\nabla \times A) \cdot \check{n} = \epsilon^{abc} \partial_{n^a} A^b n^c = 1$ (for spin half). This is a monopole field. Two choices that work are

$$A^{(1)} = -\cos\theta d\varphi, \quad \text{and} \quad A^{(2)} = (1 - \cos\theta) d\varphi.$$

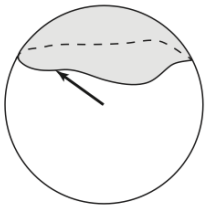
These two expressions differ by the gauge transformation $\mathrm{d}\varphi$, which is locally a total derivative. The first is singular at the N and S poles, $\check{n} = \pm\check{z}$. The second is singular only at the S pole. Considered as part of a 3d field configuration, this codimension two singularity is the 'Dirac string'. The demand of invisibility of the Dirac string quantizes the Berry flux. The gauge transformations that move around the singularities of A are accomplished by adding total derivatives to the action, *i.e.* by choosing $\psi(t)$. For example, by choosing $\psi(t) = \pm\varphi(t)$ we find the gauge that is nonsingular away from the north and south poles, respectively.

If we redo the above coherent-state quantization for a spin-s system we'll get the expression with general s (see below). Notice that this only makes sense when $2s \in \mathbb{Z}$.

We can add a nonzero Hamiltonian for our spin; for example, we can put it in an external Zeeman field $\vec{h}$, which adds $\mathbf{H} = -\vec{h} \cdot \vec{\mathbf{S}}$. This will pass innocently through the construction of the path integral, adding a term to the action $S = S_B + S_h$,

$$S_h = \int \mathrm{d}t \left(s\vec{h} \cdot \vec{n} \right)$$

where s is the spin.



We are back at the system (3.26). We see that the system we get by 'geometric quantization' of the sphere is a quantum spin. The quantized coefficient of the area is $2s$: it determines the dimension of the spin space to be $2s + 1$. Here the quantization of the WZW term is just quantization of angular momentum. (In higher-dimensional field theories, it is something else.)

Deep statement: the purpose in life of the WZW term is to enforce the commutation relation of the SU(2) generators, $[\mathbf{S}^i, \mathbf{S}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{S}^k$. It says that the different components of the spin don't commute, and it says precisely what they don't commute to.

Incidentally, another way to realize this system whose action is proportional to the area of the sphere is to take a particle on the sphere, put a magnetic monopole in the center, and take the limit that the mass of the particle goes to zero. In that context, the quantization of $2s$ is Dirac quantization of magnetic charge. And the degeneracy of $2s + 1$ states is the degeneracy of states in the lowest Landau level for a charged particle in a magnetic field; the $m \to 0$ limit gets rid of the higher Landau levels (which are separated from the lowest by the cylotron frequency, $\frac{eB}{mc}$).

In the crucial step, we assumed the path $z(t)$ was smooth enough in time that we could do calculus, $z(t + \epsilon) - z(t) = \epsilon \dot{z}(t) + \mathcal{O}(\epsilon^2)$. Is this true of the important contributions to the path integral? Sometimes not, and we'll come back to this later.

I've written the path integral for a single spin. The generalization to a many body spin system is simple in principle: just do the above for each site.

Digression on $s > \frac{1}{2}$. [Auerbach, *Interacting Electrons and Quantum Magnetism*] I want to say something about larger-spin representations of $\mathsf{SU}(2)$, partly to verify the claim above that it results in a factor of $2s$ in front of the Berry phase term. Also, large s allows us to approximate the integral by stationary phase.

In general, a useful way to think about the coherent state $|\check{n}\rangle$ is to start with the maximal-spin eigenstate $|s, s\rangle$ of $\mathbf{S}^z$ (the analog of spin up for general s), and *rotate it* by the rotation that takes $\mathbf{S}^z$ to $\mathbf{S} \cdot \check{n}$:

$$|\check{n}\rangle = \mathcal{R}(\chi, \theta, \varphi) \, |s, s\rangle \, .$$

The form of $\mathcal{R}$ involves Euler angles; let's find a better route than remembering about Euler angles.

Schwinger bosons. The following is a helpful device for spin matrix elements. Consider two copies of the harmonic oscillator algebra, with modes a, b satisfing $[a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0$. Then the objects

$$\mathbf{S}^+ = a^\dagger b,\ \mathbf{S}^- = b^\dagger a,\ \mathbf{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$

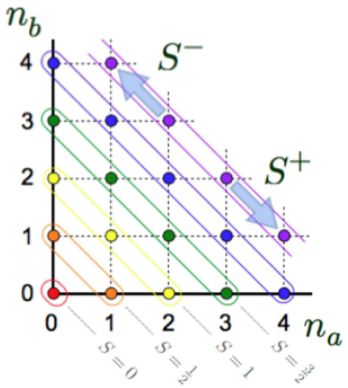
satisfy the $\mathsf{SU}(2)$ algebra. The no-boson state $|0\rangle$ is a singlet of this $\mathsf{SU}(2)$, and the one-boson states $\begin{pmatrix} a^\dagger|0\rangle \\ b^\dagger|0\rangle \end{pmatrix}$ form a spin-half doublet.

More generally, the states

$$\mathcal{H}_s \equiv \text{span}\{|n_a, n_b\rangle \, | a^\dagger a + b^\dagger b \equiv n_a + n_b = 2s\}$$

form a spin-s representation. Algebraic evidence for this is the fact that $\vec{S}^2 P_s = s(s+1)P_s$, where P_s is the projector onto $\mathcal{H}_s$. The spin-s eigenstates of $\mathbf{S}^z$ are

$$|s, m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} |0\rangle \, .$$



[nice figure from Arovas and Auerbach, 0809.4836.]

The fact that $\begin{pmatrix} a^\dagger|0\rangle \\ b^\dagger|0\rangle \end{pmatrix} = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}|0\rangle$ forms a doublet means that $\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ itself must be a doublet. But we know how a doublet transforms under a rotation, and this means we know how to write the coherent state:

$$|\check{n}\rangle = \mathcal{R}\,|s, s\rangle = \mathcal{R}\frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle = \mathcal{R}\frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}}\mathcal{R}^{-1}\mathcal{R}\,|0\rangle = \frac{(a'^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle = \frac{(z_1 a^\dagger + z_2 b^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle \, .$$

Here $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{\mathbf{i}\varphi/2}\cos\frac{\theta}{2}e^{\mathbf{i}\psi/2} \\ e^{-\mathbf{i}\varphi/2}\sin\frac{\theta}{2}e^{\mathbf{i}\psi/2} \end{pmatrix}$ as above[22].

But now we can compute the crucial ingredient in the coherent state path integral, the overlap of successive coherent states:

$$\langle\check{n}|\check{n}'\rangle = \frac{e^{-\mathbf{i}s(\psi-\psi')}}{(2s)!}\underbrace{\langle 0|\,(z_1^\star a + z_2^\star b)^{2s}(z_1' a^\dagger + z_2' b^\dagger)^{2s}\,|0\rangle}_{\overset{\text{Wick}}{=}(2s)!\left([z_1^\star a + z_2^\star b, z_1' a^\dagger + z_2' b^\dagger]\right)^{2s}} = e^{-\mathbf{i}s(\psi-\psi')}(z_1^\star z_1' + z_2^\star z_2')^{2s} = \left(e^{-\mathbf{i}(\psi-\psi')/2} z^\dagger \cdot z'\right)^{2s} .$$

[22] Sometimes you may see the notation $z_1 \equiv u, z_2 \equiv v$.

Here's the point: this is the same as the spin-half answer, raised to the $2s$ power. This means that the Berry phase just gets multiplied by $2s$, $S_B^{(s)}[n] = 2sS_B^{(\frac{1}{2})}[n] = 4\pi s W_0[n]$, as we claimed.

Semi-classical spectrum. Above we found a path integral representation for the Green's function of a spin as a function of time, $G(n_t, n_0; t)$. The information this contains about the spectrum of the hamiltonian can be extracted by Laplace transforming

$$G(n_t, n_0; E) \equiv -\mathbf{i}\int_0^\infty dt G(n_t, n_0; t) e^{\mathbf{i}(E+\mathbf{i}\epsilon)t}$$

and taking the trace

$$\Gamma(E) \equiv \int \frac{d^2 n_0}{2\pi} G(n_0, n_0; E) = \mathrm{Tr}\ \frac{1}{E - \mathbf{H} + \mathbf{i}\epsilon}.$$

This function has poles at the eigenvalues of $\mathbf{H}$. Its imaginary part is the spectral density, $\rho(E) = \frac{1}{\pi}\mathrm{Im}\Gamma(E) = \sum_\alpha \delta(E - E_\alpha)$.

Its path integral representation is then

$$\Gamma(E) = -\mathbf{i}\int dt \oint D\check{n}\ e^{\mathbf{i}((E+\mathbf{i}\epsilon)t + sS[n])}.$$

The $\oint$ indicates periodic boundary conditions, $\check{n}(0) = \check{n}(t)$, and $S[n] = S_B[n] - \int^t dt' H_{\rm cl}[n]/s$. Here $H_{\rm cl}[n] \equiv \langle \check{n} | \mathbf{H} | \check{n} \rangle$.

At large s, field configurations that vary too much in time are cancelled out by the rapidly oscillating phase, that is: we can try to do these integrals by stationary phase. The stationarity condition for the n integral is the equations of motion $0 = \dot{n} \times n - \partial_n H_{\rm cl}$. If $\mathbf{H} = \vec{h}\cdot\mathbf{S}$, this gives the Landau-Lifshitz equation (3.27) for precession. We keep only solutions periodic with $t = nT$ an integer multiple of the period T. The stationarity condition for the t integral is

$$0 = E + \partial_t S[n] = E - H_{\rm cl}[n].$$

In the second equality we used the fact that the Berry phase is geometric, it depends only on the trajectory, not on t (how long it takes to get there). So the semiclassical trajectories are periodic solutions to the EOM with energy $E = H_{\rm cl}[n^E]$. The exponent evaluated on such a trajectory is then just the Berry term. Denoting by n_1^E such trajectories that traverse once ('prime' orbits),

$$\Gamma(E) \sim \sum_{n_1^E} \sum_{m=0}^\infty e^{\mathbf{i}msS_B[n]} = \sum_{n_1^E} \frac{e^{\mathbf{i}sS_B[n]}}{1 - e^{\mathbf{i}sS_B[n]}}.$$

This is an instance of the Gutzwiller trace formula. The locations of poles of this function approximate the eigenvalues of $\mathbf{H}$. They occur at $E = E^m_{sc}$ such that $S_B[\vec{n}^{E_m}] = \frac{2\pi m}{s}$, with $m \in \mathbb{Z}$. The actual eigenvalues are $E^m = E^m_{sc} + \mathcal{O}(1/s)$.

If the path integral in question were a 1d particle in a potential, with $S_B = \int p dx$, and $H_{\text{cl}} = p^2 + V(x)$, the semiclassical condition would reduce to

$$2\pi m = \oint_{x^{E_m}} p(x)dx = \int_{\text{turning points}} \sqrt{E_m - V(x)}$$

the Bohr-Sommerfeld condition.

3.4 Topological terms from integrating out fermions

[Abanov ch 7] Here is a quick application of both fermionic path integrals and spin coherent-state path integrals. Consider a 0+1 dimensional model of *spinful* fermions $\mathbf{c}_\alpha, \alpha = \uparrow, \downarrow$ coupled to a single spin s, $\vec{\mathbf{S}}$. Let's couple them in an $\mathsf{SU}(2)$-invariant way:

$$H_K = M\left(\mathbf{c}^\dagger \vec{\sigma} \mathbf{c}\right) \cdot \vec{\mathbf{S}}$$

by coupling the spin of the fermion $\mathbf{c}^\dagger_\alpha \vec{\sigma}_{\alpha\beta} \mathbf{c}_\beta$ to the spin. 'K' is for 'Kondo'. Notice that M is an energy scale. $M > 0$ is an antiferromagnetic interaction between the spin of the fermion mode and the spin $\vec{\mathbf{S}}$. (Exercise: find the spectrum of H_K.)

Now apply both of the previous coherent state path integrals that we've learned to write the (say euclidean) partition sum as

$$Z = \int [D\psi D\bar{\psi} D\check{n}] e^{-S_0[n] - \int_0^T \mathrm{d}t \bar{\psi}(\partial_t - M\vec{n}\cdot\vec{\sigma})\psi}$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)$ is a two-component Grassmann spinor, and $\vec{\sigma}$ are Pauli matrices acting on its spinor indices. $\check{n}^2 = 1$. Let $S_0[n] = \int K\dot{n}^2 + (2s)2\pi W_0[n]$, where I've added a second-order kinetic term for reasons we'll see below. [End of Lecture 10]

First of all, consider a fixed, slowly-varying configuration of $\check{n}$. What does this do to the propagation of the fermion? I claim that it gaps out the fermion excitations, in the sense that

$$\left\langle \mathcal{T} \mathbf{c}^\dagger_\alpha(t) \mathbf{c}_\beta(0) \right\rangle = \left\langle \bar{\psi}_\alpha(t) \psi_\beta(0) \right\rangle$$

will be short-ranged in time. Let's see this using the path integral.

We can do the (gaussian) integral over the fermion, to get:

$$Z = \int [D\vec{n}] e^{-S_{\text{eff}}[\vec{n}]}$$

with

$$S_{\text{eff}}[\vec{n}] = S_0[\vec{n}] - \log\det\left(\partial_t - M\vec{n}\cdot\vec{\sigma}\right) \equiv S_0 - \log\det D \equiv S_0 + S_1.$$

The variation of the new term in the effective action under a variation of $\vec{n}$ is:

$$\delta S_1 = -\mathrm{tr}\left(\delta D D^{-1}\right) = -\mathrm{tr}\left(\delta D D^\dagger \left(D D^\dagger\right)^{-1}\right)$$

where $D^\dagger \equiv -\partial_t - M\vec{n}\cdot\vec{\sigma}$. This is

$$\delta S_{\text{eff}} = M\mathrm{tr}\left(\delta\vec{n}\cdot\vec{\sigma}\left(\partial_t + M\vec{n}\cdot\vec{\sigma}\right)\left(\underbrace{-\partial_t^2 + M^2 - M\dot{\vec{n}}\cdot\vec{\sigma}}_{=DD^\dagger}\right)^{-1}\right). \tag{3.34}$$

We can expand the denominator in $\dot{\vec{n}}/M$ (and use $n^2 = 1$) to get

$$\delta S_1 = \int \mathrm{d}t \left(-\frac{M}{|M|}\frac{1}{2}\delta\vec{n}\cdot\left(\vec{n}\times\dot{\vec{n}}\right) + \frac{1}{4M}\delta\dot{\vec{n}}\dot{\vec{n}} + ... \right)$$

where ... is higher order in the expansion and we ignore it. But we know this is the variation of

$$S_1 = -2\pi\frac{M}{|M|}W_0 + \int_0^T \mathrm{d}t \left(\frac{1}{8M}\dot{\vec{n}}^2\right) + \mathcal{O}\left(\frac{\dot{n}}{M}\right)^2$$

where W_0 is the WZW term. Integrating out the fermions has shifted the coefficient of the WZW term from $s \to s \mp \frac{1}{2}$ depending on the sign of M. This is satisfying: we are adding angular momenta, $s \otimes \frac{1}{2} = \left(s - \frac{1}{2}\right) \oplus \left(s + \frac{1}{2}\right)$. If $M > 0$, it is an antiferromagnetic interaction whose groundstates will be the ones with smaller eigenvalue of $\vec{S}^2$. If $M < 0$, it is ferromagnetic, and the low-energy manifold grows. This agrees precisely with the coefficient of the WZW term in our effective action, which is $4\pi\left(s - \frac{1}{2}\mathrm{sign}(M)\right)$.

Here is a more direct (?) calculation of the fermion determinant S_1 (also from Abanov).

$$S_1 = -\ln\det D = -\mathrm{Tr}\ \ln D \stackrel{?}{=} -\mathrm{Tr}\ \ln\tilde{D} \tag{3.35}$$

where $\tilde{D} \equiv U^\dagger D U = \partial_t - \mathbf{i}a - M\sigma^3$ where we've defined the unitary transformation U so that

$$\sigma^3 \stackrel{!}{=} U^\dagger\vec{n}\cdot\vec{\sigma}U, \qquad \text{and} \qquad a \equiv U^\dagger\mathbf{i}\partial_t U.$$

In terms of the free propagator $G_0^{-1} \equiv \partial_t - M\sigma^3$, we can write

$$\tilde{D} = G_0^{-1}(1 - G_0\mathbf{i}a).$$

Then we can expand in powers of a

$$S_1 = -\mathrm{Tr}\ \ln\tilde{D} = \mathrm{Tr}\ \left(\ln G_0 + G_0\mathbf{i}a + \frac{1}{2}\left(G_0\mathbf{i}a\right)^2 + \cdots\right) \equiv S_{(0)} + S_{(1)} + \cdots .$$

The first term is some constant which we ignore. The term linear in a is

$$S_{(1)} = \mathrm{tr}G_0\mathbf{i}a = \quad = \mathrm{tr}_\sigma \int dsdtG_0(s-t)a(t)\delta(t-s) \tag{3.36}$$

$$= \mathrm{tr}_\sigma \underbrace{\int \text{đ}\omega \frac{e^{\mathbf{i}\omega dt}}{-\mathbf{i}\omega - M\sigma^3}}_{=\theta(M\sigma^3)} \mathbf{i}a_{\omega=0} = -\mathrm{sign}(M)\mathbf{i}\int dta^3(t). \tag{3.37}$$

Here $a^3 \equiv \frac{1}{2}\mathrm{tr}_\sigma a\sigma^3 = \frac{1}{2}\cos\theta\dot{\varphi}$. In evaluating $G(t=0)$, I used the point-splitting regularization that we found in doing the path integral above. From this we conclude

$$S_{(1)} = -2\pi\mathrm{sign}(M)W_0[n].$$

Similarly, the next term is

$$S_{(2)} = \frac{1}{2}\mathrm{tr}(G_0 \mathbf{i}a)^2 = \quad (3.38)$$

$$= \frac{1}{2}\int \text{đ}\omega_1 \int \text{đ}\omega_2 \mathrm{tr}_\sigma \left(\frac{1}{-\mathbf{i}\omega_1 - M\sigma^3} \mathbf{i}a_{-\omega_2} \frac{1}{-\mathbf{i}(\omega_1+\omega_2) - M\sigma^3} \mathbf{i}a_{\omega_2} \right) \quad (3.39)$$

$$= \frac{1}{8M}\int \text{đ}\omega \mathrm{tr}_\sigma \left(a_{-\omega}a_\omega - \sigma^3 a_{-\omega}\sigma^3 a_\omega \right) \left(1 + \mathcal{O}\left(\frac{1}{M}\right)\right) \quad (3.40)$$

$$= \frac{1}{2M}\int dt \left(a_1^2 + a_2^2\right)\left(1+\mathcal{O}\left(\frac{1}{M}\right)\right) = \frac{1}{8M}\int dt \left(\partial_t \vec{n}\right)^2 \left(1+\mathcal{O}\left(\frac{1}{M}\right)\right). \quad (3.41)$$

To see (3.40), note that unless there is a σ^1 or σ^2 in between the two propagators, their poles are on the same side of the frequency contour, and so we get zero by closing the contour on the opposite side:

$$\int_{-\infty}^{\infty} \text{đ}\omega \frac{1}{-\mathbf{i}\omega_1 - s_1 M} \frac{1}{-\mathbf{i}(\omega_1+\omega) - s_2 M} = \begin{cases} 0, & \text{if } s_1 = s_2 \\ \frac{1}{2M - \mathbf{i}\omega s_1} = \frac{1}{2M}\left(1 + \mathcal{O}\left(\frac{\omega}{M}\right)\right), & \text{if } s_1 = -s_2 \end{cases} \quad (3.42)$$

We could also do the integral by the methods we used for fermion loops in QED, like Feynman parameters.

The second term in S_1 is a shift of K. Higher-order terms are suppressed by more powers of $\frac{\dot{n}}{M}$, so for $\dot{n} \ll M$, this is a local action. That means that the coupling to n must have gapped out the fermions. That the term proportional to M is a funny mass term for the fermions is clear from the expression for $DD^\dagger$ in (3.34): when n is static, $DD^\dagger = -\partial_t^2 + M^2$, so that the fermion propagator is

$$\left\langle \bar{\psi}_\alpha(t)\psi_\beta(0)\right\rangle = \left(\frac{1}{D}\right)_t = \left(\frac{D^\dagger}{DD^\dagger}\right)_t = \int \text{đ}\omega \frac{e^{\mathbf{i}\omega t}\left(\omega + \mathbf{i}M\vec{n}\cdot\sigma_{\alpha\beta}\right)}{\omega^2 + M^2} \sim e^{-Mt}$$

which is short-ranged in time. So indeed the fermions are fast modes in the presence of the coupling to the n-field.

Why did I put a question mark in (3.35)? If we redefine U by $U \to Ue^{\mathbf{i}\sigma^3\psi(t)}$, $a \to e^{-\mathbf{i}\sigma^3\psi}(a - \mathbf{i}\partial_t)e^{\mathbf{i}\sigma^3\psi}$ transforms like a gauge field, and the action S_1 changes by $\int dt\dot{\psi}$, a total derivative.

Such topological terms are one way in which some (topological) information from short distances can persist in the low energy effective action. Being quantized, they can't change under the continuous RG evolution. The WZW term manages to be

independent of M, the mass scale of the fermions. Here the information is that the system is made of fermions (or at least a half-integer spin representation of $\mathsf{SU}(2)$).

The above calculation generalizes well to higher dimensions. The general idea is that integrating out fermions with Yukawa terms involving bosons ϕ produces WZW terms for ϕ. This is how the theory of ϕ remembers that the system is made of fermions. For many examples of its application, see this paper. (The context for this paper will become clearer in §5.5). Next, we'll see an example of this in particle physics.

3.5 Pions

[Schwartz §28.1] Below the scale of electroweak symmetry breaking, we can forget the W and Z bosons. Besides the 4-Fermi interactions, the remaining drama is QCD and electromagnetism:

$$\mathcal{L}_{QCD_2} = -\frac{1}{4}F_{\mu\nu}^2 + \mathbf{i}\sum_{\alpha=L,R}\sum_f \bar{q}_{\alpha f}\not{D}q_{\alpha f} - \bar{q}Mq.$$

Here f is a sum over quark flavors, which includes the electroweak doublets, u and d. Let's focus on just these two lightest flavors, u and d. We can diagonalize the mass matrix by a field redefinition (this is what makes the CKM matrix meaningful): $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$. If it were the case that $m_u = m_d$, we would have *isospin* symmetry

$$\begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix}, \quad U \in \mathsf{SU}(N_f = 2).$$

If, further, there were no masses $m = 0$, then L and R decouple and we would also have chiral symmetry, $q \to e^{\mathbf{i}\gamma_5\alpha}q$, *i.e.*

$$q_L \to Vq_L, q_R \to V^{-1}q_R, \; V \in \mathsf{SU}(N_f = 2).$$

Why do I restrict to $\mathsf{SU}(2)$ and not $\mathsf{U}(2)$? The central bit of the axial symmetry $\mathsf{U}(1)_A$ is anomalous – its divergence is proportional to the *gluon* theta term operator $F \wedge F$, which has all kinds of nonzero matrix elements. It's not a symmetry (see Peskin page 673 for more detail). The missing non-Goldstone boson is called the η'. The central bit of the vectorlike transformation $q \to e^{\mathbf{i}\alpha}q$ is *baryon number*, B. (Actually this is anomalous in the presence of electroweak gauge fields, but $B - L$ is not).

The groundstate of QCD is mysterious, because of infrared slavery. Here's one piece of input from experiment and numerical simulation. Apparently it is the case that in the groundstate

$$\langle \bar{q}_f q_f \rangle = V^3 \tag{3.43}$$

independent of flavor f. This condensate spontaneously breaks

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R \to \mathsf{SU}(2)_{\text{isospin}}, \tag{3.44}$$

the diagonal combination. $\begin{pmatrix} u \\ d \end{pmatrix}$ is a doublet. Since $p = u_\alpha u_\beta d_\gamma \epsilon_{\alpha\beta\gamma}, n = u_\alpha d_\beta d_\gamma \epsilon_{\alpha\beta\gamma}$, this means that $\begin{pmatrix} p \\ n \end{pmatrix}$ is also a doublet. This symmetry is (explicitly) weakly broken by

the difference of the masses $m_d = 4.7\text{MeV} \neq m_u = 2.15\text{MeV}$ and by the electromagnetic interactions, since $q_d = -1/3 \neq q_u = 2/3$.

This symmetry-breaking structure enormously constrains the dynamics of the color singlets which are the low-energy excitations above the QCD vacuum (hadrons). Let us use the EFT strategy. We know that the degrees of freedom must include (pseudo-)Goldstone bosons for the symmetry breaking (3.44) ('pseudo' because of the weak explicit breaking).

Effective field theory. Since QCD is strongly coupled in this regime, let's use the knowing-the-answer trick: the low energy theory must include some fields that represent the breaking of the symmetry (3.44). One way to do this is to introduce a field Σ which transforms like

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R : \Sigma \to g_L \Sigma g_R^\dagger, \quad \Sigma^\dagger \to g_R \Sigma^\dagger g_L^\dagger$$

(this will be called a *linear* sigma model, because Σ transforms linearly) – we have in mind $\bar{q}_\alpha q_\beta \sim \Sigma_{\alpha\beta}$, like the Hubbard-Stratonovich variable. We can make singlets (hence an action) out of $\Sigma_{\alpha\beta}\Sigma^\dagger_{\beta\alpha} = \text{tr}\Sigma\Sigma^\dagger \equiv |\Sigma|^2$:

$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2 \text{tr}\Sigma\Sigma^\dagger - \frac{\lambda}{4}\left(\text{tr}\Sigma\Sigma^\dagger\right)^2 - g\text{tr}\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger + \cdots \tag{3.45}$$

which is designed to have a minimum at $\langle\Sigma\rangle = \frac{V}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, with (when $g \to 0$) $V = 2m/\sqrt{\lambda}$ (here V is from (3.43)), which preserves $\mathsf{SU}(2)_{\text{isospin}}$ (under which $\Sigma \to g\Sigma g^\dagger$). We can parametrize the fluctuations about this configuration as

$$\Sigma(x) = \frac{V + \sigma(x)}{\sqrt{2}} e^{\frac{2\mathbf{i}\pi^a(x)\tau^a}{F_\pi}}$$

where $F_\pi = V = \frac{2m}{\sqrt{\lambda}}$ is be chosen to give $\pi^a(x)$ canonical kinetic terms. The π^a parametrize the directions of field space in which the potential is flat (like the field θ that goes around the minimum of a wine-bottle potential). Under $g_{L/R} = e^{\mathbf{i}\theta^a_{L/R}\tau^a}$, the pion field transforms as

$$\pi^a \to \pi^a + \underbrace{\frac{F_\pi}{2}\left(\theta^a_L - \theta^a_R\right)}_{\text{nonlinear realization of } \mathsf{SU}(2)_{\text{axial}}} - \underbrace{\frac{1}{2} f^{abc}\left(\theta^a_L + \theta^a_R\right)\pi^c}_{\text{linear realiz'n (adj rep) of } \mathsf{SU}(2)_{\text{isospin}}} .$$

The fields $\pi^\pm, \pi^0$ create pions, they transform in the adjoint representation of the diagonal $\mathsf{SU}(2)_{\text{isospin}}$, and they *shift* under the broken symmetry. This shift symmetry forbids mass terms π^2. The radial excitation σ, on the other hand, is a fiction which we've introduced in (3.45), and which has no excuse to stick around at low energies

(and does not). We can put it out of its misery by taking $m \to \infty, \lambda \to \infty$ fixing F_π. In the limit, the useful field to use is

$$U(x) \equiv \frac{\sqrt{2}}{V}\Sigma(x)|_{\sigma=0} = e^{\frac{2\mathbf{i}\pi^a \tau^a}{F_\pi}}$$

which is unitary $UU^\dagger = U^\dagger U = \mathbb{1}$. This last identity means that all terms in an action for U require derivatives, so (again) no mass for π. The most general Lagrangian for U can be written as an expansion in derivatives, and is called the *chiral Lagrangian*:

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4}\mathrm{tr} D_\mu U\, D^\mu U^\dagger + L_1 \mathrm{tr}\left(D_\mu U\, D^\mu U^\dagger\right)^2 + L_2 \mathrm{tr} D_\mu U\, D_\nu U^\dagger \mathrm{tr} D^\nu U^\dagger D^\mu U + L_3 \mathrm{tr} D_\mu U\, D^\mu U^\dagger D_\nu U\, D^\nu U^\dagger + \cdots \tag{3.46}$$

In terms of π, the leading term expands into

$$L_\chi = \frac{1}{2} D_\mu \pi^a D^\mu \pi^a + \frac{1}{F_\pi^2}\left(-\frac{1}{3}\pi^0\pi^0 D_\mu \pi^+ D^\mu \pi^- + \cdots\right) + \frac{1}{F_\pi^4}\left(\frac{1}{18}\left(\pi^-\pi^+\right)^2 D_\mu \pi^0 D^\mu \pi^0 + \cdots\right)$$

This fixes the relative coefficients of many irrelevant interactions, all with two derivatives, suppressed by powers of F_π. The expansions of the L_i terms have four derivatives, and are therefore suppressed by further powers of E/F_π, the promised small parameter of this EFT.

Pion masses and the spurion method. The pions aren't actually massless: $m_{\pi^\pm} \sim 140\mathrm{MeV}$. In terms of quarks, one source for such a thing is the quark mass term $\mathcal{L}_{QCD} \ni \bar{q} M q$. This explicitly breaks the isospin symmetry if the eigenvalues of M aren't equal. But an *invariance* of $\mathcal{L}_{QCD}$ is

$$q_{L/R} \to g_{L/R} q_{L/R},\ M \to g_L M g_R^\dagger. \tag{3.47}$$

Think of M as a background field (such a thing is sometimes called a *spurion*). If M were an actual dynamical field, then (3.47) would be a symmetry. In the effective action that summarizes all the drama of strong-coupling QCD in terms of pions, the field M must still be there, and if we transform it as in (3.47), it should still be an invariance. Maybe we're going to do the path integral over M later. (This 'spurion' trick has applications all over physics.)

So the chiral lagrangian $\mathcal{L}_\chi$ should depend on M and (3.47) should be an invariance. We can play the EFT game again, but now with both π_a and M as our dofs. This determines

$$\Delta\mathcal{L}_\chi = \frac{V^3}{2}\mathrm{tr}\left(MU + M^\dagger U^\dagger\right) + \cdots = V^3(m_u + m_d) - \frac{V^3}{2F_\pi^2}(m_u + m_d)\sum_a \pi_a^2 + \mathcal{O}(\pi^3).$$

The coefficient V^3 is chosen so that the first term matches $\langle \bar{q} M q\rangle = V^3(m_u + m_d)$. The second term then gives

$$m_\pi^2 \simeq \frac{V^3}{F_\pi^2}(m_u + m_d)$$

which is called the Gell-Mann Oakes Renner relation. [End of Lecture 11]

Electroweak interactions. You may have noticed that I used covariant-looking Ds in (3.46). That's because the $\mathsf{SU}(2)_L$ symmetry we've been speaking about is actually gauged by W^a_μ. (The electroweak gauge boson kinetic terms are in the $\cdots$ of (3.46).) Recall that

$$\mathcal{L}_{\text{Weak}} \ni gW^a_\mu \left(\underbrace{J^a_\mu - J^{5a}_\mu}_{\text{'V' - 'A'}} \right) = gW^a_\mu \left(V_{ij}\bar{Q}_i\gamma^\mu \frac{1-\gamma^5}{2}\tau^a Q_j + \bar{L}_i\gamma^\mu\tau^a\frac{1-\gamma^5}{2}L_i \right)$$

where $Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}, L_1 = \begin{pmatrix} e \\ \nu_e \end{pmatrix}$ are doublets of $\mathsf{SU}(2)_L$.

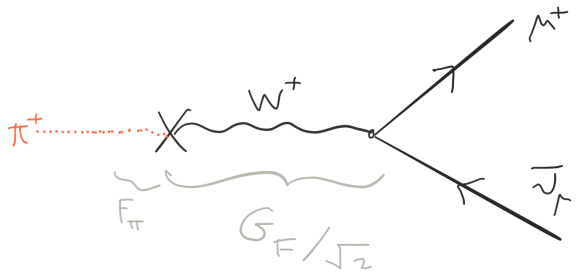
Now, in equations, the statement "a pion is a Goldstone boson for the axial $\mathsf{SU}(2)$" is:

$$\langle 0| J^{5a}_\mu(x) \left|\pi^b(p)\right\rangle = \mathbf{i}p_\mu F_\pi e^{-\mathbf{i}p\cdot x}\delta^{ab}$$

where the state $\left|\pi^b(p)\right\rangle$ is a one-pion state of momentum p. If the vacuum were invariant under the symmetry transformation generated by J_μ, the BHS would vanish. The momentum dependence implements the fact that a global rotation ($p_\mu = 0$) does not change the energy. Contracting the BHS with p^μ and using current conservation (ignoring the explicit breaking just mentioned) would give $0 = p^2F_\pi^2 = m_\pi^2 F_\pi^2$, a massless dispersion for the pions.

Combining the previous two paragraphs, we see that the following process can happen

$$\pi \overset{\text{Goldstone}}{\to} J^5_\mu \overset{\text{electroweak interaction}}{\to} \text{leptons}$$



(3.48)

and in fact is responsible for the dominant decay channel of charged pions. (Time goes from left to right in these diagrams, sorry.)

$$\mathcal{M}(\pi^+ \to \mu^+\nu_\mu) = \frac{G_F}{\sqrt{2}}F_\pi p^\mu \bar{v}_{\nu_\mu}\gamma^\mu(1-\gamma^5)u_\mu$$

where the Fermi constant $G_F \sim 10^{-5}GeV^{-2}$ (known from *e.g.* $\mu^- \to e^-\bar{\nu}_e\nu_\mu$) is a good way to parametrize the Weak interaction amplitude. Squaring this and integrating

over two-body phase space gives the decay rate

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 F_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 .$$

(You can see from the answer why the decay to muons is more important than the decay to electrons, since $m_\mu/m_e \sim 200$. This is called *helicity suppression* – the decay of the helicity-zero π^+ into back-to-back spin-half particles by the weak interaction (which only produces L particles and R antiparticles) can't happen if helicity is conserved – the mass term is required to flip the e_L into an e_R.) This contributes most of $\tau_{\pi^+} = \Gamma^{-1} = 2.6 \cdot 10^{-8} s$.

Knowing further the mass of the muon $m_\mu = 106\text{MeV}$ then determines $F_\pi = 92\text{MeV}$ which fixes the leading terms in the chiral Lagrangian. This is why F_π is called the *pion decay constant*. This gives a huge set of predictions for *e.g.* pion scattering $\pi^0\pi^0 \to \pi^+\pi^-$ cross sections.

Note that the neutral pion can decay by an anomaly into two photons:

$$q_\mu \langle p_1\epsilon_1; p_2\epsilon_2| J_\mu^{5,a=3}(q) |0\rangle = -c\frac{e^2}{4\pi^2}\epsilon^{\nu\lambda\alpha\beta} p_1^\nu \epsilon_1^\lambda p_2^\alpha \epsilon_2^\beta$$

where $\langle p_1\epsilon_1; p_2\epsilon_2|$ is a state with two photons of polarizations $\epsilon_{1,2}$. We know this because it is a matrix element of the $J_e J_e J_{\mathsf{SU}(2)-\text{axial}}$ anomaly,

$$\partial_\mu J^{\mu 5a} = -\frac{e^2}{16\pi^2}\epsilon^{\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta} \text{tr}\left(\tau^a Q^2\right)$$

where $Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$ is the quark charge matrix. Comments: (1) the $\mathsf{U}(1)$ symmetry generated by $J^{\mu 5,a=3}$ acts by $u \to e^{\mathrm{i}\theta\gamma^5} u, d \to e^{-\mathrm{i}\theta\gamma^5}$, and is *not* the same as the anomalous $\mathsf{U}(1)_A$ (which does $q_i \to e^{\mathrm{i}\theta\gamma^5} q_i$ for every flavor), and it's also not the same as isospin $u \to e^{\mathrm{i}\theta} u, d \to e^{-\mathrm{i}\theta}$, which is not chiral, and not spontaneously broken. Confusing! (2) Since the trace involves a sum over colors, the rate of π^0 decay (known since the 1940s) gives a measurement of the number of colors of QCD! (3) This effect can and must be encoded in the Lagrangian for the pions by a term

$$L \ni N_c \frac{e^2}{16\pi^2} \pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{3.49}$$

where $N_c = 3$ is the number of colors. The effective field theory consistently realizes the anomalies of the microscopic theory. This is another example of *'t Hooft anomaly matching*. How did we miss this term in our list of all terms that manifestly respect the symmetries?

SU(3) and baryons. A few more comments before we answer the previous question. The strange quark mass is also pretty small $m_s \sim 95\text{MeV}$, and $\langle \bar{s}s \rangle \sim V^3$. This means the approximate invariance and symmetry breaking pattern is actually $\mathsf{SU}(3)_L \times \mathsf{SU}(3)_R \to \mathsf{SU}(3)_{\text{diag}}$, meaning that there are $16 - 8 = 8$ pseudo NGBs. Besides $\pi^{\pm,0}$, the others are the kaons $K^{\pm,0}$ and η. It's still only the $\mathsf{SU}(2)_L$ that's gauged.

We can also include baryons $B = \epsilon_{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma$. Since $q = (u, d, s) \in 3$ of the flavor $\mathsf{SU}(3)$, the baryons are in the representation

$$\begin{aligned} 3 \otimes 3 \otimes 3 &= (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1 \\ \square \otimes \square \otimes \square &= (\square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}) \otimes \square = \square\square\square \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \end{aligned} \tag{3.50}$$

The proton and neutron are in one of the octets. This point of view brought some order (and some predictions) to the otherwise-bewildering zoo of hadrons.

Returning to the two-flavor $\mathsf{SU}(2)$ approximation, we can include the nucleons $N_{L/R} = \begin{pmatrix} p \\ n \end{pmatrix}_{L/R}$ and couple them to pions by the symmetric coupling

$$\mathcal{L} \ni \lambda_{NN\pi} \bar{N}_L \Sigma N_R.$$

The expectation value for Σ gives a nucleon mass: $m_N = \lambda_{NN\pi} F_\pi$, where $\lambda_{NN\pi}$ can be measured by scattering. This is a cheap version of the *Goldberger-Treiman relation*; for a better one see Peskin pp. 670-672.

WZW terms in the chiral Lagrangian. Finally, I would be remiss not to mention that the chiral Lagrangian must be supplemented by WZW terms to have the correct realization of symmetries (in order to encode all the effects of anomalies, and in order to violate $\pi \to -\pi$ which is not a symmetry of QCD). This is an important additional ingredient in the EFT recipe book: although we wrote all the local Lagrangian terms that were *manifestly* consistent with the symmetries, this actually did *not* account for all the symmetric terms that we can add to the action! The WZW term can only be written in a manifestly-symmetric way at the expense of introducing some extra dimension (or gauge redundancy).

The chiral Lagrangian governs a *non-linear sigma model* (NLσM)– a QFT whose fields are maps from spacetime into some *target space*. In this case the target space is the coset space G/H, where G is the full symmetry group ($\mathsf{SU}(N_f)_L \times \mathsf{SU}(N_f)_R$) and H is the *unbroken* subgroup $\mathsf{SU}(N_f)_{\text{diagonal}}$. We can parametrize this space by $U = e^{\mathbf{i}\pi^a T^a \frac{2}{F_\pi}}$ where the T^a includes only generators of the broken part of the group, so the π^a are coordinates on G/H.

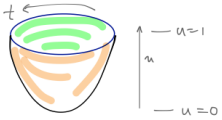
A WZW term is a term which we can sometimes add to a NLσM action; it is defined by the fact that it is symmetric under some group G, but *isn't* the integral of a

symmetric local Lagrangian density in D dimensions. Making it manifestly symmetric requires the introduction of a fictitious extra dimension. This has the dramatic and surprising consequence that its coefficient is quantized.

To get the idea, consider again a model in $D = 0 + 1$ where the field variable $\check{n}$ takes values on the unit sphere S^2, $1 = \sum_{a=1,2,3} \check{n}_a^2$. This is a special case of a coset space with $\mathsf{G}/\mathsf{H} = \mathsf{SU}(2)/\mathsf{U}(1)$.

In order to write the WZW term in a manifestly symmetric way (under the $\mathsf{SO}(3)$ of rotations of the sphere, we have to extend the field into a (possibly fictitious) extra dimension whose coordinate is u.
We do this in such a way that the real system lives at $u = 1$:

$$\check{n}(t, u = 1) \equiv \check{n}(t), \quad \check{n}(t, u = 0) \equiv (0, 0, 1)$$



it goes to the north pole at the other end of the extra dimension for all t. Consider periodic boundary conditions in time $\check{n}(2\pi) = \check{n}(0)$. Then this means that the full space is really a disk with the origin at $u = 0$, and the boundary at $u = 1$. Call this disk B, its boundary $\partial B = \mathcal{M}$ is the real spacetime (here a circle).

We can write the WZW term in terms of the S^2-valued field $\check{n}^{1,2,3}$ as

$$\mathcal{W}_0[\check{n}] = \frac{2\pi}{\Omega_2} \int_{B_2} \check{n}^a \mathrm{d}\check{n}^b \wedge \mathrm{d}\check{n}^c \epsilon_{abc} = \frac{1}{4\pi} \int_{\mathcal{M}} \mathrm{d}t \, (1 - \cos\theta) \, \partial_t \phi.$$

The integrand here is the volume element of the image of a chunk of spacetime in the target S^2. If we integrate over the union of two balls with cancelling boundaries $B_2 \cup \bar{B}_2$, we get an integer multiple of 2π (the integer is the winding number of the map).

The coefficient k of $\mathcal{W}_1$ in the action $\Delta S[\check{n}] = k\mathcal{W}_1[\check{n}]$ must be an integer since B_1 and $\bar{B}_1$ give equally good definitions of $\mathcal{W}_2$, which differ by $2\pi k$. So this ambiguity will not affect the path integral if $k \in \mathbb{Z}$.

A simple generalization of this is a model in $D = d + 1$ dimensions with a field variable $\check{n}$ taking values on S^{d+2}. Then we can write a WZW term as

$$\mathcal{W}_d[\check{n}] = \frac{2\pi}{\Omega_{d+2}} \int_{B_{d+2}} \check{n}^{a_0} d\check{n}^{a_1} \wedge \cdots \wedge dn^{a_{d+2}} \epsilon_{a_0 \cdots a_{d+2}}. \tag{3.51}$$

The integrand is the volume element on the image of the chunk of spacetime. This term is manifestly $\mathsf{O}(d + 3)$-symmetric. Again the EOM depend only on the fields at the boundary, and again the coefficient must be quantized.

The generalization to a group-valued variable U in any dimension is of the form

$$\mathcal{W}_{D-1} = c\int_{B_{D+1}} \mathrm{tr}\underbrace{U^{-1}dU\wedge U^{-1}dU\wedge\cdots\wedge U^{-1}dU}_{D+1\text{ of these}}.$$

Such terms are interesting when $\pi_{D+1}(\mathcal{N})$ is nontrivial, where $\mathcal{N}$ is the space where the fields live (the target space, $\mathcal{N}=\mathsf{G}/\mathsf{H}$ when it arises by symmetry breaking), that is, there are maps from S^{D+1} to $\mathcal{N}$ that cannot be smoothly deformed to the trivial map where every point in the base space goes to the same point in the target. The variation of $\mathcal{W}_{D-1}$ with respect to U is (for even D)[23]:

$$\delta\mathcal{W}_{D-1} = (D+1)c\int_{B_{D+1}}\mathrm{tr}\left\{\left(U^{-1}dU\right)^D\underbrace{\delta\left(U^{-1}dU\right)}_{=U^{-1}d(\delta UU^{-1})U}\right\} \tag{3.53}$$

$$= (D+1)c\int_{B_{D+1}}\mathrm{tr}\left\{\left(dUU^{-1}\right)^D d(\delta UU^{-1})\right\} \tag{3.54}$$

$$= (D+1)c\int_{B_{D+1}} d\mathrm{tr}\left\{\left(U^{-1}dU\right)^D U^{-1}\delta U\right\} \tag{3.55}$$

$$\overset{\text{Stokes}}{=} (D+1)c\int_{\mathcal{M}}\mathrm{tr}\left\{\left(U^{-1}dU\right)^D U^{-1}\delta U\right\}$$

which only depends on the field configuration on the actual spacetime $\mathcal{M}$, not on the extension to B_{D+1}. Again there can be topologically distinct ways to make the extension; demanding that they always give the same answer determines c in terms of

[23] Why do I restrict to even D?

$$\mathrm{tr}\left(U^{-1}dU\right)^{D+1} = \epsilon^{\mu_1\cdots\mu_{D+1}}\mathrm{tr}\left(U^{-1}\partial_{\mu_1}U\cdots U^{-1}\partial_{\mu_{D+1}}U\right)$$

but $\epsilon^{\mu_1\cdots\mu_{D+1}} = -(-1)^{D+1}\epsilon^{\mu_{D+1}\mu_1\cdots\mu_D}$ so $\mathcal{W}_{D-1} = (-1)^D\mathcal{W}_{D-1}$ vanishes for odd D. The step from (3.54) to (3.55) also relies on this fact. Using $1 = U^{-1}U$ and hence $0 = \delta(U^{-1}U) = d(U^{-1}U)$, so that

$$dU^{-1} = -U^{-1}dUU^{-1}, \tag{3.52}$$

the term by which (3.54) and (3.55) differ is

$$\mathrm{tr}\left\{\left(d\left(U^{-1}dU\right)^D\right)\delta UU^{-1}\right\}$$

$$\overset{\text{product rule}}{=}\mathrm{tr}\left\{\left(dU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-1} - \left(U^{-1}dU\wedge dU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-2}+\cdots\right)\delta UU^{-1}\right\}$$

$$\overset{(3.52)}{=} -\mathrm{tr}\left\{\left(U^{-1}dU\wedge U^{-1}dU\wedge\left(U^{-1}dU\right)^{D-1} - U^{-1}dUU^{-1}\wedge dUU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-2}+\cdots\right)\delta UU^{-1}\right\}$$

$$= \mathrm{tr}\left\{\underbrace{(1-1+1-1...)}_{D-1\text{ of these}}\left(U^{-1}dU\right)^{D-1}\delta UU^{-1}\right\} \overset{D-1\text{ even}}{=} 0.$$

See Weinberg, vol 2, §23.4 for more.

volumes of spheres (so that $c\int_{S^{D+1}} \mathrm{tr}(U^{-1}dU)^{D+1} \in \mathbb{Z}$ is the winding number), and the coefficient must be an integer multiple of 2π. (In $D = 4$, we have $c = \frac{\mathbf{i}}{240\pi^2}$.)

This WZW term is *less* topological than the theta term we discussed above, in the sense that it affects the equations of motion for $\check{n}(t)$. The variation of $\mathcal{W}$ *is* local in D dimensions. The following table gives a comparison between theta terms and WZW terms for a field theory in D spacetime dimensions, on a spacetime $\mathcal{M}_D$:

theta term	WZW term
$\mathcal{H} = \int_{\mathcal{M}_D} h$ $h = \mathrm{d}q$ Doesn't affect EOM	$\mathcal{W}_{D-1} = \int_{B_{D+1}} w, \quad \partial B_{D+1} = \mathcal{M}_D$ $\delta w = \mathrm{d}v$ Affects EOM
Invisible in perturbation theory	Appears in perturbation theory, *e.g.* in beta functions
$\mathcal{H} \in \mathbb{Z}$ for $\mathcal{M}_D$ closed	Coefficient of $\mathcal{W} \in 2\pi\mathbb{Z}$ in order for path integral to be well-defined.

[End of Lecture 12]

Pion physics is the context where these terms were first discovered, and where it was realized that their coefficients are quantized. In particular the coefficient of the WZW term $W_3[U]$ here is N_c, the number of colors, as Witten shows by explicitly coupling to electromagnetism, and finding the term (3.49) that encodes $\pi^0 \to \gamma\gamma$. Apparently Witten realized that such a term was required because without it the chiral Lagrangian had an extra symmetry under $\pi \to -\pi$ which is absent in QCD; the WZW term also produces a 5-pion amplitude which violates this symmetry.

One dramatic consequence here is that the chiral Lagrangian (with some higher-derivative terms) has a topological soliton solution (the skyrmion) which is a *fermion* if the number of colors of QCD is odd. The field configuration $U(x,t)$ is constant in time and approaches the vacuum at infinity, so we can regard it as a map

$$U : (\text{space} \cup \infty \sim S^d) \to G/H, \tag{3.56}$$

where G is the full symmetry group and H is the unbroken subgroup, so G/H is the space of Goldstones (in the chiral Lagrangian, $G/H = \mathsf{SU}(3) \times \mathsf{SU}(3)/\mathsf{SU}(3)_{\text{preserved}} \simeq \mathsf{SU}(3)_{\text{broken}}$). The configuration is topological in the sense that as a map from $S^3 \to G/H$, it cannot be smoothly deformed to the trivial map – it represents a nontrivial element of $\pi_3(G/H)$. Its nontriviality is witnessed by a winding number, which

can be written as the integral of a local density. In fact, the baryon number of this configuration comes from the anomalous (WZW) contribution to the baryon number current[24]

$$B_\mu = \frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}U^{-1}\partial_\nu U U^{-1}\partial_\alpha U U^{-1}\partial_\beta U \tag{3.57}$$

whose conserved charge $\int_{\text{space}} B_0$ is exactly the winding number of the map from space (plus the point at infinity) to the space of goldstones. And finally this object a fermion because the WZW term evaluates to π on a spacetime trajectory where the soliton makes a 2π rotation. So this object is a fermionic particle which carries baryon number. It also carries isospin. It's a nucleon! Above we added nucleon fields to the chiral Lagrangian, but we actually didn't need to – they were already there as solitonic excitations. Note that the *size* of the soliton (the region of space over which the fields vary) is determined by the higher-derivative terms in the chiral lagrangian, so we shouldn't take too seriously the substructure of the proton predicted by this picture. But it doesn't do too badly.

As a preview, I should also mention that WZW terms are important in the study of interacting spin systems, for example in our understanding of the dependence on the s of Heisenberg spin-s chains (§4.1), and in phase transitions beyond the Landau-Ginzburg (symmetry-breaking) paradigm (*i.e.* deconfined quantum criticality, §5.5).

Constraints on solitons in scalar field theories. The fact that the chiral Lagrangian has nontrivial, stable, static solitonic particle solutions merits some further comment. The irrelevant terms actually play an important role. Without them, we can show that no such stable solutions exist.

Derrick's argument: Consider a field theory of scalars with 0-derivative and 2-derivative terms. For purposes of finding static solutions, extremizing the action is the same as extremizing the energy:

$$E[\phi] = \int d^dx \left(g(\phi) \left(\vec{\nabla}\phi\right)^2 + V(\phi)\right)^2 \equiv I_1 + I_2.$$

There could be multiple scalars, so for example, the argument applies to the leading term in the chiral lagrangian $L = \mathrm{tr}\,(U^{-1}\partial U)^2$. We'll assume $I_1 > 0$, since otherwise there is an obvious gradient instability of the theory.

Suppose we have a solution $\underline{\phi}$ that extremizes E. To describe a particle excitation of the vacuum, it must approach the vacuum value far away, $\underline{\phi}(x) \overset{x\to\infty}{\to} \phi_0$.

[24] Witten gives two arguments for this. One is by including the couplings to the $\mathsf{SU}(2)_L$ electroweak gauge bosons, he shows that this term is related by a gauge transformation to terms responsible for the $\mathsf{U}(1)_B\mathsf{SU}(2)^2_{EW}$ anomaly. The second is an appeal to a generalization of the calculation of Goldstone and Wilczek described on the first homework.

Now consider a dilated configuration $\phi_\lambda(x) \equiv \underline{\phi}(\lambda x)$. Plugging in and changing integration variables gives

$$E[\phi_\lambda] = \frac{I_1}{\lambda^{d-2}} + \frac{I_2}{\lambda^d}.$$

Demanding that $\underline{\phi}$ is a stationary point implies

$$0 = \partial_\lambda E[\phi_\lambda]|_{\lambda=1} = (2-d)I_1 - dI_2 \quad \Longrightarrow \quad I_2 = \frac{2-d}{d} I_1$$

and then

$$\partial_\lambda^2 E[\phi_\lambda]|_{\lambda=1} = (2-d)(1-d)I_1 + d(d+1)I_2 = -2(d-2)I_1 < 0.$$

So the solution is unstable to dilations for $d > 2$.

If we add a term with more derivatives, like $I_3 = \frac{1}{M^4}\int(\vec{\nabla}\phi)^6$, it will contribute positively to $\partial_\lambda^2 E[\phi_\lambda]|_{\lambda=1}$ and the argument is no longer valid. The length scale $1/M$ in front of this higher-derivative term then determines the size of the soliton.

4 Field theory of spin systems

Where do spin systems come from? Just as the classical ising model arises in many ways (lattice gas, double well potential....), quantum spin systems arise in many ways. for example:

1. Spinful fermions at half-filling, with charge dofs quenched.
2. Similarly, spinless fermions hopping around on a lattice also realize a spin system: there are two states per site. How to relate fermion operators and spin operators? In 1+1 dimensions, we can answer this in complete explicitness, below.
3. If we view down-spin as vacuum and up-spin as the presence of a boson, we can view a hard-core boson system as a spin-1/2 system. Here it is natural to preserve a $\mathsf{U}(1) \subset \mathsf{SU}(2)$ symmetry which rotates the phase of $\mathbf{X} + \mathbf{iY}$; the conserved quantity is $\mathbf{Z} = (2\mathbf{n}_B - 1)$ where $\mathbf{n}_B$ is the boson number.

So the Hamiltonian for a spin system can preserve some or all of the $\mathsf{SU}(2)$ spin rotation symmetry. First we'll study situations that preserve the whole $\mathsf{SU}(2)$.

4.1 Ferromagnets and antiferromagnets

[Zee §6.5] Now we'll try a spin system in $D \geq 1+1$. Consider a lattice of spins, each of spin $s \in \mathbb{Z}/2$, interacting via the Heisenberg hamiltonian:

$$\mathbf{H} = \sum_{\langle jj' \rangle} J \vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j'}.$$

This hamiltonian is invariant under global spin rotations, $\mathbf{S}_j^a \to \mathcal{R}\mathbf{S}_j^a\mathcal{R}^{-1} = R_b^a \mathbf{S}_j^b$ for all j. For $J < 0$, this interaction is ferromagnetic, so it favors a state like $\left\langle \vec{\mathbf{S}}_j \right\rangle = s\hat{z}$. For $J > 0$, the neighboring spins want to anti-align; this is an antiferromagnet, which favors a state like $\left\langle \vec{\mathbf{S}}_j \right\rangle = (-1)^j s\hat{z}$. Whether the spins actually order is in general a difficult question: the consequence of short-range interactions of some particular sign for the groundstate is not so obvious. For example, antiferromagnetic interactions may be *frustrated*: If I want to disagree with both Kenenisa and Lasse, and Kenenisa and Lasse want to disagree with each other, then some of us will have to agree, or maybe someone has to withhold their opinion, $\langle S \rangle = 0$. So you see frustration and symmetry breaking are in conflict.

In the special case of $D = 1+1$ dimensions with short-range interactions, spontaneous breaking of a continuous symmetry does not happen. Really there is only short-

range order because of the Coleman-Mermin-Wagner theorem. But that is enough for the calculation we want to do.

We can write down the action that we get by coherent-state quantization – it's just many copies of the above, where each spin plays the role of the external magnetic field for its neighbors:

$$L = \mathbf{i}s \sum_j z_j^\dagger \partial_t z_j - Js^2 \sum_{\langle jj' \rangle} \vec{n}_j \cdot \vec{n}_{j'}.$$

Spin waves in ferromagnets. Let's use this to find the equation of motion for small fluctuations $\delta\vec{n}_i \equiv \vec{n}_i - \hat{z}$ about the ferromagnetic state. Once we recognize the existence of the Berry phase term, this is the easy case. In fact the discussion is not restricted to $D = 1 + 1$.

Assume the system is translation invariant, so we should Fourier transform. The condition that $\vec{n}_j^2 = 1$ means that $\delta n_z(k) = 0$.[25] The equations of motion are (using (3.28))

$$0 = \frac{\delta S}{\delta \vec{n}_j(t)} = s\vec{n}_j \times \partial_t \vec{n}_j - s^2 \sum_{\langle j|l \rangle} \vec{n}_l$$

where $\sum_{\langle j|l \rangle}$ is an instruction to sum over the neighbors l of the fixed site j. Taking $s^{-1}\vec{n}_j \times$ (BHS) gives

$$0 = -\partial_t \vec{n}_j + \vec{n}_j \times \sum_{\langle j|l \rangle} sJ\vec{n}_l.$$

Linearizing in $\delta\vec{n}$ and fourier transforming, we find

$$0 = \begin{pmatrix} h(k) & -\frac{\mathbf{i}}{2}\omega \\ \frac{\mathbf{i}}{2}\omega & h(k) \end{pmatrix} \begin{pmatrix} \delta n_x(k) \\ \delta n_y(k) \end{pmatrix} \tag{4.1}$$

with $h(k)$ determined by the exchange (J) term. It is (proportional to) the lattice laplacian in k-space. For example for the square lattice, it is $h(k) = 4s|J|\left(2 - \cos k_x a - \cos k_y a\right) \overset{k\to 0}{\simeq} 2s|J|a^2k^2$, with a the lattice spacing. For small k, $h(k) \sim \vec{k}^2$, so the solutions of (4.1) have $\omega \sim k^2$, a $z = 2$ dispersion (meaning that there is scale invariance near $\omega = k = 0$, but space and time scale differently: $k \to \lambda k, \omega \to \lambda^2\omega$. The two spin polarizations have their relative phases locked $\delta n_x(k) \propto \mathbf{i}\delta n_y(k)$ with a $\pi/2$ phase shift, and so these modes describe precession of the spin about the ordering vector. These low-lying spin excitations are visible in neutron scattering and they dominate the low-temperature

[25] $1 = n_j^2 \ \forall j \implies n_j \cdot \delta n_j = 0, \forall j$ which means that for any k,

$$0 = \sum_j e^{\mathbf{i}kja} n_j \cdot \delta n_j = \sum_j e^{\mathbf{i}kja}(\hat{z} + \delta n_j) \cdot \delta n_j = \delta n_k^z + \mathcal{O}(\delta n^2).$$

thermodynamics. Their thermal excitations produce a version of the blackbody spectrum with $z = 2$. We can determine the generalization of the Stefan-Boltzmann law by dimensional analysis: the free energy (or the energy itself) is extensive, so $F \propto L^d$, but it must have dimensions of energy, and the only other scale available is the temperature. With $z \neq 1$, temperature scales like $[T] = [L^{-z}]$. Therefore $F = cL^d T^{\frac{d+z}{z}}$. (For $z = 1$ this is the ordinary Stefan-Boltzmann law).

Notice that a ferromagnet is a bit special because the order parameter $Q^z = \sum_i \mathbf{S}_i^z$ is actually *conserved*, $[Q^z, \mathbf{H}] = 0$. This is the origin of the funny $z = 2$ dispersion of the goldstones, and the fact that although the groundstate breaks *two* generators Q^x and Q^y, there is only one gapless mode. If you are impatient to understand this connection, take a look at this paper. [End of Lecture 13]

Antiferromagnets. [Fradkin, 2d ed, p. 203] Now, let's study instead the equation of motion for small fluctuations about the antiferromagnetic (Neel) state, $S(\vec{x}) \sim (-1)^{\sum_i x_i} m$. The conclusion will be that there is a linear dispersion relation. This would be the conclusion if we simply erased the WZW/Berry phase term and replaced it with an ordinary kinetic term

$$\frac{1}{2g^2} \sum_j \partial_t \vec{n}_j \cdot \partial_t \vec{n}_j \ .$$

How this comes about is actually a bit more involved! An important role will be played[26] by the *ferro*magnetic fluctuation $\vec{\ell}_j$ in

$$\vec{n}_j = (-1)^j \vec{m}_j + a\vec{\ell}_j \ .$$

$\vec{m}_j$ is the AF order parameter; a is the lattice spacing; $s \in \mathbb{Z}/2$ is the spin. The constraint $\vec{n}^2 = 1$ tells us that $\vec{m}^2 = 1$ and $\vec{m} \cdot \vec{\ell} = 0$.

Why do we have to include both variables? Because $\vec{m}$ are the AF order-parameter fluctuations, but the total spin is conserved, and therefore its local fluctuations $\vec{\ell}$ still constitute a slow mode. This is an illustration of a general point: amongst the low-energy modes in our effective field theory, we should make sure we keep track of the conserved quantities, which can often move around but can never disappear. The name for this principle is *hydrodynamics*.

The exchange (J) term in the action is

$$S_J[\vec{n}_j = (-1)^j \vec{m}_j + a\vec{\ell}_j] = -aJs^2 \int \mathrm{d}x\mathrm{d}t \left(\frac{1}{2} (\partial_x \vec{m})^2 + 2\ell^2 \right) \ .$$

[26] A pointer to the past: this story is very similar to the origin of the second order kinetic term for the Goldstone mode in a superfluid arises, which we discussed as an application of coherent state quantization of bosons in §3.1. The role of $\vec{\ell}$ here is played there by ρ, the density.

Here we used $\vec{n}_i \cdot \vec{n}_j = \frac{1}{2}(n_i + n_j)^2 - 1$ and

$$\vec{n}_{2r} + \vec{n}_{2r-1} \approx a\left(\partial_x \vec{m}_{2r} + 2\ell_{2r}\right) + \mathcal{O}(a^2). \tag{4.2}$$

Now let's focus on $D = 1+1$. The WZW terms evaluate to[27]

$$S_W = 4\pi s \sum_{j=1}^{N} W_0[(-1)^j m_j + \ell_j] \overset{N\to\infty, a\to 0, Na \text{ fixed}}{\simeq} \int \mathrm{d}x\mathrm{d}t \left(\frac{s}{2}\vec{m}\cdot(\partial_t\vec{m}\times\partial_x\vec{m}) + s\vec{\ell}\cdot(\vec{m}\times\partial_t\vec{m})\right). \tag{4.3}$$

Altogether, we find that ℓ is an auxiliary field with no time derivative:

$$L[m,\ell] = -2aJs^2\vec{\ell}^2 + s\vec{\ell}\cdot(\vec{m}\times\partial_t\vec{m}) + L[m]$$

so we can integrate out ℓ (this is the step analogous to what we did for ρ in the EFT of SF in §2.3.3) to find

$$S[\vec{m}] = \int \mathrm{d}x\mathrm{d}t \left(\frac{1}{2g^2}\left(\frac{1}{v_s}\left(\partial_t\vec{m}\right)^2 - v_s\left(\partial_x\vec{m}\right)^2\right) + \frac{\theta}{8\pi}\epsilon_{\mu\nu}\vec{m}\cdot(\partial_\mu\vec{m}\times\partial_\nu\vec{m})\right), \tag{4.4}$$

with $g^2 = \frac{2}{s}$ and $v_s = 2aJs$, and $\theta = 2\pi s$. The equation of motion for small fluctuations of $\vec{m}$ therefore gives linear dispersion with velocity v_s. Notice that there are two independent gapless modes (recall that $\vec{m}^2 = 1$). These fluctuations have wavenumber k close to π, since they are fluctuations of the AF order ($k = \pi$ means changing sign between each site), that is, $\omega \sim |k - \pi|$. (For a more microscopic treatment, see the book by Auerbach.)

So the conclusion is that the EFT for an antiferromagnet in $D = 1+1$ is a NLSM with target space S^2, and with a theta term. The last ('theta') term in (4.4) is a total derivative. This means it doesn't affect the EOM, and it doesn't affect the Feynman rules. It is even more topological than the WZW term – its value only depends on the topology of the field configuration, and not on local variations. It is like the $\theta F \wedge F$ term in 4d gauge theory, or the $\dot{\phi}$ term for a particle on a ring. You might think then

[27] The essential ingredient is

$$\delta W_0[n] = \frac{1}{4\pi}\int \mathrm{d}t\,\delta\vec{n}\cdot(\vec{n}\times\partial_t\vec{n}) \ .$$

So

$$4\pi W_0\left[\underbrace{n_{2r}}_{=-n_{2r-1}+\Delta n}\right] + 4\pi W_0[n_{2r-1}] = 4\pi\left(W_0[-n_{2r-1}] + \int dt \frac{\delta W_0}{\delta n^i_{2r-1}(t)} \underbrace{\Delta n}_{\overset{(4.2)}{=}\partial_x \hat{n}^i_{2r-1} a} + W_0[n_{2r-1}]\right) = \int dt a \hat{n}\times\partial_t\hat{n}\cdot\partial_x\hat{n}.$$

The factor of $\frac{1}{2}$ in (4.3) comes from $a\sum_r = \frac{1}{2}\int dx$.

that it doesn't matter. Although it doesn't affect small fluctuations of the fields, it does affect the path integral. Where have we seen this functional before? The integrand is the same as in our 2d representation of the WZW term in 0+1 dimensions: the object multiplying theta counts the winding number of the field configuration $\vec{m}$, the number of times Q the map $\vec{m} : \mathbb{R}^2 \to S^2$ covers its image (we can assume that the map $\vec{m}(|x| \to \infty)$ approaches a constant, say the north pole). We can break up the path integral into sectors, labelled by this number $Q \equiv \frac{1}{8\pi} \int \mathrm{d}x\mathrm{d}t \; \epsilon_{\mu\nu} \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m})$:

$$Z = \int [D\vec{m}] e^{\mathbf{i}S} = \sum_{Q \in \mathbb{Z}} \int [D\vec{m}]_Q e^{\mathbf{i}S_{\theta=0}} e^{\mathbf{i}\theta Q} \ .$$

θ determines the relative phase of different topological sectors (for $\theta = \pi$, this a minus sign for odd Q).

Actually, the theta term makes a huge difference. (Perhaps it is not so surprising if you think about the quantum mechanics of a particle constrained to move on a ring with magnetic flux through it?) The model with $2s$ even flows to a trivial theory in the IR, while the model with $2s$ odd flows to a nontrivial fixed point, called the $\mathrm{SU}(2)_1$ WZW model. It can be described in terms of one free relativistic boson. If you are impatient to understand more about this, the 2nd edition of the book by Fradkin continues this discussion. Perhaps I can be persuaded to say more.

Nonlinear sigma models in perturbation theory. Let us discuss what happens in perturbation theory in small g. A momentum-shell calculation integrating out fast modes (see the next subsection, §4.2) shows that

$$\frac{dg^2}{d\ell} = (D-2)g^2 + (n-2)K_D g^4 + \mathcal{O}(g^5) \tag{4.5}$$

where ℓ is the logarithmic RG time, and $\ell \to \infty$ is the IR. n is the number of components of $\hat{n}$, here $n = 3$, and $K_D = \frac{\Omega_{D-1}}{(2\pi)^D}$ as usual. Cultural remark: the second term is proportional to the *curvature* of the target space, here S^{n-1}, which has positive curvature for $n > 1$. For $n = 2$, we get S^1 which is one-dimensional and hence flat and there is no perturbative beta function. In fact, for $n = 2$, it's a free massless scalar. (But there is more to say about this innocent-looking scalar!)

The fact that the RHS of (4.5) is positive at small g in $D = 2$ says that this model is *asymptotically free* – the coupling is weak in the UV (though this isn't so important if we are starting from a lattice model) and becomes strong in the IR. This is opposite what happens in QED; the screening of the charge in QED makes sense in terms of polarization of the vacuum by virtual charges. Why does this antiscreening happen here? There's a nice answer: the effect of the short-wavelength fluctuations is to make the spin-ordering vector $\vec{n}$ effectively *smaller*. It is like what happens when you do the

block spin procedure, by just averaging the spins. But rescaling the variable $\vec{n} \to a\vec{n}$ with $a \lesssim 1$ is the same as rescaling the coupling $g \to g/a$ – the coupling gets bigger. (Beware Peskin's comments about the connection between this result and the Coleman-Mermin-Wagner theorem: it's true that the logs in 2d enhance this effect, but in fact the model can reach a fixed point at finite coupling; in fact, this is what happens when $\theta = \pi$.)

Beyond perturbation theory. Like in QCD, this *infrared slavery* (the dark side of asymptotic freedom) means that we don't really know what happens in the IR from this calculation. Dimensional transmutation says that the coupling is of order one at an energy scale of order

$$\Lambda_H = \Lambda_0 e^{-\frac{c}{g_0^2}},$$

for some constant c, analogous to the QCD scale (H is for 'Haldane', who did the relevant work at UCSD). From other viewpoints (Bethe ansatz solutions, numerical methods, and some analytic methods to be described below), we know that (for integer s) there is an energy gap above the groundstate of order Λ_H. Here g_0 is the value of g at the scale Λ_0; so Λ_H is roughly the energy scale where g becomes large.

For $s \in \mathbb{Z}$, for studying bulk properties like the energy gap, we can ignore the theta term since it only appears as $e^{2\pi \mathbf{i} n}$, with $n \in \mathbb{Z}$ in the path integral. [28] For half-integer s, there is destructive interference between the topological sectors. Various results (such as the paper by Read and Shankar, Nuclear Physics B336 (1990) 457-474, which contains an amazingly apt Woody Allen joke) show that this destroys the gap. This last sentence was a bit unsatisfying; more satisfying would be to understand the origin of the gap in the $\theta = 2\pi n$ case, and show that this interference removes that mechanism. This strategy is taken in this paper by Affleck.

In the section on large-N, we'll get some intuition for these claims.

4.2 The beta function for 2d non-linear sigma models

[Polyakov §3.2; Peskin §13.3; Auerbach chapter 13] I can't resist explaining the result (4.5). Consider this action for a $D = 2$ non-linear sigma model with target space S^{n+1}, of radius R:

$$S = \int \mathrm{d}^2 x R^2 \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} \equiv \int \mathrm{d}^2 x R^2 \mathrm{d}n^2.$$

[28] $\theta = 2\pi n$ does, however, affect other properties, such as the groundstate wavefunction and the behavior in the presence of a boundary. $\theta = 2\pi$ is actually a different phase of matter than $\theta = 0$. It is an example of a SPT (symmetry-protected topological) phase, the first one discovered. See the homework for more on this.

Notice that R is a coupling constant (it's what I called $1/g$ earlier). In the second step I just made some compact notation.

Since not all of the components of $\hat{n}$ are independent (recall that $\hat{n}\cdot\hat{n}=1$!), the expansion into slow and fast modes here is a little trickier than in our previous examples. Following Polyakov, let

$$n^i(x) \equiv n^i_<(x)\sqrt{1-\phi^2_>} + \sum_{a=1}^{n-1}\phi^>_a(x)e^i_a(x). \tag{4.6}$$

Here the slow modes are represented by the unit vector $n^i_<(x)$, $\hat{n}_<\cdot\hat{n}_<=1$; the variables e^i_a are a basis of unit vectors spanning the $n-1$ directions perpendicular to $\vec{n}_<(x)$

$$n_<\cdot\hat{e}_a=0, \hat{e}_a\cdot\hat{e}_a=1; \tag{4.7}$$

they are not independent dynamical variables and how we choose them does not matter.

The fast modes are encoded in $\phi^>_a(x)\equiv\int_{\Lambda/s}^{\Lambda}đk e^{\mathbf{i}kx}\phi_k$, which only has fourier modes in a shell of momenta, and $\phi^2_>\equiv\sum_{a=1}^{n-1}\phi^>_a\phi^>_a$. Notice that differentiating the relations in (4.7) gives

$$\hat{n}_<\cdot\mathrm{d}\hat{n}_<=0, \quad \hat{n}_<\cdot\mathrm{d}\hat{e}_a+\mathrm{d}\hat{n}_<\cdot\hat{e}_a=0. \tag{4.8}$$

Below when I write ϕs, the $>$ symbol is implicit.

We need to plug the expansion (4.6) into the action, whose basic ingredient is

$$\mathrm{d}n^i=\mathrm{d}n^i_<\left(1-\phi^2\right)^{\frac{1}{2}}-n^i_<\frac{\phi\cdot\mathrm{d}\phi}{\sqrt{1-\phi^2}}+\mathrm{d}\phi\cdot e^i+\phi\cdot\mathrm{d}e^i.$$

So $S=\int d^2x\ \mathcal{L}$ with

$$\begin{aligned}\mathcal{L}&=\frac{1}{2g^2}\left(\mathrm{d}\vec{n}\right)^2\\&=\frac{1}{2g^2}\Bigg(\left(\mathrm{d}n_<\right)^2\left(1-\phi^2\right)+\underbrace{\mathrm{d}\phi^2}_{\text{kinetic term for }\phi}+2\phi_a\mathrm{d}\phi_b\vec{e}_a\cdot\mathrm{d}\vec{e}_b\\&\qquad+\underbrace{\mathrm{d}\phi_a\mathrm{d}\vec{n}_<\cdot\vec{e}_a}_{\text{source for }\phi}+\phi_a\phi_b\mathrm{d}\vec{e}_a\cdot\mathrm{d}\vec{e}_b+\mathcal{O}(\phi^3)\Bigg)\end{aligned} \tag{4.9}$$

So let's do the integral over ϕ, by treating the $\mathrm{d}\phi^2$ term as the kinetic term in a gaussian integral, and the rest as perturbations:

$$e^{-S_{\text{eff}}[n_<]}=\int[D\phi_>]^{\Lambda}_{\Lambda/s}e^{-\int L}=\int[D\phi_>]^{\Lambda}_{\Lambda/s}e^{-\frac{1}{2g^2}\int(\mathrm{d}\phi)^2}\left(\text{all the rest}\right)\equiv\langle\text{all the rest}\rangle_{>,0}\,Z_{>,0}\ .$$

The $\langle ... \rangle_{>,0}$s that follow are with respect to this measure.

$$\implies L_{\text{eff}}[n_<] = \frac{1}{2g^2} \left(\mathrm{d}n_<\right)^2 \left(1 - \left\langle \phi^2 \right\rangle_{>,0}\right) + \langle \phi_a \phi_b \rangle_{>,0} \, \mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b + \text{terms with more derivatives}$$

$$\langle \phi_a \phi_b \rangle_{>,0} = \delta_{ab} g^2 \int_{\Lambda/s}^{\Lambda} \frac{\mathrm{d}^2 k}{k^2} = g^2 K_2 \log(s) \delta_{ab}, \qquad K_2 = \frac{1}{2\pi}.$$

What to do with this $\mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_b$ nonsense? Remember, $\vec{e}_a$ are just some arbitrary basis of the space perpendicular to $\hat{n}_<$; its variation can be expanded in our ON basis at x, $(n_<, e_c)$ as

$$\mathrm{d}\vec{e}_a = \underbrace{\left(\mathrm{d}e_a \cdot \hat{n}_<\right)}_{\overset{(4.8)}{=} -\mathrm{d}\hat{n}_< \cdot \vec{e}_a} \hat{n}_< + \sum_{c=1}^{n-1} \left(\mathrm{d}\vec{e}_a \cdot \vec{e}_c\right) \vec{e}_c$$

Therefore

$$\mathrm{d}\vec{e}_a \cdot \mathrm{d}\vec{e}_a = + \left(\mathrm{d}n_<\right)^2 + \sum_{c,a} \left(\vec{e}_c \cdot \mathrm{d}\vec{e}_a\right)^2$$

where the second term is a higher-derivative operator that we can ignore for our present purposes. Therefore

$$\begin{aligned} L_{\text{eff}}[n] &= \frac{1}{2g^2} \left(\mathrm{d}\hat{n}_<\right)^2 \left(1 - \left((N-1) - 1\right) g^2 K_2 \log s\right) + ... \\ &\simeq \frac{1}{2} \left(g^2 + \frac{g^4}{2\pi} (N-2) \log s + ...\right)^{-1} \left(\mathrm{d}\hat{n}_<\right)^2 + ... \end{aligned} \tag{4.10}$$

Differentiating this running coupling with respect to s gives the one-loop term in the beta function quoted above. The tree-level (order g^2) term comes from engineering dimensions.

4.3 $\mathbb{CP}^1$ representation and large-N

[Auerbach, *Interacting Electrons and Quantum Magnetism*, Polyakov, *Gauge fields and strings*] Above we used large spin as our small parameter to try to control the contributions to the path integral. Here we describe another route to a small parameter, which can be just as useful if we're interested in small spin (like spin-$\frac{1}{2}$).

Recall the relationship between the coherent state vector $\check{n}$ and the spinor components z: $n^a = z^\dagger \sigma^a z$. Imagine doing this at each point in space and time:

$$n^a(x) = z^\dagger(x)\sigma^a z(x). \tag{4.11}$$

We saw that the Berry phase term could be written nicely in terms of z as $\mathbf{i}z^\dagger \dot{z}$; what about the rest of the path integral?

First, some counting: $1 = \check{n}^2 \Leftrightarrow 1 = z^\dagger \cdot z = \sum_{m=\uparrow,\downarrow} |z_m|^2$. But this leaves only two components of n, and three components of z_m. The difference is made up by the fact that the rephasing

$$z_m(x) \to e^{\mathbf{i}\chi(x)} z_m(x) \tag{4.12}$$

doesn't change $\check{n}$. So it can't act on the physical Hilbert space. This is a (local, since $\chi(x)$ depends on x) $\mathsf{U}(1)$ gauge redundancy of the description in terms of z.

There two ways to proceed from here. One is via exact path integral tricks which are relatively straightforward in this case, but generally unavailable. The second is by the Landau method of knowing the answer: what else could it be.

Path integral manipulations. [Auerbach, chapter 14] First notice that the AF kinetic term is

$$\partial_\mu n^a \partial^\mu n^a = 4\left(\partial_\mu z^\dagger \partial^\mu z - \mathcal{A}_\mu \mathcal{A}^\mu\right) = 4\left(\partial_\mu z^\dagger \partial^\mu z - \mathcal{A}_\mu \mathcal{A}^\mu z^\dagger z\right). \tag{4.13}$$

where $\mathcal{A}_\mu \equiv -\frac{\mathbf{i}}{2}\left(z^\dagger \partial_\mu z - \partial_\mu z^\dagger z\right)$ is a connection one-form made from z itself. Notice that $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial\chi$ and the BHS of (4.13) is gauge invariant under (4.12). We must impose the constraint $|z(x)|^2 = 1$ at each site, which let's do it by a lagrange muliptlier $\delta[|z|^2 - 1] = \int D\lambda\, e^{\mathbf{i}\int d^D x \lambda(x)\left(|z|^2 - 1\right)}$. In the action, the $\mathcal{A}^2$ term is a self-interaction of the zs, which makes it difficult to do the integral. The standard trick for ameliorating this problem is the Hubbard-Stratonovich identity:

$$e^{c\mathcal{A}_\mu^2} = \sqrt{\frac{c}{\pi}} \int dA_\mu\, e^{-cA_\mu^2 + 2cA_\mu \mathcal{A}^\mu}.$$

The saddle point value of A is $\mathcal{A}$. This gives

$$e^{-\#\int dn^2} = \int [dA] e^{-\#\int |(\partial - \mathbf{i}A)z|^2}.$$

Finally, let's think about the measure at each point: $\int d^3n\delta(n^2-1)... = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi....$ Compare this to the integral over zs, parametrized as $z = \begin{pmatrix}\rho_1 e^{\mathbf{i}\phi_1}\\ \rho_2 e^{\mathbf{i}\phi_2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{\mathbf{i}\varphi/2}e^{\mathbf{i}\chi/2}\\ \sin\frac{\theta}{2}e^{-\mathbf{i}\varphi/2}e^{\mathbf{i}\chi/2}\end{pmatrix}$:

$$\int \prod_{m=1,2} dz_m dz_m^\dagger \delta(|z_1|^2+|z_2|^2-1)... = = c'\int \sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta d\varphi d\chi...$$

which is the same as $\int dn$ except for the extra integral over χ: that's the gauge direction. The integral over χ is just a number at each point, as long as we integrate invariant objects (otherwise, it gives zero). Thinking of z as parametrizing an arbitrary normalized spinor $z = \mathcal{R}(\theta,\varphi,\chi)\begin{pmatrix}1\\0\end{pmatrix}$, so that $\mathcal{R}$ is an arbitrary element of $\mathsf{SU}(2)$, we've just shown the geometric equivalence between the round S^2 and $\mathbb{CP}^1 = \mathsf{SU}(2)/\mathsf{U}(1)$.

Therefore, we can rewrite the path integral for the nonlinear sigma model (NLSM) on S^2 as

$$Z_{S^2} \simeq \int [dzdz^\dagger dAd\lambda] e^{-\int d^Dx\left(\frac{2\Lambda^{D-2}}{g^2}|(\partial-\mathbf{i}A)z|^2-\mathbf{i}\lambda\left(|z|^2-1\right)\right)}. \tag{4.14}$$

This is a $\mathsf{U}(1)$ gauge theory with $N=2$ charged scalars. It is called the $\mathbb{CP}^1$ sigma model. There are two slightly funny things: (1) the first is that the gauge field A lacks a kinetic term: in the microscopic description we are making here, it is infinitely strongly coupled. We'll see what the interactions with matter have to say about the coupling in the IR. (2) The second funny thing is that the scalars z have a funny interaction with this field λ which only appears linearly. If we add a $\lambda^2/(4\kappa)$ quadratic term, we can do the lambda integral and find $V(|z|^2) = \kappa(|z|^2-1)^2$, an ordinary quartic potential for $|z|$. This has the effect of replacing the delta function imposition with an energetic recommendation that $|z|^2 = 1$. This is called a *soft* constraint, and it shouldn't change the universal physics.

Alternatively, we could have arrived at this point

$$Z_{S^2} \simeq \int [dzdz^\dagger dA] e^{-\int d^Dx\left(\frac{2\Lambda^{D-2}}{g^2}|(\partial-\mathbf{i}A)z|^2-\kappa\left(|z|^2-1\right)^2\right)}$$

by regarding (4.11) as a *slave-particle* or *parton* ansatz for a new set of variables. The demand of gauge invariance (4.12) is a strong constraint on the form of the interactions, and requires the inclusion of the gauge field A. Relative to the most general 2-derivative Lagrangian consistent with the symmetries and gauge redundancy, we are missing kinetic terms for A; we'll see below that these are generated by the fluctuations of the zs.

Other such ansatze are possible, such as one in terms of slave fermions $\vec{S} = \psi^\dagger\vec{\sigma}\psi$. In this case, this turns out to be *also* correct – more later. More generally, any given change of variables may or may not be useful to describe the relevant physics.

[End of Lecture 14]

Large N. This representation allows the introduction of another possible small parameter (besides the number of components of $\vec{n}$ (which gives $\mathsf{SO}(n)$ symmetry) or the size of the spin (which takes a large representation of $\mathsf{SU}(2)$), namely the number of components of z (which will give a theory with $\mathsf{SU}(N)$ symmetry). Suppose instead of two components, it has N

$$\sum_{m=1}^{N} |z_m|^2 = \frac{N}{2},$$

and let's think about the resulting $\mathbb{CP}^{N-1}$ sigma model (notice that $\mathbb{CP}^{N-1}$ and S^N are different generalizations of S^2, in the sense that for $N \to 2$ they are both S^2):

$$\begin{aligned} Z_{\mathbb{CP}^{N-1}} &= \int [dz dz^\dagger dA d\lambda] e^{-\int d^D x \left(\frac{2\Lambda^{D-2}}{g^2} |(\partial - \mathbf{i}A)z|^2 - \mathbf{i}\lambda \left(|z|^2 - N/2 \right) \right)} \\ &= \int [dA d\lambda] e^{-NS[A,\lambda]} \overset{N \gg 1}{\simeq} Z' e^{-NS[\underline{A},\underline{\lambda}]}. \end{aligned}$$

The z-integral is gaussian in the representation (4.14) even for $N = 2$, but the resulting integrals over A, λ are then horrible, with action

$$S[A,\lambda] = \mathrm{Tr}\ \ln\left(-(\partial - \mathbf{i}A)^2 + \mathbf{i}\lambda \right) - \frac{\Lambda^{D-2}}{g^2} \int \mathbf{i}\lambda. \tag{4.15}$$

(In the last step, I absorbed some factors into a redefinition of λ.) The role of large N is to make those integrals well-peaked about their saddle point. The saddle point equations are solved by $\underline{A} = 0$ (though there may sometimes be other saddles where $\underline{A} \neq 0$, which break various discrete symmetries). This leaves us with

$$S[0,\lambda] = V \int \text{đ}^D k \ln(k^2 + \mathbf{i}\lambda) - \frac{\Lambda^{D-2}}{g^2} V \mathbf{i}\lambda$$

(where V is the number of sites, the volume of space, and I've assumed constant λ), which is solved by $\lambda = -\mathbf{i}\underline{\lambda}$ satisfying

$$\int \frac{\text{đ}^D k}{k^2 + \underline{\lambda}} = \frac{\Lambda^{D-2}}{g^2}.$$

This is an equation to determine $\underline{\lambda}$, which in turn appears as the coefficient of $z^\dagger z$ in the action – a mass for the charged fields (so this equation is sometimes called the *gap equation*). The solution of this equation depends on the number of dimensions D.

$$D = 1: \quad \frac{1}{g^2 \Lambda} = \int \frac{\text{đ}k}{k^2 + \underline{\lambda}} = \frac{1}{\sqrt{\underline{\lambda}}} \underbrace{\int \frac{\text{đ}\underline{k}}{\underline{k}^2 + 1}}_{= \frac{1}{2}} \quad \Longrightarrow \ \underline{\lambda} = \frac{g^4 \Lambda^2}{4}\ .$$

Since the action for z contains a term $\lambda|z|^2$, this says that the mass of the excitations is $m = \Lambda g^2/\sqrt{2}$. Where did that come from? $D = 1$ means we are studying the quantum mechanics of a particle constrained to move on $\mathbb{CP}^{N-1}$:

$$H = -\frac{g^2\Lambda}{2}\partial_z\partial_{\bar{z}} + \infty\left(|z|^2 - N/2\right)^2.$$

The groundstate is the uniform state $\langle z|\text{groundstate}\rangle = \Psi(z) = \frac{1}{\sqrt{\text{vol}}}$. Quantum mechanics of a finite number of particles on a compact space has a gap above the groundstate. This gap is determined by the kinetic energy and naturally goes like $g^2\Lambda$.

$$D = 2: \quad g^{-2} = \int \frac{\text{đ}^2k}{k^2 + \underline{\lambda}} = -\frac{1}{4\pi}\ln\frac{\underline{\lambda}}{\Lambda^2} \implies \underline{\lambda} = \Lambda^2 e^{-\frac{4\pi}{g^2}}.$$

This is the case with asymptotic freedom; here we see again that asymptotic freedom is accompanied by dimensional transmutation: the interactions have generated a mass scale

$$m = \Lambda e^{-\frac{2\pi}{g^2}}$$

which is parametrically (in the bare coupling g) smaller than the cutoff. This is the promised (Haldane) gap.

$$D = 3: \quad \frac{\Lambda}{g^2} = \int \frac{\text{đ}^3k}{k^2 + \underline{\lambda}} = \frac{1}{2\pi^2}\left(\Lambda - \sqrt{\lambda}\arctan\frac{\Lambda}{\sqrt{\lambda}}\right) \implies 1 - \frac{2\pi^2}{g^2} = x\arctan\frac{1}{x} \overset{x\ll 1}{\simeq} \frac{\pi}{2}x$$

where $x \equiv \frac{\sqrt{\lambda}}{\Lambda}$. Notice that for $D \geq 3$ there is a critical value of g below which there is no solution of this equation. The dominant contribution is at $\underline{\lambda} = m^2 = 0$ (for more explanation see Polyakov page 129), and the z-fields are gapless Goldstone modes. That means symmetry breaking. This doesn't happen in $D \leq 2$ (by the Hohenberg-Coleman-Mermin-Wagner theorem it cannot). The critical coupling occurs when $g_c^{-2} = \int \frac{\text{đ}^D k}{k^2} \simeq \frac{\Lambda^{D-2}}{D-2}$. The rate at which the mass goes to zero as $g \to g_c$ from above is

$$m^2 \simeq \Lambda^2\left(\frac{g^2 - g_c^2}{g_c^2}\right)^{\frac{2}{D-2}}.$$

This is a universal exponent. (For more on critical exponents from large-N calculations, see Peskin p. 464-465.)

Correlation functions. A quantity we'd like to be able to compute for $N = 2$ is $S^{+-}(x) \equiv \langle S^+(0)S^-(x)\rangle$. For example, this will allow us to see more quantitatively how the saddle point value for λ produces a gap. We can write this in terms of the coherent state variables using the identity

$$\mathbf{S}^a = \mathcal{N}_s \int dn\, |\check{n}\rangle\langle\check{n}|\, n^a, \quad (\mathcal{N}_s = \frac{(s+1)(2s+1)}{4\pi}).$$

(Up to the constant factor, this identity follows from $\mathsf{SU}(2)$ invariance. The constant can be checked by looking at a convenient matrix element of the BHS.) Then:

$$S^{+-}(x) = \langle (n^x + \mathbf{i}n^y)(0)(n^x - \mathbf{i}n^y)(x) \rangle \,.$$

Recalling that $n^x + \mathbf{i}n^y = z^\dagger \sigma^+ z = z_1^\star z_2$, we can generalize this to large N as the four-point function

$$S^{m\neq m'}(x) = \left\langle z_m^\star(0) z_{m'}(0) z_m(x) z_{m'}^\star(x) \right\rangle \overset{N\gg 1}{\simeq} |G(x)|^2$$

which *factorizes* at leading order in large N. This phenomenon (large-N factorization) that at large-N the correlations are dominated by the disconnected bits is general. (We'll see a diagrammatic argument momentarily.) The factors are correlators of zs in the fixed saddle-point configuration of A, λ:

$$\begin{aligned} G(x) &= \frac{1}{Z} \int [dz] z^\dagger(0) z(x) e^{-\frac{2\Lambda^{D-2}}{g^2} \int \mathrm{đ}^D k \left(|k|^2 + \underline{\lambda}\right) z_k^\dagger z_k - \frac{NV\Lambda^{D-2}}{g^2}\underline{\lambda}} \\ &\propto \int \mathrm{đ}^D k \frac{e^{-\mathbf{i}kx}}{|k|^2 + \underline{\lambda}} \simeq \frac{1}{|x|^{\frac{D-1}{2}}} e^{-|x|\sqrt{\underline{\lambda}}}. \end{aligned}$$

This says that the correlation length for the spins in $S^{m\neq m'}(x) \overset{x>\xi}{\simeq} \frac{1}{|x|^{D-1}} e^{-|x|/\xi}$ is $\xi = \frac{1}{\sqrt{\underline{\lambda}}}$ depends variously on D. In $D = 1$, it is $\xi = \frac{1}{\Lambda g^2}$, so large-$N$ predicts a gap, growing with g. In $D = 2$, the correlation length is $\xi = \Lambda^{-1} e^{+\frac{2\pi}{g^2}}$ In $D = 3$, the correlation length diverges as $g \to g_c$: $2\xi = \Lambda^{-1} \left(\frac{2}{\pi} - \frac{4\pi}{g^2}\right)^{-1}$, signaling the presence of gapless modes, which we interpret as Goldstones.

Exercise. Check that the other components of the spin such as $S^z = |z^m|^2 - |z^{m'}|^2$ have the same falloff, as they must by $\mathsf{SU}(N)$ symmetry.

A dynamical gauge field emerges. Finally, let me show you that a dynamical Maxwell gauge field emerges. Let's expand the action $S_{\text{eff}}[A, \lambda]$ in (4.15) about the saddle point at $A = 0, \lambda = \underline{\lambda} \equiv m^2$:

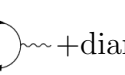
$$S[A = 0 + a, \lambda = m^2 + v] = W_0 + \underbrace{W_1}_{=0 \text{ by def}} + W_2 + \mathcal{O}(\delta^3)$$

where the interesting bit is the terms quadratic in the fluctuations:

$$W_2 = \frac{N}{2} \int \mathrm{đ}^D q \left(v(q) \Pi(q) v(-q) + a_\mu(q) \Pi_{\mu\nu}(q) a_\nu(-q) \right)$$

where

$$\Pi(q) = \ldots = \int \mathrm{đ}^D k \frac{1}{(k^2 + m^2)((k+q)^2 + m^2)} \tag{4.16}$$

$\Pi_{\mu\nu} =$  +diamagnetic diagram $= \int \text{đ}^D k \frac{(2k+q)_\mu (2k+q)_\nu}{(k^2+m^2)((k+q)^2+m^2)} - 2g_{\mu\nu} \int \frac{\text{đ}^D k}{k^2+m^2}.$ (4.17)

Familiarly, gauge invariance implies that $q^\mu \Pi_{\mu\nu}(q) = 0$ – it prevents a mass for the gauge field. For example, in $D = 2$, the long wavelength behavior is

$$\Pi_{\mu\nu}(q) \overset{q\to 0}{\sim} \frac{c}{m^2}\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)$$

which means that the effective action for the gauge fluctuation is

$$W_2 \sim \frac{N}{m^2}\int d^2x F_{\mu\nu}F^{\mu\nu} + \text{more derivatives.}$$

It is a dynamical gauge field.

Another term we can add to the action for a 2d gauge field is

$$\theta \int \frac{F}{2\pi}$$

where we regard $F = dA$ as a two-form. This is the 2d theta term, analogous to $\int F \wedge F$ in $D = 4$ in that $F = dA$ is locally a total derivative, it doesn't affect the equations of motion, and it integrates to an integer on smooth configurations. This integer is called the *Chern number* of the gauge field configuration. What integer is it? On the homework you'll show that $F \propto \epsilon^{abc} n^a dn^b dn^c$. It's the skyrmion number! So the coefficient is $\theta = 2\pi s$.

4.3.1 Large-N diagrams.

I think it will help to bring home some of the previous ideas by rederiving them using diagrams in a familiar context. So let's study the $\mathsf{O}(N)$ model:

$$L = \frac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} + \frac{g}{4N}\left(\vec{\varphi}\cdot\vec{\varphi}\right)^2 + \frac{m^2}{2}\vec{\varphi}\cdot\vec{\varphi}. \tag{4.18}$$

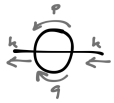
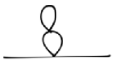
Let's do euclidean spacetime, D dimensions. The bare propagator is

$$\langle\varphi_b(x)\varphi_a(0)\rangle = \delta_{ab}\int \text{đ}^D k\ \frac{e^{-\mathbf{i}kx}}{k^2+m^2} \equiv \delta_{ab}\int \text{đ}^D k\ e^{-\mathbf{i}kx}\Delta_0(k).$$

The bare vertex is $-\frac{2g}{N}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$. With this normalization, the leading correction to the propagator is

 $= -\frac{g}{4N}(4N+8)\delta_{ab}\int \frac{\text{đ}q}{q^2+m^2} \overset{N\gg 1}{\simeq} -g\delta_{ab}\int \text{đ}q\Delta_0(q)$

of order N^0. This is the motivation for the normalization of the coupling in (4.18).

Which diagrams dominate at large N (and fixed g)? Compare two diagrams at the same order in λ with different topology of the index flow: eyeball  and cactus  . The former has one index loop, and the latter has two, and therefore dominates. The general pattern is that: at large N cacti dominate the 1PI self-energy. Each extra pod we add to the cactus costs a factor of g/N but gains an index loop N. So the sum of cacti is a function of gN^0.

The full propagator, by the usual geometric series, is then

$$\Delta_F(k) = \frac{1}{k^2 + m^2 + \Sigma(k)}. \tag{4.19}$$

We can sum all the cacti by noticing that cacti are self-similar: if we replace Δ_0 by Δ_F in the propagator:

$$\Sigma(p) = g \int \mathrm{đ}^D k \Delta_F(k) + \mathcal{O}(1/N). \tag{4.20}$$

The equations (4.19), (4.20) are integral equations for Δ_F; they are called Schwinger-Dyson equations,

OK, now notice the p-dependence in (4.20): the RHS is independent of p to leading order in N, so $\Sigma(p) = \delta m^2$ is just a mass shift.

Look at the position-space propagator

$$\langle \varphi_b(x) \varphi_a(y) \rangle = \delta_{ab} \int \mathrm{đ}^D k e^{-\mathbf{i}k(x-y)} \Delta_F(k). \tag{4.21}$$

Let

$$y^2 \equiv \left\langle \frac{\sum_a \varphi_a(x)\varphi_a(x)}{N} \right\rangle = \left\langle \frac{\varphi^2}{N} \right\rangle;$$

it is independent of x by translation invariance. Now let $y \to x$ in (4.21):

$$y^2 = \int \mathrm{đ}^D k \Delta_F(k) \overset{(4.20)}{=} g^{-1}\Sigma.$$

Now integrate the BHS of (4.19):

$$\int \mathrm{đ}^D p \Delta_F(p) = \int \mathrm{đ}^D p \frac{1}{p^2 + m^2 + \Sigma}$$

$$y^2 = \int \mathrm{đ}^D p \frac{1}{p^2 + m^2 + g y^2}.$$

This is an equation for the positive number y^2. Notice its similarity to the gap equation for λ we found from saddle point.

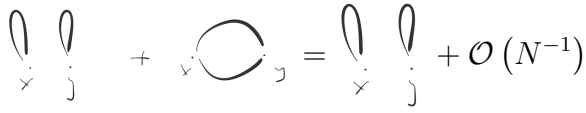
Large-N factorization. [Halpern] The fact that the fluctuations about the saddle point are suppressed by powers of N has consequences for the structure of the correlation functions in a large-N field theory. A basic example is

$$\langle \mathcal{I}(x)\mathcal{I}(y)\rangle = \langle \mathcal{I}(x)\rangle \langle \mathcal{I}(y)\rangle + \mathcal{O}\left(N^{-1}\right)$$

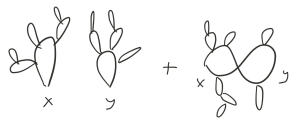
where $\mathcal{I}$ are any invariants of the large-N group (*i.e.* $\mathsf{O}(N)$ in the $\mathsf{O}(N)$ model (naturally) and $\mathsf{SU}(N)$ in the $\mathbb{CP}^{N-1}$ model), and $\langle ...\rangle$ denotes either euclidean vacuum expectation value or time-ordered vacuum expectation value. Consider, for example, in the $\mathsf{O}(N)$ model, normalized as above

$$\left\langle \frac{\varphi^2(x)}{N}\frac{\varphi^2(y)}{N}\right\rangle .$$

In the free theory, $g=0$, there are two diagrams

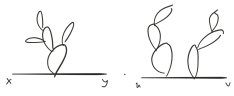
$$\left\langle \frac{\varphi^2(x)}{N}\frac{\varphi^2(y)}{N}\right\rangle_{\text{free}} =$$

$$+\mathcal{O}\left(N^{-1}\right)$$

– the disconnected diagram dominates, because it has one more index loop and the same number of interactions (zero). With interactions, representative diagrams are

$$\left\langle \frac{\varphi^2(x)}{N}\frac{\varphi^2(y)}{N}\right\rangle =$$

$$= \left\langle \frac{\varphi^2(x)}{N}\right\rangle \left\langle \frac{\varphi^2(y)}{N}\right\rangle + \mathcal{O}\left(N^{-1}\right) = y^4 + \mathcal{O}\left(N^{-1}\right)$$

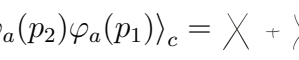
– it is independent of $x-y$ to leading order.

The same phenomenon happens for correlators of non-local singlet operators:

$$\left\langle \frac{\varphi(x)\cdot\varphi(y)}{N}\frac{\varphi(u)\cdot\varphi(v)}{N}\right\rangle =$$

$$= \left\langle \frac{\varphi(x)\cdot\varphi(y)}{N}\right\rangle \left\langle \frac{\varphi(u)\cdot\varphi(v)}{N}\right\rangle + \mathcal{O}\left(N^{-1}\right)$$

The basic statement is that mean field theory works for singlets. At large N, the entanglement follows the flavor lines.

We can still ask: what processes dominate the connected (small) bit at large N? And what about non-singlet operators? Consider (no sum on b, a):

$$G^{b\neq a}_{4,c} = \langle \varphi_b(p_4)\varphi_b(p_3)\varphi_a(p_2)\varphi_a(p_1)\rangle_c =$$

$$+\mathcal{O}\left(N^{-2}\right)$$

The answer is: *bubbles*. More specifically chains of bubbles, propagating in the s-channel. What's special about the s-channel, here? It's the channel in which we can

make $\mathsf{O}(N)$ singlets. In the other channels, we don't get loops with free flavor indices. The diagrams I've displayed all go like $f(g)/N$. Other candidates are eyeballs:  and ladders:  but as you can see, these go like N^{-2}. However, bubbles can have cactuses growing on them, like this:  These all go like $1/N$. To sum all of these, we just use the full propagator in the internal lines of the bubbles, $\Delta_0 \to \Delta_F$.

I claim that the bubble sum is a geometric series:

$$G_{4,c}^{b\neq a} = -\left(\Delta_0(\text{external})\right)^4 \frac{2}{N}\frac{g}{1+gL(p_1+p_2)} + \mathcal{O}\left(N^{-2}\right) \tag{4.22}$$

where L is the loop integral $L(p) \equiv \int \mathrm{d}^D k \Delta_F(k)\Delta_F(p+k)$. You can see this by being careful about the symmetry factors.

 $= \Delta_0\,(\text{external})^4 \left(\frac{g}{4N}\right)\cdot 2\cdot 4$

 $= \Delta_0\,(\text{external})^4 \left(\frac{g}{4N}\right)^2 \cdot 2\cdot 4\cdot 8\cdot \underbrace{\frac{1}{2!}}_{Dyson} L = \Delta_0\,(\text{external})^4 \frac{2}{N}(g)^2 L.$

Similarly, the chain of two bubbles is $\frac{2}{N}g^3L^2$, etc.

Here's how we knew this had to work without worrying about the damn symmetry factors: the bubble chain *is* the σ propagator! At the saddle, $\sigma \simeq \varphi^a\varphi^a$, which is what is going in and out of this amplitude. And the effective action for sigma (after integrating out φ) is

$$S_{\text{eff}}[\sigma] = \int \frac{\sigma^2}{g} + \operatorname{tr}\ln\left(\partial^2 + m^2 + \sigma\right).$$

The connected two-point function means we subtract off $\langle\underline{\sigma}\rangle\langle\underline{\sigma}\rangle$, which is the same as considering the two point function of the deviation from saddle value. This is

$$\langle\sigma_1\sigma_2\rangle = \left(\frac{\delta^2}{\delta\sigma_1\delta\sigma_2}S_{\text{eff}}[\sigma]\right)^{-1} = \left(\frac{1}{g^{-1}+\left(\frac{1}{\partial^2+m^2+\sigma}\right)^2}\right)^{-1}$$

which becomes exactly the expression above if we write it in momentum space.

Two comments: (1) We were pretty brash in integrating out all the φ variables and keeping the σ variable: how do we know which are the slow ones and which are the

fast ones? This sort of non-Wilsonian strategy is common in the literature on large-N, where physicists are so excited to see an integral that they can actually do that they don't pause to worry about slow and fast. But if we did run afoul of Wilson, at least we'll know it, because the action for σ will be nonlocal.

(2) $\sigma \sim \varphi^2$ is a composite operator. Nevertheless, the sigma propagator we've just derived can have poles at some $p^2 = m^2$ (likely with complex m). These would produce particle-like resonances in a scattering experiment (such as $2-2$ scattering of φs of the same flavor) which involved sigmas propagating in the s-channel. Who is to say what is fundamental.

Now that you believe me, look again at (4.22); it is of the form

$$G_{4,c}^{b\neq a} = -\left(\Delta_0(\text{external})\right)^4 \frac{2}{N} g_{\text{eff}}(p_1+p_2) + \mathcal{O}\left(N^{-2}\right)$$

where now

$$g_{\text{eff}}(p) = \frac{g}{1 + g\int \mathrm{d}^D k \Delta_F(k)\Delta_F(p+k))}$$

is a momentum-dependent effective coupling, just like one dreams of when talking about the RG.

[End of Lecture 15]

5 Duality

The word 'duality' in field theory refers to the situation when there are two distinct descriptions of the physical system. Distinct here generally means that when one description is weakly-coupled, the other is strongly coupled. Often, the elementary perturbative quanta in one description are realized as solitons in the other description. The two descriptions must agree about unambiguous quantities, like symmetries and anomalies and (more ambitiously) the energy spectrum. The two descriptions are generally related by some non-local change of variables, though (in all the examples below) each is itself a local QFT. We don't have a systematic understanding of this phenomenon, so we proceed by a series of examples.

5.1 Transverse-Field Ising Model

Next, I want to spend some talking about situations where only $\mathbb{Z}_2 \subset \mathsf{SU}(2)$ of the spin rotations is preserved, the Ising model.

The Ising model has many guises. There is this from statistical mechanics:

$$Z = \sum_{\{s_j\}} e^{-K \sum_{\langle jl \rangle} s_j s_l}.$$

There is this quantum spin system:

$$\mathbf{H}_{TFIM} = -J \sum_j \left(g_x \mathbf{X}_j + g_z \mathbf{Z}_j \mathbf{Z}_{j+1} \right) \ .$$

And there is this 2d conformal field theory:

$$S[\chi] = \int \mathrm{d}^2 z \left(\chi \overline{\partial}_z \chi \right) \tag{5.1}$$

which I first encountered on the worldsheet of a superstring. An important part of our job is to understand the connections between these things. One thing they have in common is a $\mathbb{Z}_2$ symmetry, $s_j \to -s_j$ or $\mathbf{Z}_j \to -\mathbf{Z}_j$ or $\chi \to -\chi$.

Whether or not you liked the derivation above of its relation to the euclidean statistical mechanics Ising model, we are going to study the quantum system whose hamiltonian is

$$\mathbf{H}_{\text{TFIM}} = -J \left(g \sum_j \mathbf{X}_j + \sum_{\langle jl \rangle} \mathbf{Z}_j \mathbf{Z}_l \right). \tag{5.2}$$

Some of the things we say next will be true in one or more spatial dimensions.

Notice that J has units of energy; we could choose units where it's 1. In 1d (or on bipartite lattices), the sign of J does not matter for determining what state of matter we realize: if $J<0$, we can relabel our operators: $\tilde{Z}_j = (-1)^j \mathbf{Z}_j$ and turn an antiferromagnetic interaction into a ferromagnetic one. So let's assume $g, J > 0$.

This model is interesting because of the competition between the two terms: the $\mathbf{X}_j$ term wants *each* spin (independently of any others) to be in the state $|{\rightarrow}\rangle_j$ which satisfies

$$\mathbf{X}_j |{\rightarrow}\rangle_j = |{\rightarrow}\rangle_j \,. \quad |{\rightarrow}\rangle_j = \frac{1}{\sqrt{2}} \Big(|{\uparrow}\rangle_j + |{\downarrow}\rangle_j \Big) \,.$$

In conflict with this are the desires of $-\mathbf{Z}_j \mathbf{Z}_{j+1}$, which is made happy (*i.e.* smaller) by the more cooperative states $|{\uparrow_j}{\uparrow_{j+1}}\rangle$, or $|{\downarrow_j}{\downarrow_{j+1}}\rangle$. In fact, it would be just as happy about any linear combination of these $a\,|{\uparrow_j}{\uparrow_{j+1}}\rangle + b\,|{\downarrow_j}{\downarrow_{j+1}}\rangle$ and we'll come back to this point.

Another model which looks like it might have some form of competition is

$$\mathbf{H}_{\text{boring}} = \cos\theta \sum_j \mathbf{Z}_j + \sin\theta \sum_j \mathbf{X}_j \ , \quad \theta \in [0, \frac{\pi}{2}]$$

Why is this one boring? Notice that we can continuously interpolate between the states enjoyed by these two terms: the groundstate of $\mathbf{H}_1 = \cos\theta \mathbf{Z} + \sin\theta \mathbf{X}$ is

$$|\theta\rangle = \cos\frac{\theta}{2} |{\uparrow}\rangle + \sin\frac{\theta}{2} |{\downarrow}\rangle$$

– as we vary θ from 0 to $\pi/2$ we just smoothly rotate from $|{\uparrow_z}\rangle$ to $|{\uparrow_x}\rangle$. And it is always a product state.

How do we know the same thing can't happen in the transverse-field Ising chain? Symmetry. We've already seen that the Ising model has a $\mathsf{G} = \mathbb{Z}_2$ symmetry which acts by $\mathbf{Z}_j \to \mathbf{S}\mathbf{Z}_j\mathbf{S}^\dagger = -\mathbf{Z}_j, \mathbf{X}_j \to \mathbf{S}\mathbf{X}_j\mathbf{S}^\dagger = +\mathbf{X}_j$, where the unitary $\mathbf{S}$ commutes with $\mathbf{H}_{\text{TFIM}}$: $\mathbf{S}\mathbf{H}_{\text{TFIM}}\mathbf{S}^\dagger = \mathbf{H}_{TFIM}$. Here $\mathbf{S} = \prod_i \mathbf{X}_i$. The difference with $\mathbf{H}_{\text{boring}}$ is that $\mathbf{H}_{\text{TFIM}}$ has two phases in which G is realized differently on the groundstate.

$\boxed{g=\infty}$: First, let's take g so big that we may ignore the ZZ ferromagnetic term, so

$$\mathbf{H}_{g\to\infty} = -\sum_j \mathbf{X}_j \ .$$

(The basic idea of this discussion will apply in any dimension, on any lattice.) Since all terms commute, the groundstate is the simultaneous groundstate of each term:

$$\mathbf{X}_j |\text{gs}\rangle = +|\text{gs}\rangle \,, \quad \forall j, \qquad \Longrightarrow \ |\text{gs}\rangle = \otimes_j |{\rightarrow}\rangle_j \,.$$

Notice that this state *preserves the symmetry* in the sense that $\mathbf{S}|\text{gs}\rangle = |\text{gs}\rangle$. Such a symmetry-preserving groundstate is called a *paramagnet*.

$\boxed{g=0}$: Begin with $g = 0$.

$$\mathbf{H}_0 = -J \sum_j \mathbf{Z}_j \mathbf{Z}_{j+1}$$

has groundstates

$$|+\rangle \equiv |\uparrow\uparrow \cdots \uparrow\rangle\,, \quad |-\rangle \equiv |\downarrow\downarrow \cdots \downarrow\rangle\,,$$

or any linear combination. Note that the states $|\pm\rangle$ are not symmetric: $\mathbf{S}\,|\pm\rangle = |\mp\rangle$, and so we are tempted to declare that the symmetry is broken by the groundstate.

You will notice, however, that the states

$$\left| \text{  }_{\pm} \right\rangle \equiv \frac{1}{\sqrt{2}} \left(|+\rangle \pm |-\rangle\right)$$

are symmetric – they are $\mathbf{S}$ eigenstates, so $\mathbf{S}$ maps them to themselves up to a phase. It gets worse: In fact, in finite volume (finite number of sites of our chain), with $g \neq 0$, $|+\rangle$ and $|-\rangle$ are *not* eigenstates, and $\left| \text{  }_{+} \right\rangle$ is the groundstate. **BUT:**

1. The two states $|+\rangle$ and $|-\rangle$ only mix at order N in perturbation theory in g, since we have to flip all N spins using the perturbing hamiltonian $\Delta\mathbf{H} = -gJ \sum_j \mathbf{X}_j$ to get from one to the other. The tunneling amplitude is therefore

$$T \sim g^N \langle -|\, \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_N \,|+\rangle \overset{N\to\infty}{\to} 0.$$

2. There's a reason for the symbol I used to denote the symmetric states: at large N, these 'cat states' are superpositions of macroscopically distinct quantum states. Such things don't happen, because of decoherence: if even a single dust particle in the room measures the spin of a single one of the spins, it measures the value of the whole chain. In general, this happens very rapidly.

3. Imagine we add a small symmetry-breaking perturbation: $\Delta\mathbf{H} = -\sum_j h\mathbf{Z}_j$; this splits the degeneracy between $|+\rangle$ and $|-\rangle$. If $h > 0$, $|+\rangle$ is for sure the groundstate. Consider preparing the system with a tiny $h > 0$ and then setting $h = 0$ after it settles down. If we do this to a finite system, $N < \infty$, it will be in an excited state of the $h = 0$ Hamiltonian, since $|+\rangle$ will not be stationary (it will have a nonzero amplitude to tunnel into $|-\rangle$). But if we take the thermodynamic limit *before* taking $h \to 0$, it will stay in the state we put it in with the 'training

field' h. So beware that there is a singularity of our expressions (with physical significance) that means that the limits do not commute:

$$\lim_{N\to\infty}\lim_{h\to 0} Z \neq \lim_{h\to 0}\lim_{N\to\infty} Z.$$

The physical one is to take the thermodynamic limit first.

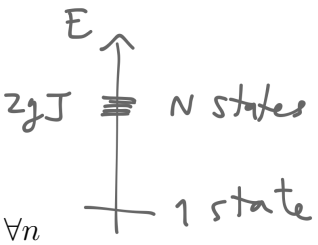
The conclusion of this brief discussion is that spontaneous symmetry breaking actually happens in the $N\to\infty$ limit. At finite N, $|+\rangle$ and $|-\rangle$ are approximate eigenstates which become a better approximation as $N\to\infty$.

This state of a $\mathbb{Z}_2$-symmetric system which spontaneously breaks the $\mathbb{Z}_2$ symmetry is called a ferromagnet.

So the crucial idea I want to convey here is that there must be a sharp phase transition at some finite g: the situation cannot *continuously* vary from one unique, symmetric groundstate $\mathbf{S}\left|\mathrm{gs}_{g\ll 1}\right\rangle = \left|\mathrm{gs}_{g\ll 1}\right\rangle$ to two symmetry-breaking groundstates: $\mathbf{S}\left|\mathrm{gs}_{\pm}\right\rangle = \left|\mathrm{gs}_{\mp}\right\rangle$.

Quasiparticles. Next, let's ask what are the low-lying excitations, and see what happens when we try to vary the coupling away from the extreme points.

$\boxed{g\gg 1}$ An excited state of the paramagnet, deep in the phase, is achieved by flipping one spin. With $\mathbf{H}=\mathbf{H}_\infty = -gJ\sum_j \mathbf{X}_j$, this costs energy $2gJ$ above the groundstate. There are N such states, labelled by which spin we flipped:



$$|n\rangle \equiv \Big|\to\cdots\to\underbrace{\leftarrow}_{nth\ \text{site}}\to\cdots\Big\rangle, \qquad (\mathbf{H}_\infty - E_0)\,|n\rangle = 2gJ\,|n\rangle\,,\ \forall n$$

When g is not infinite, we can learn a lot from (1st order) degenerate perturbation theory in the ferromagnetic term. The key information is the matrix elements of the perturbing hamiltonian between the degenerate manifold of states. Using the fact that $\mathbf{Z}_j\,|{\to_j}\rangle = |{\leftarrow_j}\rangle$, so,

$$\mathbf{Z}_j\mathbf{Z}_{j+1}\,|{\to_j}{\leftarrow_{j+1}}\rangle = |{\leftarrow_j}{\to_{j+1}}\rangle$$
$$\langle n\pm 1|\sum_j \mathbf{Z}_j\mathbf{Z}_{j+1}\,|n\rangle = 1,$$

the ferromagnetic term hops the spin flip by one site. Within the degenerate subspace, it acts as

$$\mathbf{H}_{\text{eff}}\,|n\rangle = -J\left(|n+1\rangle + |n-1\rangle\right) + (E_0+2gJ)\,|n\rangle\,.$$

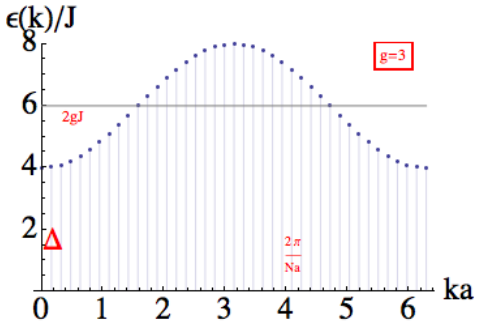
It is a kinetic, or 'hopping' term for the spin flip.

Let's see what this does to the spectrum. Assume periodic boundary conditions and N sites total. This is a translation invariant problem (in fact the same one as the masses connected by springs), which we solve by Fourer transform:

$$|n\rangle \equiv \frac{1}{\sqrt{N}} \sum_j e^{-\mathbf{i}kx_j} |k\rangle \,, \quad \begin{cases} x_j \equiv ja, \\ k = \frac{2\pi m}{Na}, \quad m = 1..N \end{cases}$$

On the momentum states, we have

$$(H - E_0)\,|k\rangle = (-2J\cos ka + 2gJ)\,|k\rangle \,.$$



The dispersion of these spinon particles is

$$\epsilon(k) = 2J(g - \cos ka) \overset{k\to 0}{\sim} \Delta + J(ka)^2 \qquad (5.3)$$

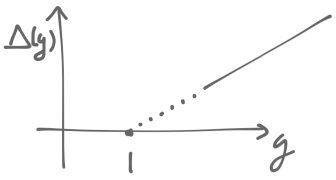
with $\Delta = 2J(g-1)$ – there is an energy gap (notice that Δ does not depend on system size). So these are *massive* particles, with dispersion $\epsilon = \Delta + \frac{k^2}{2M} + ...$ where Δ is the energy to create one at rest (notice that the rest energy is not related to its inertial mass $M^{-1} = 2Ja^2$).

A particle at j is created by the creation operator $\mathbf{Z}_n$:

$$|n\rangle = \mathbf{Z}_n\,|\mathrm{gs}_\infty\rangle \,.$$

And it is annihilated by the annihilation operator $\mathbf{Z}_n$ – you can't have two spin flips at the same location! These particles are their own antiparticles.

The *number* of such particles is counted by the operator $\sum_j (-\mathbf{X}_j)$. The number of particles is only conserved modulo two, however.



What happens as g gets smaller? The gap to creating a spin flip at large g looks like $2J(g-1)$. If we take this formula seriously, we predict that at $g = g_c = 1$ it costs zero energy to create spin flips: they should condense in the vacuum. Condensing spin flips means that the spins point in all directions, and the state is paramagnetic. (We shouldn't take the value of g_c too seriously because it's just first order in perturbation theory, but it turns out to be exactly right.)

It's possible to develop some more evidence for this picture and understanding of the physics of the paramagnetic phase in the Ising chain by doing more perturbation theory, and including states with two spin flips. Notice that for a state with two spin-flip particles, the total momentum k no longer uniquely determines the energy,

since the two spin-flips can have a relative momentum; this means that there is a *two-particle continuum* of states, once we have enough energy to make two spin flips. For more on this, see *e.g.* Sachdev (2d ed) §5.2.2. In particular the two spin-flips can form boundstates, which means the lowest-energy two-particle state is actually slightly below 2Δ.

$\boxed{g \ll 1}$ Now let's consider excitations of the ferromagnet, about the state $|+\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$. Here $D = 1+1$ will matter. $|+\rangle$ is an eigenstate of $\mathbf{H}_0 = -J\sum_j \mathbf{Z}_j\mathbf{Z}_{j+1}$ and its (groundstate) energy is $E_0 = -JN$. We can make an excitation by flipping one spin:

$$|\cdots\uparrow\ \uparrow\ \uparrow\cdot\downarrow\cdot\uparrow\ \uparrow\ \uparrow\cdots\rangle$$

This makes two bonds unhappy, and costs $2J + 2J = 4J$. But once we make it there are many such states: the hamiltonian is the same amount of unhappy if we also flip the next one.

$$|\cdots\uparrow\ \uparrow\ \uparrow\cdot\downarrow\ \downarrow\cdot\uparrow\ \uparrow\cdots\rangle$$

The actual elementary excitation is a *domain wall* (or *kink*), which only costs $2J$. The domain wall should be regarded as living between the sites. It is not entirely a local object, since with periodic boundary conditions, we must make two, which can then move independently. To create two of them far apart, we must change the state of many spins.

At $g = 0$ the domain walls are *localized* in the sense that a domain wall at a fixed position is an energy eigenstate (just like the spinons at $g = \infty$), with the same energy for any position. But now the paramagnetic term $-\sum_j g\mathbf{X}_j$ is a kinetic term for the domain walls:

$$\mathbf{X}_{j+1}\underbrace{|\cdots\uparrow\uparrow\uparrow_j\cdot\downarrow_{j+1}\downarrow\downarrow\cdots\rangle}_{|\bar{j}\rangle} = \underbrace{|\cdots\uparrow\uparrow\uparrow_j\uparrow_{j+1}\cdot\downarrow_{j+2}\downarrow\cdots\rangle}_{=|\bar{j}+1\rangle}.$$

Just like in our $g \gg 1$ discussion, acting on a state with a single domain wall[29]

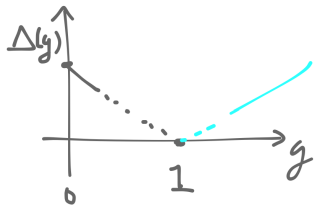
$$(H_{\text{eff}} - E_0)\,|\bar{j}\rangle = -gJ\,(|\bar{j}+1\rangle + |\bar{j}-1\rangle) + 2J\,|\bar{j}\rangle$$

where the diagonal term is the energy cost of one domain wall at rest. Again this is diagonalized in k-space with energy

$$\epsilon_{\text{one dwall}}(k) = 2J(1 - g\cos ka)$$

[29] A state with a single domain wall can't happen with periodic boundary conditions, but with open boundary conditions, it can.

Again, this calculation is almost ridiculously successful at predicting the location of the phase transition:



$$\Delta_{DW} = 2J(1-g) \overset{g\to 1}{\to} 0.$$

Although our discussion of the paramagnetic state $g \gg 1$ can be applied in any $d \geq 1$, the physics of domain walls is very dimension-dependent.

Pause to reassess goals. You may be wondering: why in this class about QFT are we lavishing so much attention on this spin chain? It may seem like this discussion involves lots of lattice-scale details which are divorced from our fancy Lorentz-invariant continuum QFT considerations. First: Recall that my definition of QFT is the study of extensive quantum systems, *i.e.* (quantum) degrees of freedom distributed over space. A spin chain exactly fits this definition. Second: It's true that this kind of system (and the TFIM model in particular) arises in many discussions of condensed matter. But the real reason is: In this example, we can understand everything. In particular, we can understand everything about how QFT emerges from a completely well-defined quantum mechanical system, in which there are obviously no divergences or infinities or anomalies. And yet, the system exhibits a quantum phase transition, described by a relativistic conformal field theory (CFT)! (In particular, Lorentz invariance emerges.) And moreover, we will discover that a description in terms of gauge theory is useful. So I think the TFIM is particularly useful as an answer to the question 'where do QFTs come from?'

Interpretation of the stability of the SSB state in terms of domain walls. If at finite N, with periodic boundary conditions, we prepare the system in the state $|+\rangle$, tunneling to $|-\rangle$ requires creation of a pair of domain walls $\Delta E = 4J$, which then *move* all the way around the circle, giving the tunneling rate

$$\prod_{j=1}^{N} \left(\frac{\langle \bar{j}+1 | \, \mathbf{H}_{\text{eff}} \, | \bar{j} \rangle}{\Delta E} \right) \sim \frac{(gJ)^N}{J^N} \sim g^N \sim e^{-N \log \frac{1}{g}}.$$

(For $g < 1$, $\log \frac{1}{g} > 0$.) The tunneling rate goes like e^{-N} – it is exponentially small in the system size.

Duality between spin flips and domain walls. The discussion we've just made of the small-g physics has a lot in common with the large-g physics. More quantitatively, the dispersion relation $\epsilon_{\text{one dwall}}(k)$ for a single domain wall looks nearly the same as that of one spin flip (5.3). In fact they are mapped to each other by the replacement

$$g \to \frac{1}{g}, \quad J \to Jg. \tag{5.4}$$

Notice that this takes small g (weak coupling of domain walls, strong coupling of spin flips) to large g (strong coupling of domain walls, weak coupling of spin flips).

In fact, there is a change of variables that (nearly) interchanges the two sides of the phase diagram. Suppose the system is on an interval – open boundaries – the chain just stops at $j = 1$ and $j = N$. (We do this to avoid the constraint of an even number of domain walls.) We can specify a basis state in the Z-basis by the direction (up or down along Z) of the first spin and the locations of domain walls.

Consider the operator, diagonal in this basis, which measures whether there is a domain wall between j and $j + 1$:

$$\boldsymbol{\tau}^x_{\bar{j}} \equiv \mathbf{Z}_{\bar{j}-\frac{1}{2}}\mathbf{Z}_{\bar{j}+\frac{1}{2}} = \begin{cases} +1, & \text{if } z_{\bar{j}-\frac{1}{2}} = z_{\bar{j}+\frac{1}{2}} \\ -1, & \text{if } z_{\bar{j}-\frac{1}{2}} = -z_{\bar{j}+\frac{1}{2}} \end{cases} = (-1)^{\text{disagreement}}.$$

Notice that $\boldsymbol{\tau}^2_{\bar{j}} = 1, \boldsymbol{\tau}^\dagger_{\bar{j}} = \boldsymbol{\tau}_{\bar{j}}$. Similarly, consider the operator that *creates* a domain wall at $\bar{j}$:

$$\boldsymbol{\tau}^z_{\bar{j}} \equiv \mathbf{X}_{\bar{j}+\frac{1}{2}}\mathbf{X}_{\bar{j}+\frac{3}{2}}\cdots = \prod_{j>\bar{j}} \mathbf{X}_j.$$

This operator flips all the spins to the right of the link in question (and fixes our reference first spin). It, too, is hermitian and squares to one. Finally, notice that

$$\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\tau}^x_{\bar{j}'} = (-1)^{\delta_{\bar{j}\bar{j}'}}\boldsymbol{\tau}^x_{\bar{j}'}\boldsymbol{\tau}^z_{\bar{j}}$$

just like $\mathbf{Z}$ and $\mathbf{X}$ (since when $\bar{j} = \bar{j}'$, they contain a single $\mathbf{Z}$ and $\mathbf{X}$ at the same site). The domain walls can be represented in terms of two-state systems living on the links.

Notice that the inverse of the map from $\mathbf{X}, \mathbf{Z}$ to $\boldsymbol{\tau}^x, \boldsymbol{\tau}^z$ is

$$\mathbf{X}_j = \boldsymbol{\tau}^z_{j-\frac{1}{2}}\boldsymbol{\tau}^z_{j+\frac{1}{2}} \ .$$

(The right hand side is an inefficient way to flip a single spin at j: namely, flip all the spins right of $j - 1$, and then flip back all the spins to the right of j.)

So the 1d TFIM hamiltonian in bulk is

$$\begin{aligned} \mathbf{H}_{\text{TFIM}} &= -J\sum_j \left(g\mathbf{X}_j + \mathbf{Z}_j\mathbf{Z}_{j+1}\right) \\ &= -J\sum_{\bar{j}} \left(g\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\tau}^z_{\bar{j}+1} + \boldsymbol{\tau}^x_{\bar{j}}\right). \end{aligned} \tag{5.5}$$

This is the TFIM hamiltonian again with $\mathbf{Z} \to \boldsymbol{\tau}^z$ and $\mathbf{X} \to \boldsymbol{\tau}^x$ and the couplings mapped by (5.4).

This is in fact the same map as Kramers-Wannier duality (or rather it is mapped to it by the quantum-to-classical map). As K&W argued, if there is a single phase transition it must occur at the self-dual point $g = 1$.

Notice that the paramagnetic (disordered) groundstate of the original system is a *condensate* of domain walls, in the following sense. The operator that creates a domain wall has an expectation value:

$$\left\langle \boldsymbol{\tau}^z_{\bar{j}} \right\rangle = \left\langle \mathrm{gs}_{g=\infty} \right| \boldsymbol{\tau}^z_{\bar{j}} \left| \mathrm{gs}_{g=\infty} \right\rangle = \left\langle \mathrm{gs}_{g=\infty} \right| \prod_{j>\bar{j}} \mathbf{X}_j \left| \mathrm{gs}_{g=\infty} \right\rangle = 1 \quad \forall \bar{j} \ .$$

(For $g \in (1, \infty)$, this expectation value is less than one but nonzero, just like how $|\langle \mathbf{Z} \rangle|$ decreases from 1 as g grows from zero.) Although there is a condensate, there is no order, in the sense that an expectation value of $\mathbf{X}$ does not break any symmetry of $\mathbf{H}_{\mathrm{TFIM}}$. (There is another state where $\langle \boldsymbol{\tau}^x \rangle = -1$, but (at large g) it's a high-energy state.)

An important point (and the reason 'duality' is a dangerous word): the two sides of the phase diagram are *not the same*. On one side there are two groundstates related by the broken symmetry, on the other side there is a unique symmetric groundstate. That's how we knew there had to be a phase transition! I will say more about this mismatch.

Open boundaries. Let us make sure we can reproduce the correct number of groundstates in the two phases. To get this right, we have to be precise about the endpoint conditions. Let's study the case where we have N sites in a row; the first and last sites have only one neighbor. The Hamiltonian is

$$\mathbf{H}_{\mathrm{TFIM}} = -J \left(\sum_{j=1}^{N-1} (g\mathbf{X}_j + \mathbf{Z}_j \mathbf{Z}_{j+1}) + g\mathbf{X}_N \right) \ .$$

The duality map is

$$\mathbf{Z}_j \mathbf{Z}_{j+1} = \boldsymbol{\tau}^x_{j+\frac{1}{2}}, j = 1, 2 \ldots N-1, \quad \mathbf{X}_j = \boldsymbol{\tau}^z_{j-\frac{1}{2}} \boldsymbol{\tau}^z_{j+\frac{1}{2}}, \quad j = 1 \ldots N \quad .$$

In terms of the domain-wall variables, the hamiltonian is

$$\mathbf{H}_{\mathrm{TFIM}} = -Jg \left(\sum_{j=1}^{N-1} \left(\boldsymbol{\tau}^z_{j-\frac{1}{2}} \boldsymbol{\tau}^z_{j+\frac{1}{2}} + \frac{1}{g} \boldsymbol{\tau}^x_{j+\frac{1}{2}} \right) + \boldsymbol{\tau}^z_{N-\frac{1}{2}} \boldsymbol{\tau}^z_{N+\frac{1}{2}} \right) \ .$$

But now there are two special cases:
• $\boldsymbol{\tau}^z_{N+\frac{1}{2}} = 1$: this operator flips all the spins with $j > N$; but there are no spins with $j > N$. So it is the identity operator.

• $\boldsymbol{\tau}^x_{\frac{1}{2}}$: this operator measures whether or not there is a domain wall between $j = 1$ and $j = 0$, $\boldsymbol{\tau}^x_{\frac{1}{2}} = \mathbf{Z}_0\mathbf{Z}_1$. But there is no spin at $j = 0$. One way to think about this is to put a "ghost spin" at $j = 0$ which is always in the state $\mathbf{Z}_0 = 1$. So $\boldsymbol{\tau}^x_{\frac{1}{2}} = \mathbf{Z}_1$: it measures the value of our reference spin.

At $g = 0$: $\mathbf{H}_{g=0} = -J\sum_{j=2}^{N} \boldsymbol{\tau}^x_{j-\frac{1}{2}}$ and the groundstate is $\boldsymbol{\tau}^x_{j-\frac{1}{2}} = 1$ for $j = 2...N$. But $\boldsymbol{\tau}^x_{\frac{1}{2}}$ does not appear, so there are two degenerate groundstates, eigenstates of $\boldsymbol{\tau}^x_{\frac{1}{2}}$ with eigenvalue $\pm$, which are just $|\pm\rangle$, the states with no domain walls: all the other spins agree with the first one in a state where $\boldsymbol{\tau}^x_{j>1} = 1$.

At $g = \infty$, $\mathbf{H}_{g=\infty} = -Jg\left(\sum_{j=1}^{N-1} \boldsymbol{\tau}^z_{j-\frac{1}{2}}\boldsymbol{\tau}^z_{j+\frac{1}{2}} + \boldsymbol{\tau}^z_{N-\frac{1}{2}}\right)$. The first term requires agreement between $\boldsymbol{\tau}^z_{j-\frac{1}{2}} = \boldsymbol{\tau}^z_{j+\frac{1}{2}}$ for $j = 1...N-1$, but the second term requires $\boldsymbol{\tau}^z_{N-\frac{1}{2}} = 1$. So all the others must be $+1$, too. This is the unique groundstate of the paramagnet.

Comment: Recall that the classical stat mech model we get by trotterizing the TFIM is the ordinary 2d Ising model, with anisotropic couplings,

$$H_c = -\sum_{\vec{n}} \left(K s(\vec{n} + \check{\tau})s(\vec{n}) + K_x s(\vec{n} + \check{x})s(\vec{n})\right).$$

In these variables, the critical curve between the two phases is at

$$\sinh(2K_x)\sinh(2K) = 1\ ;$$

this relationship follows by *Kramers-Wannier duality*. In the continuous-time limit, $K_x \sim \Delta\tau \ll 1, K \gg 1$, so the critical condition becomes

$$1 = \sinh(2K_x)\sinh(2K) \simeq 2K_x\frac{e^{2K}}{2} = J\Delta\tau\frac{1}{Jg\Delta\tau} = \frac{1}{g}.$$

This determines the location of the quantum phase transition, in agreement with all the other stuff we've said about it.

5.1.1 Solution of Ising chain in terms of Majorana fermions

The TFIM in $d = 1$ is actually exactly solvable, for all g, including the quantum critical point. Let us see how this works and how the critical field theory emerges near $g = 1$.

Jordan-Wigner in 0+1 dimensions. As a warmup, consider a single fermionic operator $\mathbf{c}$ satisfying

$$\mathbf{c}^2 = 0, \quad \{\mathbf{c}, \mathbf{c}^\dagger\} = 1\ .$$

This constructs the Hilbert space of a qbit as follows:

$$\mathbf{c}\,|{\rightarrow}\rangle = 0, \quad \mathbf{c}^\dagger\,|{\rightarrow}\rangle = |{\leftarrow}\rangle\,, \quad \mathbf{c}^\dagger\,|{\leftarrow}\rangle = 0, \quad \mathbf{c}\,|{\leftarrow}\rangle = |{\rightarrow}\rangle\ . \tag{5.6}$$

The two states of the qbit just describe the presence or absence of the fermion in this state. We can rewrite the Pauli operators as

$$\mathbf{Z} = \mathbf{c} + \mathbf{c}^\dagger,$$

which flips the spin,

$$\mathbf{Y} = \frac{1}{\mathbf{i}}\left(\mathbf{c} - \mathbf{c}^\dagger\right),$$

$$\mathbf{X} = \mathbb{1} - 2\mathbf{c}^\dagger\mathbf{c}.$$

Here $\mathbf{c}^\dagger\mathbf{c}$ counts the number of fermions. Also note that the raising and lowering operators are $\boldsymbol{\sigma}^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = c^\dagger, \boldsymbol{\sigma}^- = c$ and the number operator $\mathbf{c}^\dagger\mathbf{c}$ (whose eigenvalues are 0 and 1) is $\mathbf{c}^\dagger\mathbf{c} = \frac{1}{2}\left(\mathbb{1} - \mathbf{X}\right)$.

The story is less simple with more than one fermion mode operator, because they must satisfy $\{\mathbf{c}_\alpha, \mathbf{c}_\beta\} = 0$, rather than just commuting, as the would if we just used the above map for each mode.

[0904.2771, Fradkin] Let's look at the TFIM again:

$$\mathbf{H}_{\text{TFIM}} = -J\sum_j \left(g\mathbf{X}_j + \mathbf{Z}_j\mathbf{Z}_{j+1}\right)$$

has a phase transition between large-g and small-g phases.

(Disordered) large g: excitations are created by $\mathbf{Z}_j$ – they are spin flips. The groundstate is a condensate of domain walls: $\langle\boldsymbol{\tau}^z\rangle \neq 0$.

(Ordered) small g: excitations are created by the 'disorder' operator $\boldsymbol{\tau}^z_{\tilde{j}} = \prod_{j>\tilde{j}}\mathbf{X}_j$ – they are domain walls. The groundstate is a condensate of spins $\langle\mathbf{Z}_j\rangle \neq 0$, *i.e.* a ferromagnet.

So we understand what are the 'correct variables' (in the sense that they create the elementary excitations above the groundstate) at large and small g. I claim that the Correct Variables *everywhere* in the phase diagram are obtained by "attaching a spin to a domain wall". These words mean the following: let

$$\begin{aligned} \boldsymbol{\chi}_j &\equiv \mathbf{Z}_j\boldsymbol{\tau}^z_{j+\frac{1}{2}} = \mathbf{Z}_j\prod_{j'>j}\mathbf{X}_{j'} \\ \tilde{\boldsymbol{\chi}}_j &\equiv \mathbf{Y}_j\boldsymbol{\tau}^z_{j+\frac{1}{2}} = -\mathbf{i}\mathbf{Z}_j\prod_{j'\geq j}\mathbf{X}_{j'} \end{aligned} \tag{5.7}$$

The first great virtue of this definition is that these operators agree with the creators of the elementary excitations in both regimes we've studied: When $g \ll 1$, $\langle \mathbf{Z}_j \rangle \simeq 1$ and more strongly, $\mathbf{Z}_j = \langle \mathbf{Z}_j \rangle + \text{small}$, so $\boldsymbol{\chi}_j \simeq \langle \mathbf{Z}_j \rangle \boldsymbol{\tau}^z_{j+\frac{1}{2}} \simeq \boldsymbol{\tau}^z_{j+\frac{1}{2}}$, the domain wall creation operator. Similarly, when $g \gg 1$, $\boldsymbol{\tau}^z_j \simeq 1 + \text{small}$, so $\boldsymbol{\chi}_j \simeq \mathbf{Z}_j \left\langle \boldsymbol{\tau}^z_{j+\frac{1}{2}} \right\rangle \simeq \mathbf{Z}_j$, which is the spin flipper on the paramagnetic vacuum.

Now let us consider the algebra of these $\boldsymbol{\chi}$s:

- They are *real*: $\boldsymbol{\chi}_j^\dagger = \boldsymbol{\chi}_j, \tilde{\boldsymbol{\chi}}_j^\dagger = \tilde{\boldsymbol{\chi}}_j$.

and

- They are *fermions*:

$$\text{if } i \neq j, \ \boldsymbol{\chi}_j \boldsymbol{\chi}_i + \boldsymbol{\chi}_i \boldsymbol{\chi}_j \equiv \{\boldsymbol{\chi}_j, \boldsymbol{\chi}_i\} = 0, \quad \{\tilde{\boldsymbol{\chi}}_j, \tilde{\boldsymbol{\chi}}_i\} = 0, \quad \{\boldsymbol{\chi}_j, \tilde{\boldsymbol{\chi}}_i\} = 0. \tag{5.8}$$

This is because the spin flip $\mathbf{Z}_j$ in $\boldsymbol{\chi}_j$ changes sign when it moves through the domain wall created by $\boldsymbol{\chi}_i$. When they are at the same site:

$$\boldsymbol{\chi}_j^2 = 1 = \tilde{\boldsymbol{\chi}}_j^2. \quad \text{In summary:} \quad \boxed{\{\boldsymbol{\chi}_i, \boldsymbol{\chi}_j\} = 2\delta_{ij}, \ \{\tilde{\boldsymbol{\chi}}_i, \tilde{\boldsymbol{\chi}}_j\} = 2\delta_{ij},}.$$

Notice that (5.8) means that $\boldsymbol{\chi}_i$ cares about $\boldsymbol{\chi}_j$ even if $|i-j| \gg 1$. Fermions are weird and non-local!

Notice that these fermions don't have a spin index. I should mention that this change of variables is called *Jordan-Wigner*.

Real fermion operators like this are called *Majorana* fermion operators. We can make more familiar-looking objects by making complex combinations:

$$\mathbf{c}_j \equiv \frac{1}{2}\left(\boldsymbol{\chi}_j - \mathbf{i}\tilde{\boldsymbol{\chi}}_j\right) \qquad \Longrightarrow \quad \mathbf{c}_j^\dagger = \frac{1}{2}\left(\boldsymbol{\chi}_j + \mathbf{i}\tilde{\boldsymbol{\chi}}_j\right)$$

These satisfy the more familiar anticommutation relations:

$$\{\mathbf{c}_i, \mathbf{c}_j^\dagger\} = \delta_{ij}, \quad \{\mathbf{c}_i, \mathbf{c}_j\} = 0, \quad \{\mathbf{c}_i^\dagger, \mathbf{c}_j^\dagger\} = 0,$$

and in particular, $\left(\mathbf{c}_i^\dagger\right)^2 = 0$, like a good fermion creation operator should.

We can write $\mathbf{H}_{\text{TFIM}}$ in terms of the fermion operators. We need to know how to write $\mathbf{X}_j$ and $\mathbf{Z}_j\mathbf{Z}_{j+1}$.

The operator which *counts* spin flips in the paramagnetic phase is

$$\mathbf{X}_j = -\mathbf{i}\tilde{\boldsymbol{\chi}}_j \boldsymbol{\chi}_j = -2\mathbf{c}_j^\dagger \mathbf{c}_j + 1 = (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j}.$$

(To get this we can use (5.7) and $\mathbf{YZ} = \mathbf{iX}$ and $\left(\boldsymbol{\tau}^z\right)^2 = 1$. Even better: notice that $\tilde{\boldsymbol{\chi}}_j = +\mathbf{i}X_j\boldsymbol{\chi}_j$.) Here $\mathbf{c}_j^\dagger \mathbf{c}_j = \mathbf{n}_j$ measures the number of fermions at the site j and is either 0 or 1, since they are fermions. At each site

$$|{\rightarrow_j}\rangle = |n_j = 0\rangle\,, \qquad |{\leftarrow_j}\rangle = |n_j = 1\rangle$$

like in the one-mode case, (5.6). The number of spin flips is the number of fermions.

The operator which counts domain walls is

$$\mathbf{Z}_j\mathbf{Z}_{j+1} = \mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j.$$

(Check: $\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j = \mathbf{i}\mathbf{Y}_{j+1}\prod_{k\geq j+2}\mathbf{X}_k\mathbf{Z}_j\prod_{l\geq j+1}\mathbf{X}_l = \left(\underbrace{\mathbf{i}\mathbf{Y}_{j+1}\mathbf{X}_{j+1}}_{=\mathbf{Z}_{j+1}}\right)\mathbf{Z}_j$.)

$$\mathbf{H}_{\text{TFIM}} = -J\sum_j\left(\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j + g\mathbf{i}\boldsymbol{\chi}_j\tilde{\boldsymbol{\chi}}_j\right)$$

is quadratic in these variables, for any g! Free at last! (It is quadratic in the $\mathbf{c}$s, too, since they are linear in the $\boldsymbol{\chi}$s.)

[End of Lecture 16]

Comments:

- Notice that the relation
$$\mathbf{X}_j = 1 - 2\mathbf{c}_j^\dagger\mathbf{c}_j$$
is exactly implementing the simple idea that *spinless* fermions on a lattice produce two-states per site which we can regard as spin up or spin down (in this case it's up or down along x): The states $\mathbf{X} = \pm 1$ correspond to $\mathbf{c}^\dagger\mathbf{c} = 0$ and 1 respectively.

- The symmetry operator is $\mathbf{S} = \prod_j \mathbf{X}_j = \prod_j(-1)^{\mathbf{c}_j^\dagger\mathbf{c}_j} = (-1)^F$ where F is the total fermion number.

- The description in terms of majoranas is preferred over the complex fermions because the phase rotation symmetry generated by the fermion number $\mathbf{c}^\dagger\mathbf{c}$ is not a symmetry of $\mathbf{H}_{\text{TFIM}}$ – in terms of the $\mathbf{c}$s, it contains terms of the form $\mathbf{c}_j\mathbf{c}_{j+1}$ which change the total number of $\mathbf{c}$ fermions (by ± 2). It is the hamiltonian for a *superconductor*, in which the continuous fermion number symmetry is broken down to a $\mathbb{Z}_2$ subgroup. Fermion number is still conserved mod two, and this is the $\mathbb{Z}_2$ symmetry of the Ising model, which acts by $\mathbf{Z} \to -\mathbf{Z}$.

- A useful thing to remember about majorana operators $\{\chi, \gamma\} = 0$ is that $(\mathbf{i}\chi\gamma)^\dagger = -\mathbf{i}\gamma\chi = +\mathbf{i}\chi\gamma$ is hermitian.

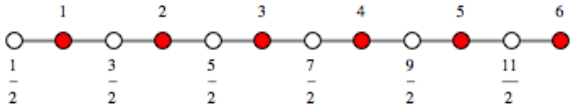
- Another useful fact:
$$\mathbf{c}(-1)^{\mathbf{c}^\dagger\mathbf{c}} = -\mathbf{c} \tag{5.9}$$
which is true because the BHS only nonzero if the number is nonzero *before* the annihilation operator acts, in which case we get $(-1)^1$. Similarly (the conjugate equation), $(-1)^{\mathbf{c}^\dagger\mathbf{c}}\mathbf{c}^\dagger = -\mathbf{c}^\dagger$, and $(-1)^{\mathbf{c}^\dagger\mathbf{c}}\mathbf{c} = \mathbf{c}$ and so on.

- This procedure of "attaching spin to a domain wall" led to fermions. This maybe isn't so surprising in one dimension. But there are analogs of this procedure in higher dimensions. In 2+1 dimensions, an analog is to attach charge to a vortex (or to attach magnetic flux to charge). This leads to transmutation of statistics from bosons to fermions and more generally to anyons and the fractional quantized Hall effect. In 3+1 dimensions, an analog is attaching charge to a magnetic monopole to produce a 'dyon'; in this case, the angular momentum carried by the EM fields is half-integer.

Dual fermions

Let

$$\boldsymbol{\gamma}_{j+\frac{1}{2}} = -\tilde{\boldsymbol{\chi}}_{j+1}, \quad \tilde{\boldsymbol{\gamma}}_{j+\frac{1}{2}} = \boldsymbol{\chi}_j.$$



We are dividing up the pair of majorana modes that we had previously associated with the site j between the two neighboring links.

In terms of these $\mathbf{H}$ is

$$\mathbf{H}_{\text{TFIM}} = -J \sum_{\bar{j}=j+\frac{1}{2}} \left(\mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}}\boldsymbol{\gamma}_{\bar{j}} + g\mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}+1}\boldsymbol{\gamma}_{\bar{j}} \right).$$

We have interchanged the form of the two terms. The two expressions are related by $\boldsymbol{\chi} \leftrightarrow \boldsymbol{\gamma}, J \leftrightarrow Jg, g \leftrightarrow 1/g$.

Now let's think about the two phases in terms of the fermion fields.

$\boxed{g \gg 1}$ The disordered state, with a unique groundstate is governed by

$$\mathbf{H}_{g\to\infty} = -Jg \sum_j \mathbf{i}\boldsymbol{\chi}_j\tilde{\boldsymbol{\chi}}_j = -Jg \sum_j (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j}.$$

The groundstate of this is just the state $|0\rangle$ with no fermions, $\mathbf{c}_j|0\rangle = 0$ for all j, on which $(-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j} = 1$.

$\boxed{g \ll 1}$

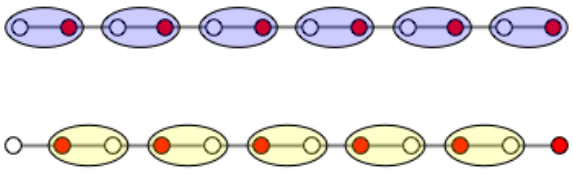
$$\mathbf{H}_{g\to 0} = -J \sum_j \mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j.$$

This is a hopping term for the χ fermions, which will be diagonalized by some band-structure, which will then be partially filled by fermions. It's easier just to use the dual fermion variables, in terms of which

$$\mathbf{H}_{g\to 0} = -J \sum_{\bar{j}=j+\frac{1}{2}} \mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}}\boldsymbol{\gamma}_{\bar{j}} = -J \sum_{\bar{j}=j+\frac{1}{2}} (-1)^{\check{\mathbf{c}}_{\bar{j}}^\dagger \check{\mathbf{c}}_{\bar{j}}}$$

where $\check{\mathbf{c}}_{\bar{j}} \equiv \frac{1}{2}\left(\boldsymbol{\gamma}_{\bar{j}} - \mathbf{i}\tilde{\boldsymbol{\gamma}}_{\bar{j}}\right)$ are the dual complex fermions. The groundstate of this is just the state with no dual fermions: $\check{\mathbf{c}}_{\bar{j}} \left|\check{0}\right\rangle = 0, \forall \bar{j}$.

Again we should think about the endpoint conditions. Consider what happens for an open chain. It's useful to account for the states as follows: think of dividing up each site into a pair of sites (white and black, as in the figure, one from the lattice, one from the dual lattice) with one majorana mode living at each. To help visualize, let's say $\tilde{\boldsymbol{\chi}}_j = -\boldsymbol{\gamma}_{j-\frac{1}{2}}$ lives at the white site (at $j - \frac{1}{2}$) and $\boldsymbol{\chi}_j = \tilde{\boldsymbol{\gamma}}_{j+\frac{1}{2}}$ lives at the red site to its right (at j). Note that $\boldsymbol{\gamma}_{\frac{1}{2}} = -\tilde{\boldsymbol{\chi}}_1, \tilde{\boldsymbol{\gamma}}_{\frac{1}{2}} = \boldsymbol{\chi}_0$ – the latter of which does not exist in a chain with open BC starting at $j = 1$. The g (para) term is a coupling between the white and red sites at fixed j. The ferro term couples a red site at $j - \frac{1}{2}$ to the next white site at j. This means that at small g, in an open chain, one site at each end will be *left out* – will not appear in the hamiltonian at all. This results in *unpaired* majorana zeromodes at the ends of the chain. Let $\mathbf{a}^\dagger \equiv \frac{1}{2}\left(\mathbf{i}\tilde{\boldsymbol{\chi}}_1 + \boldsymbol{\chi}_N\right)$. The algebra $\{\mathbf{a}, \mathbf{a}^\dagger\} = 1$ must be represented on the groundstates. This requires a pair of states



$$|0\rangle = |+\rangle\,, \;\; \mathbf{a}^\dagger\,|0\rangle = |-\rangle\,.$$

The SSB degeneracy of the ferromagnet is mapped by this fermionization map to a *topological* degeneracy in terms of the fermions.

Splitting of the energy these two states is small because the modes are separated in space and the bulk is gapped. The modes are separated by a distance much bigger than the correlation length, $\xi \sim \frac{1}{\Delta}$. The splitting comes from a term

$$\Delta\mathbf{H} = \epsilon\mathbf{a}^\dagger\mathbf{a} = \epsilon\mathbf{i}\tilde{\boldsymbol{\chi}}_1\boldsymbol{\chi}_N$$

and we (again) estimate that $\epsilon \sim e^{-N/\xi}$.

5.1.2 Continuum limit

[Sachdev, p. 139 of 2nd ed, p. 48 of 1st ed] We found above that a quantum lattice model (with continuous time and discrete space) can be Trotterized into a lattice statistical mechanics model (with discrete (euclidean) time and discrete space). We also found a reverse map, via the continuum scaling limit. In this subsection we discuss the analogous scaling limit that takes a quantum lattice model to a continuum quantum field theory (with continuous time *and* continuous space).

Label site j of the lattice by its position $x_j = ja$, where a is the lattice spacing. $L = Na$ is the length of the chain. Wavevectors lie in the interval $k \in (-\frac{\pi}{a}, \frac{\pi}{a}]$.

In terms of complex fermions

$$\mathbf{X}_j = 1 - 2\mathbf{c}_j^\dagger \mathbf{c}_j, \quad \mathbf{Z}_j = -\prod_{i>j}(1 - 2\mathbf{c}_i^\dagger \mathbf{c}_i)\left(\mathbf{c}_j + \mathbf{c}_j^\dagger\right) = -\prod_{i>j}(-1)^{\mathbf{c}_i^\dagger \mathbf{c}_i}\left(\mathbf{c}_j + \mathbf{c}_j^\dagger\right).$$

and their Fourier modes $\mathbf{c}_k \equiv \frac{1}{\sqrt{N}}\sum_j \mathbf{c}_j e^{-\mathbf{i}kx_j}$, the TFIM hamiltonian is

$$\mathbf{H}_{\text{TFIM}} = J\sum_k \left(2(g - \cos ka)\mathbf{c}_k^\dagger \mathbf{c}_k - \mathbf{i}\sin ka\left(\mathbf{c}_{-k}^\dagger \mathbf{c}_k^\dagger + \mathbf{c}_{-k}\mathbf{c}_k\right) - g\right)$$

Notice that the second set of terms violates fermion number conservation in units of two; this is the same statement that the ferromagnetic term creates spin flips in pairs. The constant at the end is irrelevant unless we are after the Onsager free energy.

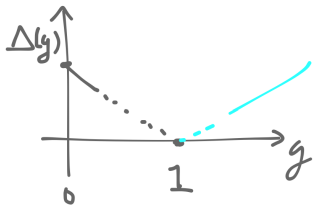
This Hamiltonian is quadratic in $\mathbf{c}_k$s, but not quite diagonal. The final solution for the spectrum involves one more operation the fancy name for which is 'Bogoliubov transformation', which is the introduction of new (complex) mode operators which mix particles and holes:

$$\boldsymbol{\gamma}_k = u_k \mathbf{c}_k - \mathbf{i}v_k \mathbf{c}_{-k}^\dagger$$

Demanding that the new variables satisfy canonical commutators $\{\boldsymbol{\gamma}_k, \boldsymbol{\gamma}_{k'}^\dagger\} = \delta_{k,k'}$ requires $u_k = \cos(\phi_k/2), v_k = \sin(\phi_k/2)$. We fix the angles ϕ_k by demanding that the hamiltonian in terms of $\boldsymbol{\gamma}_k$ be diagonal – no $\boldsymbol{\gamma}_k\boldsymbol{\gamma}_{-k}$ terms. The resulting condition is $\tan\phi_k = \frac{\epsilon_2(k)}{\epsilon_1(k)}$ with $\epsilon_1(k) = 2J(g - \cos ka), \epsilon_2(k) = -J\sin ka$, and $\mathbf{H} = \sum_k \epsilon_k\left(\boldsymbol{\gamma}_k^\dagger \boldsymbol{\gamma}_k - \frac{1}{2}\right)$, with $\epsilon_k = \sqrt{\epsilon_1^2 + \epsilon_2^2}$.

The end result is that the exact single-particle (single $\boldsymbol{\gamma}$) dispersion is

$$\epsilon_k = 2J\sqrt{1 + g^2 - 2g\cos ka}\ .$$



The argument of the sqrt is positive for $g \geq 0$. This is minimized at $k = 0$, which tells us the exact gap at all g:

$$\epsilon_k \geq \epsilon_0 = 2J|1 - g| = \Delta(g)$$

which, ridiculously, is just what we got from 1st order perturbation theory on each side of the transition.[30]

[30] Notice by the way that this 'single-particle excitation' in terms of the spins is a state with a single domain wall, and therefore only makes sense in the thermodynamic limit; in finite volume, the spin hilbert space only contains states with an even number of fermions. See §5.1.3 below for more on this point.

Comment on relation to Onsager. Notice that by the quantum-to-classical mapping, this solution has all the universal information of Onsager's solution. In particular, his exact free energy can be obtained by computing the groundstate energy of the fermions. (See the homework.)

The critical theory is scale invariant. At $g \to 1$, the fermions at $k = 0$ become gapless:

$$\epsilon_k \overset{g=g_c=1}{=} c|k|,$$

where the speed of propagation is $c = 2Ja$. There is an emergent Lorentz symmetry, at least in the spectrum, at the critical point. And the fact that the long-wavelength ($k \sim 0$) modes have the lowest energy allows a continuum description that forgets the lattice details.

Near $g \to g_c = 1$ (*i.e.* $|g - g_c| \ll g_c$),

$$\epsilon_k \overset{g\sim g_c=1}{=} c\sqrt{k^2 + \underbrace{\left(\frac{g-g_c}{a}\right)^2}_{m^2\to 0}}$$

There is a diverging length scale, which is $\xi = \frac{1}{m} = \frac{a}{|g-g_c|}$.

Some useful language:

- The *correlation length critical exponent* is ν in $\xi \sim |g - g_c|^{-\nu}$, the rate at which the correlation length diverges near the critical point. (Here $\nu = 1$.)
- The *dynamical critical exponent* is the power law in the dispersion *at* the critical point:

  $$\epsilon(k) \propto k^z. \tag{5.10}$$

  Notice that if we rescale space and time like

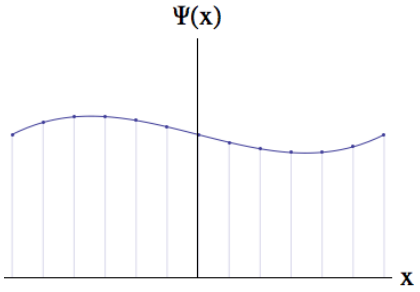
  $$x \to \lambda x, \qquad t \to \lambda^z t \tag{5.11}$$

  it will preserve the dispersion (5.10). (Here $z = 1$, as in a relativistic theory where time and space can be boosted into each other.)

We can expect the continuum description (at least the long-wavelength information) to be invariant under the scale transformation (5.11).

When $\xi \gg a$, define the continuum fermion field

$$\mathbf{\Psi}(x_j) \equiv \frac{1}{\sqrt{a}}\mathbf{c}_j$$



where the factor of $\sqrt{a}$ is designed to convert the kronecker deltas in

$$\{\mathbf{c}_j, \mathbf{c}_{j'}^\dagger\} = \delta_{jj'} \quad \text{into} \quad \{\mathbf{\Psi}(x), \mathbf{\Psi}^\dagger(x')\} = \delta(x - x').$$

The fourier transform is

$$\mathbf{c}_k = \int dx \frac{e^{-\mathbf{i}kx}}{\sqrt{L}} \mathbf{\Psi}(x) \ .$$

Let's plug this into $\mathbf{H}_{\text{TFIM}}$, focussing on (the lightest) modes with $ka \ll 1$.

$$J \sum_k (g - \cos ka) \mathbf{c}_k^\dagger \mathbf{c}_k \rightsquigarrow (g - g_c) \int dx \ \mathbf{\Psi}^\dagger(x) \mathbf{\Psi}(x)$$

Here the $\rightsquigarrow$ means we are dropping terms of higher order in the expansion in powers of $a\partial_x$.

$$-\mathbf{i}J \sum_k \sin ka \mathbf{c}_{-k}^\dagger \mathbf{c}_k^\dagger \rightsquigarrow \frac{c}{2} \int dx \ \mathbf{\Psi}(x)^\dagger \partial_x \mathbf{\Psi}(x)^\dagger$$

So

$$\mathbf{H} \rightsquigarrow \frac{c}{2} \int dx \ \left(\mathbf{\Psi}(x)^\dagger \partial_x \mathbf{\Psi}(x)^\dagger - \mathbf{\Psi}(x) \partial_x \mathbf{\Psi}(x) \right) + \Delta \int dx \mathbf{\Psi}^\dagger \mathbf{\Psi}$$

with $\Delta = 2J|g - 1|$.

[Fradkin, 2d ed p. 119] To understand the significance of this result it will be best to use the (fermion coherent state) path integral, which we'll do below. But we can learn something just by thinking about the Heisenberg equations of motion.

$$\begin{aligned} \mathbf{H}_{\text{TFIM}} &= -J \sum_j (g \mathbf{X}_j + \mathbf{Z}_j \mathbf{Z}_{j+1}) \\ &= -\mathbf{i}J \sum_j (g \boldsymbol{\chi}(j) \tilde{\boldsymbol{\chi}}(j) - \boldsymbol{\chi}(j) \tilde{\boldsymbol{\chi}}(j+1)) \ . \end{aligned} \tag{5.12}$$

The Heisenberg eom are $\mathbf{i}\partial_t \mathcal{O} = [\mathbf{H}, \mathcal{O}]$:

$$\begin{aligned} \mathbf{i}\partial_t \boldsymbol{\chi}(j) &= \mathbf{i}J \left(g \tilde{\boldsymbol{\chi}}(j) - \tilde{\boldsymbol{\chi}}(j+1) \right) \\ \mathbf{i}\partial_t \tilde{\boldsymbol{\chi}}(j) &= \mathbf{i}J \left(-g \boldsymbol{\chi}(j) + \boldsymbol{\chi}(j-1) \right) \end{aligned} \tag{5.13}$$

(Free fields means linear equations of motion.) In the continuum,

$$\boldsymbol{\chi}(j+1) \simeq \boldsymbol{\chi}(x_j) + a \partial_x \boldsymbol{\chi}(x_j) + \mathcal{O}(a^2 \partial_x^2)$$

so

$$\frac{1}{aJ} \partial_t \boldsymbol{\chi}(x) \simeq -\left(\frac{1-g}{aJ} \right) \tilde{\boldsymbol{\chi}}(x) - \partial_x \tilde{\boldsymbol{\chi}}(x)$$

$$\frac{1}{aJ}\partial_t\tilde{\boldsymbol{\chi}}(x) \simeq \left(\frac{1-g}{aJ}\right)\boldsymbol{\chi}(x) - \partial_x\boldsymbol{\chi}(x) \tag{5.14}$$

If we let $\boldsymbol{\chi}_\pm = \frac{1}{2}\left(\boldsymbol{\chi} \mp \tilde{\boldsymbol{\chi}}\right)$ and rescale the speed of propagation into the time variable, $t \equiv aJx^0$,

$$\begin{aligned}\partial_0\boldsymbol{\chi}_+ &= \partial_x\boldsymbol{\chi}_+ + m\boldsymbol{\chi}_- \\ \partial_0\boldsymbol{\chi}_- &= -\partial_x\boldsymbol{\chi}_- - m\boldsymbol{\chi}_+\end{aligned} \tag{5.15}$$

with $m = \frac{1-g}{a}$. When $m \to 0$, at the critical point, these are *chiral* fermions:

$$\left(\partial_0 \mp \partial_x\right)\boldsymbol{\chi}_\pm = 0.$$

The $\boldsymbol{\chi}$s are still real, so these are majorana fermions. In a language we will introduce soon, this is an example of a 1+1d CFT with central charge $c = (\frac{1}{2}, \frac{1}{2})$.

Even a bit away from the critical point, we can reconstruct the relativistic symmetry, and organize (5.15) into following Dirac equation

$$0 = \mathbf{i}\gamma^\mu\partial_\mu\boldsymbol{\chi} + \mathbf{i}m\boldsymbol{\chi},$$

Notice that a spin index has emerged to save the spin-statistics theorem. Here we have chosen a basis of gamma matrices where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(in the $\pm$ basis) which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ as they should. So the Dirac equation above is

$$0 = \mathbf{i}\begin{pmatrix} m & \partial_0 + \partial_x \\ -\partial_0 + \partial_x & m \end{pmatrix}\begin{pmatrix} \boldsymbol{\chi}_+ \\ \boldsymbol{\chi}_- \end{pmatrix} .$$

(The Majorana condition I am imposing is $\bar{\boldsymbol{\chi}} = \boldsymbol{\chi}^T\gamma^0$, aka $\boldsymbol{\chi}^\dagger = \boldsymbol{\chi}^T$, so $\boldsymbol{\chi}_\pm$ are real.)

5.1.3 Duality and Jordan-Wigner with periodic boundary conditions

How do I define the creation operator for a single domain wall with periodic boundary conditions on the spins? I can't. The fact that I can only make an even number means that the *sign* of the operator is not well-defined – in fact you can't measure the single-domain wall operator. And since the fermions $\boldsymbol{\chi}$ are made from this operator, you can't measure a single fermion operator either.

Although there is a sign ambiguity, we can still define the domain wall operators:

$$\boldsymbol{\tau}^x_{\bar{j}} = \mathbf{Z}_{\bar{j}-\frac{1}{2}}\mathbf{Z}_{\bar{j}+\frac{1}{2}}, \quad \boldsymbol{\tau}^z_{\bar{j}} = \prod_{N\geq j>\bar{j}} \mathbf{X}_j$$

– in the second definition, we have made a choice: even though $\mathbf{Z}_{N+1} = \mathbf{Z}_1$, we've decided to pick out the link where $\bar{j} = N + \frac{1}{2}$ to stop the string of $\mathbf{X}$s. As we saw in the case of open boundary conditions, this means that $\boldsymbol{\tau}_{\frac{1}{2}}$ and $\boldsymbol{\tau}_{N+\frac{1}{2}}$ are special:

$$\boldsymbol{\tau}^z_{N+\frac{1}{2}} = 1$$

since the product is over the empty set.

$$\boldsymbol{\tau}^z_{\frac{1}{2}} = \prod_{j=1}^{N} \mathbf{X}_j = \mathbf{S}$$

is the symmetry generator. This is significant because this operator commutes with $\mathbf{H}_{TFIM}$, it means that eigenstates of $\boldsymbol{\tau}^z_{\frac{1}{2}}$ are energy eigenstates. And notice that it determines the boundary conditions on the $\boldsymbol{\tau}$s in following sense: combining the previous two equations,

$$1 = \boldsymbol{\tau}^z_{N+\frac{1}{2}} = \begin{cases} \boldsymbol{\tau}^z_{\frac{1}{2}} & \text{if } \prod_j \mathbf{X}_j = \mathbf{S} = 1 \\ -\boldsymbol{\tau}^z_{\frac{1}{2}} & \text{if } \prod_j \mathbf{X}_j = \mathbf{S} = -1 \end{cases}$$

This operator $\boldsymbol{\tau}^z_{\frac{1}{2}}$ which says whether the $\boldsymbol{\tau}$s are periodic or antiperiodic is a dynamical variable.

5.1.4 Scaling near the fixed-point theory of the Ising phase transition.

Armed with the fermion coherent state path integral, let's go back to the continuum hamiltonian describing the TFIM near $g = 1$:

$$\mathbf{H} \rightsquigarrow \frac{c}{2}\int dx \ \left(\boldsymbol{\Psi}(x)^\dagger \partial_x \boldsymbol{\Psi}(x)^\dagger - \boldsymbol{\Psi}(x)\partial_x \boldsymbol{\Psi}(x)\right) + \Delta \int dx \boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \equiv \int dx \ \mathfrak{h}$$

with $\Delta = 2J|g-1|$.

The euclidean-time action is $S[\Psi, \Psi^\dagger] = \int d\tau \int dx \mathcal{L}$, with

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}\partial_\tau \Psi + \mathfrak{h} \\ &= \bar{\Psi}\partial_\tau \Psi + \frac{c}{2}\left(\bar{\Psi}\partial_x \bar{\Psi} - \Psi \partial_x \Psi\right) + \Delta \bar{\Psi}\Psi. \end{aligned} \tag{5.16}$$

Here I have simply replaced the field operators $\mathbf{\Psi}(x), \mathbf{\Psi}^\dagger(x)$ with their (right and left, respectively) grassmann eigenvalues $\Psi(x), \bar{\Psi}$. This lagrangian is more appealing in terms of the self-conjugate variables $\psi_\pm$:

$$\Psi = \psi_+ + \psi_- + \mathbf{i}\left(\psi_+ - \psi_-\right), \quad \bar{\Psi} = \psi_+ + \psi_- - \mathbf{i}\left(\psi_+ - \psi_-\right)$$

$$\mathcal{L} = \sum_\pm \psi_\pm \left(\partial_\tau \pm \mathbf{i}\partial_x\right)\psi_\pm + \Delta \psi_+ \psi_-.$$

If we let $z \equiv x + \mathbf{i}\tau$, and turn off the mass perturbation, this is the third avatar of the Ising model I mentioned at the beginning in (5.1).

Let's analyze its behavior under scale transformations. In order to make S invariant, we must scale time and space the same way ($z = 1$), and we must scale

$$x \to \lambda x, t \to \lambda t, \Psi \to \lambda^{-\frac{1}{2}}\Psi.$$

So just by scaling, we can see that its correlators behave as

$$\left\langle \mathbf{\Psi}(x)^\dagger \mathbf{\Psi}(0)\right\rangle \sim \frac{1}{x}.$$

(Since it's a gaussian theory, you can also calculate the coefficient.) More generally, the scaling dimension δ of an operator $\mathcal{O}(x)$ in $\mathcal{O} \to \lambda^{-\delta}\mathcal{O}$ in a scale-invariant QFT determines its vacuum autocorrelation functions to be

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0)\right\rangle \sim \frac{1}{x^{2\delta}}.$$

(Later we will interpret δ as an eigenvalue of the dilatation operator.)

The mass perturbation Δ will violate the scale invariance, since

$$\int dx d\tau \bar{\Psi}\Psi \to \lambda^{+1} \int dx d\tau \bar{\Psi}\Psi \ .$$

As we make lengths and times bigger, it gets bigger – it's a *relevant* perturbation, which determines the behavior at long distances. The scaling dimension of this relevant perturbation determines the correlation length critical exponent: the correlation length is a length scale $\xi = a\lambda$ at which the relevant coupling $\Delta = \lambda^\nu \Delta_0 \sim 1$ has evolved

to an order-one value (Δ_0 is its 'bare' value, at the lattice scale a, and ν is its scaling dimension). Eliminating $\lambda = \xi/a$ from these equations, we get

$$1 = (\xi/a)^\nu \Delta_0 \implies \xi \sim \frac{1}{\Delta_0^\nu},$$

so ν, the scaling dimension of the relevant operator, is indeed the correlation length critical exponent defined above.

Let's ask whether there are any other *relevant* operators, in the sense that they scale to larger values at larger λ (longer wavelengths). If there are more it means that our fixed point is *multicritical* – we would have to tune more than one parameter to reach it. We're going to demand that the Ising symmetry is preserved; this is the fermion number symmetry, $\Psi(x) \to -\Psi(x)$. So we should only consider operators made of an even number of fermions. And all the other operators we can make out of an even number of Ψ are *irrelevant*, in the sense that they scale to smaller values at larger λ. The marginal-looking operator $\bar\Psi\Psi\bar\Psi\Psi$ vanishes by Fermi statistics. Other operators have more derivatives and have no chance to be relevant:

$$\int dx d\tau \bar\Psi \partial_x^2 \Psi \sim \lambda^{-1}$$

$$\int dx d\tau \bar\Psi \partial_x \Psi \bar\Psi \partial_x \Psi \sim \lambda^{-2}$$

This means we have to tune only one parameter ($g - g_c \sim \Delta$) to get to the critical theory. (Since there are no classically-marginal operators, there is no need to do any perturbating to determine the effects of turning them on a little bit – small quantum corrections cannot overcome the order-1 amounts by which these operators are irrelevant.)

There *are* operators of dimension less than two, namely the operator $\mathbf{Z}$ itself; this is an operator which creates a *branch cut* in the fermions, which after all are made from the domain wall operators. To understand its dimension, it is useful to use a bosonized description, which we will do later (I hope). Its scaling dimension turns out to be 1/8. This is important because it determines the rate at which the magnetization vanishes as $g \to g_c$ from below. It's important that the $\mathbb{Z}_2$ symmetry forbids us from adding this (relevant) operator to the action. So in the presence of the symmetry, the Ising fixed point has only one relevant perturbation (the fermion mass).

Notice how different this critical theory is from the guess would have made from mean field theory: we would have guessed that the critical fluctuations should be those of the mean field θ, which when I set $g = g_c$ becomes a massless free boson, with many relevant operators which preserve the $\theta \to -\theta$ symmetry, since demanding scale

invariance of

$$S_0[\theta] = \int dx d\tau \ \left((\partial_\tau \theta)^2 + (\partial_x \theta)^2 \right)$$

implies that θ is dimensionlesss. So all even powers $\theta^2, \theta^4 ...$ (in fact, all even functions, like $\cos\theta$) have a chance to be relevant. We'll see that it's *not* actually quite so far off. (And in fact is the correct variables in $d > 1$, and the correct fixed point for $d \geq 3$ (the upper critical dimension).)

The nontrivial scaling of $\mathbf{Z}$ is an example of an *anomalous dimension*: the engineering dimension is different from the correct scaling behavior. Dimensional analysis is inviolable. What has happened is that powers of the short-distance scales are involved; that is: it is $J^{1/8}\mathbf{Z}$ (or $a^{-1/8}\mathbf{Z}$) that has finite correlation functions in the scaling limit. In relativistic QFT this phenomenon is called 'wavefunction renormalization' – we must multiplicatively renormalize our operators to get finite (not zero, not infinity) correlators in the long-wavelength limit.

5.1.5 Beyond the quantum Ising chain

The Jordan-Wigner transformation is useful in spin chains with many other choices of Hamiltonian and symmetry group, as you'll see on the problem set, and below.

5.1.6 Gauge theory formulation of the 1d duality

There is a systematic way to understand what's going on with the boundary conditions under the duality transformation, which generalizes to higher dimensions (but it is kind of overkill in this case). It is a reformulation of the TFIM as a lattice gauge theory. This is worth doing because in duality transformations in higher dimensions, the gauge theory formulation cannot be avoided. So this is a context where we can encounter the concepts of gauge theory without lots of other complications.

We begin by enlarging our system, by adding, in addition to the real qbits on the sites of the chain another set of fake qbits on the links:

$$\text{enlarge } \mathcal{H} = \otimes_j \mathcal{H}_j \quad \subset \quad \mathcal{H}_{\text{big}} \equiv (\otimes_j \mathcal{H}_j) \otimes \left(\otimes_{\bar{j}} \check{\mathcal{H}}_{\bar{j}}\right) \ .$$

Anticipating the answer, I will label the two states on a link $\bar{j}$ as $\left|\tau^z_{\bar{j}}\right\rangle = |\pm 1\rangle$. So as for any two-state system, associated with each link there are pauli operators

$$\boldsymbol{\tau}^z_{\bar{j}} \left|\tau^z_{\bar{j}}\right\rangle = \tau^z_{\bar{j}} \left|\tau^z_{\bar{j}}\right\rangle, \quad \boldsymbol{\tau}^x_{\bar{j}} \left|\tau^z_{\bar{j}}\right\rangle = \left|-\tau^z_{\bar{j}}\right\rangle \ , \ \boldsymbol{\tau}^z_{\bar{j}} \boldsymbol{\tau}^x_{\bar{j}'} = (-1)^{\delta_{\bar{j}\bar{j}'}} \boldsymbol{\tau}^x_{\bar{j}} \boldsymbol{\tau}^z_{\bar{j}'} \ .$$

We want to pick out a subspace of this larger fake Hilbert space which is the real one. One way to think about this is to declare that the following 'gauge transformation' is an equivalence:

$$\mathcal{G} : \tau^z_{\bar{j}} \to s_{\bar{j}} \tau^z_{\bar{j}}, \quad s_{\bar{j}} = \pm 1 \ . \tag{5.17}$$

Notice that this group $\mathcal{G}$ is big – we pick an element of $\mathbb{Z}_2$ (*i.e.* a sign) $s_{\bar{j}} = \pm 1$ for each link of the lattice, its order is $|\mathbb{Z}_2|^N = 2^N$. If a configuration of τ^zs is equivalent to its image under this map, we can just pick $s_{\bar{j}} = \tau^z_{\bar{j}}$ and this means that any configuration of τ^zs is equivalent to 1: we can just get rid of the τs and we are back at our original Hilbert space. This is called *unitary gauge*. Below we will implement this more concretely. But first we must decide how the gauge redundancy $\mathcal{G}$ acts on our original variables. It will be useful to choose

$$\mathcal{G} : \mathbf{X}_j \to s_{j-\frac{1}{2}} \mathbf{X}_j s_{j+\frac{1}{2}}, \quad \mathbf{Z}_j \to \mathbf{Z}_j. \tag{5.18}$$

This is the transformation law of a *link variable* in lattice gauge theory. The site of the original lattice is a link of the dual lattice. Let me use notation which emphasizes this:

$$\mathbf{X}_j \equiv \boldsymbol{\sigma}^x_{j-\frac{1}{2}, j+\frac{1}{2}}, \quad \mathbf{Z}_j \equiv \boldsymbol{\sigma}^z_{j-\frac{1}{2}, j+\frac{1}{2}} \ .$$

The *generator* of the gauge transformation is

$$\mathbf{G}_{\bar{j}} = \boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}} \boldsymbol{\tau}^x_{\bar{j}} \boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1}.$$

What I mean by 'generator' is that the action of the symmetry on states is $|\psi\rangle \to \mathbf{G}\,|\psi\rangle$ and the action on operators is

$$\mathcal{O} \to \mathbf{G}\mathcal{O}\mathbf{G}^\dagger.$$

The operator $\mathbf{G}_{\bar{j}}$ generates the transformation where $s_{\bar{j}} = -1$ and all the other $s_{\bar{j}'} = +1$. Check that this reproduces (5.17) and (5.18).

The physical hilbert space is cut out of $\mathcal{H}_{\text{big}}$ by projecting to eigenstates of these operators with eigenvalue 1:

$$\mathcal{H}_{\text{phys}} = \{|\psi\rangle \in \mathcal{H}_{\text{big}} \text{ such that } \mathbf{G}_{\bar{j}}\,|\psi\rangle = |\psi\rangle\,, \forall j\} \subset \mathcal{H}_{\text{big}}.$$

This is just the statement that physical states don't change under our made-up redundancy[31].

In order for this construction to make sense, our hamiltonian acting on the big hilbert space must be *gauge invariant*. That is, we must have

$$[\mathbf{H}, \mathbf{G}_{\bar{j}}] = 0, \quad \forall \bar{j}.$$

This way, we can simultaneously diagonalize $\mathbf{H}$ and $\mathbf{G}_{\bar{j}}$. A gauge invariant hamiltonian acting on the big hilbert space is:

$$\mathbf{H}_{\text{big}} - J\sum_{\bar{j}} \left(\boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}}\boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1} + g\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}\boldsymbol{\tau}^z_{\bar{j}+1}\right) \tag{5.19}$$

In unitary gauge, $\boldsymbol{\tau}^z = 1$, we can erase the $\boldsymbol{\tau}^z$s and this reduces to $\mathbf{H}_{\text{TFIM}}$. This is why I chose it.

On the other hand, using the local constraints

$$1 = \mathbf{G}_{\bar{j}} \;\Leftrightarrow\; \boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}}\boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1} = \boldsymbol{\tau}^x_{\bar{j}} \quad \text{on} \quad \mathcal{H}_{\text{phys}},$$

we have

$$\mathbf{H}_{\text{big}} = -J\sum_{\bar{j}} \left(\boldsymbol{\tau}^x_{\bar{j}} + g\boldsymbol{\tau}^z_{\bar{j}}\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}\boldsymbol{\tau}^z_{\bar{j}+1}\right)\ . \tag{5.20}$$

In this last expression, we still have the gauge freedom to play with – too many variables. Let's use it to get rid of the original $\boldsymbol{\sigma}$s. On an open chain, we can use the transformation

$$\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1} \to s_{\bar{j}}\boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}s_{\bar{j}+1} \tag{5.21}$$

to set $\boldsymbol{\sigma}^x = 1$ for each $\bar{j}$. In that case, the expression (5.20) precisely reduces to our dual description of the TFIM in terms of domain wall operators, (5.5).

[31] We can call this constraint the 'Gauss' Law constraint'. If you squint at this expression you can recognize it as a lattice version of the divergence of the electric flux coming out of the site $\bar{j}$.

But with periodic boundary conditions, the object

$$\mathbf{W} \equiv \prod_{\bar{j}=1}^{N} \boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1}$$

is a sign which does not change under the transformation (5.21). It is called a *Wilson line* operator. We can't get rid of it. In fact it is the familiar object

$$\mathbf{W} = \prod_j \mathbf{X}_j = \mathbf{S},$$

the $\mathbb{Z}_2$ symmetry charge.

You can see from (5.19) that if $\mathbf{W} = -1$, we have APBCs on $\boldsymbol{\tau}^z$. We can choose a gauge where we get rid of all the $\boldsymbol{\sigma}$s except the first, $\boldsymbol{\sigma}^z_{\frac{1}{2},\frac{3}{2}} = \mathbf{W}$. (This is a manifestation of the fact that we had to label a configuration of the spins by the location of the domain walls plus the value of one spin, which we are again choosing to be the first one.) Then the last link in the chain has an extra factor:

$$\mathbf{H}_{\text{dual}} = -J \sum_{\bar{j}=\frac{1}{2}}^{N-\frac{3}{2}} \left(\boldsymbol{\tau}^x_{\bar{j}} + g\boldsymbol{\tau}^z_{\bar{j}} \boldsymbol{\tau}^z_{\bar{j}+1} \right) + \boldsymbol{\tau}^x_{N-\frac{1}{2}} + \mathbf{W} g \boldsymbol{\tau}_{N-\frac{1}{2}} \boldsymbol{\tau}_{\frac{1}{2}}$$

Gauge theory without metaphysics: Kitaev's toric code in 1d

Since there seemed to be some discontent about the sudden appearance of gauge theory in the previous discussion, I will try to make amends here by giving an example of whence this construction can come. It is a 1d version of Kitaev's toric code. The analogous construction in higher dimensions is extremely important as a demonstration of the emergence of gauge theory from a local lattice model. This silly reduction of it also does that, but gauge theory in 1d is not quite as exciting as it is in higher dimensions.

Suppose I tell you that the following is a high-energy description of a lattice quantum system: The Hilbert space is

$$\mathcal{H}_{\rm big} \equiv (\otimes_j \mathcal{H}_j) \otimes \left(\otimes_{\bar{j}} \check{\mathcal{H}}_{\bar{j}}\right)$$

(nothing is fake, there is no redundancy, no nonsense, this is really the Hilbert space), and the Hamiltonian is

$$\mathbf{H}_{\rm TC} = \mathbf{H}_{\mathbf{G}} + \mathbf{H}_{\rm big}$$

where $\mathbf{H}_{\rm big} = -J \sum_{\bar{j}} \left(\boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}} \boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1} + g \boldsymbol{\tau}^z_{\bar{j}} \boldsymbol{\sigma}^x_{\bar{j},\bar{j}+1} \boldsymbol{\tau}^z_{\bar{j}+1} \right)$ as before, and the new term relative to the previous discussion is

$$\mathbf{H}_{\mathbf{G}} \equiv -E_{\rm giant} \sum_{\bar{j}} \mathbf{G}_{\bar{j}}$$

with $\mathbf{G}_{\bar{j}} = \boldsymbol{\sigma}^z_{\bar{j}-1,\bar{j}} \boldsymbol{\tau}^x_{\bar{j}} \boldsymbol{\sigma}^z_{\bar{j},\bar{j}+1}$ as before. Here $E_{\rm giant}$ is an energy scale which is a gajillion times bigger than any other energy in the problem ($E_{\rm giant} \gg J \gg gJ$).

A few relevant facts:

$$[\mathbf{H}_{\rm big}, \mathbf{G}_{\bar{j}}] = 0, \qquad [\mathbf{G}_{\bar{j}}, \mathbf{G}_{\bar{j}'}] = 0, \quad \forall \bar{j}, \bar{j}'.$$

This means that we can diagonalize $\mathbf{H}_{\rm big}$ by first diagonalizing $\mathbf{H}_{\mathbf{G}}$ (which is a good idea, since its coefficient is so giant), and further, we can do *that* by diagonalizing each $\mathbf{G}_{\bar{j}}$ at the same time.

So who are these $\mathbf{G}_{\bar{j}}$s? $\mathbf{G}_{\bar{j}}$ is unitary (so should be thought of as enacting a transformation on $\mathcal{H}_{\rm big}$), and, in this example, $\mathbf{G}^2_{\bar{j}} = 1$, it is an element of $\mathbb{Z}_2$, so its eigenvalues are ± 1. For each $\mathbf{G}_{\bar{j}}$, the eigenstate with eigenvalue $+1$ is the low-energy one.

Let us pause to note that this model actually has a *local symmetry*: the symmetry generated by $\mathbf{G}_{\bar{j}}$ acts only on the dofs at or next to $\bar{j}$, and this is a symmetry for any $\bar{j}$. *However*, with the sign of $E_{\rm giant}$ as indicated, all states on which this symmetry

actually acts nontrivially have an enormous energy $\sim E_{\text{giant}}$. This means that we may forget about them and focus on the low-energy subspace of the Hilbert space

$$\mathcal{H}_{\text{phys}} \equiv \{|\psi\rangle \in \mathcal{H}_{\text{big}} \text{ such that } \mathbf{G}_{\bar{j}} |\psi\rangle = |\psi\rangle \, , \forall j\} \subset \mathcal{H}_{\text{big}}.$$

You have seen this before, on the previous page: it is the Hilbert space of the $\mathbb{Z}_2$ gauge theory. We have emerged a gauge theory.

Now let me explain better what I was saying about "choosing a gauge". What I meant is the following. States of $\mathcal{H}_{\text{phys}}$ correspond to *orbits* of the action of $\{\mathbf{G}_{\bar{j}}\}$ on $\mathcal{H}_{\text{big}}$.[32] We can think of the projection to the $\mathbf{G} = 1$ subspace as *choosing a representative of each orbit* of the action of (each) $\mathbf{G}$.

Consider an (oversimplified) example of a single two-state system with $\mathbf{G} = \boldsymbol{\sigma}^z$. In the $\mathbf{X}$-basis, this symmetry action has two orbits:

$$\frac{1}{\sqrt{2}} (|\rightarrow\rangle + |\leftarrow\rangle) = |\uparrow\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}} (|\rightarrow\rangle - |\leftarrow\rangle) = |\downarrow\rangle$$

which are determined by the eigenvalue $\sigma^z = \uparrow, \downarrow$. If we *know* that we only want to keep the state with eigenvalue $+1$ then we can label an orbit by the first term. In this case there is only one orbit, and the projection is one-dimensional. We can describe this by saying that we have used the gauge redundancy to set $\sigma^x = 1$, and simply set $\boldsymbol{\sigma}^x = 1$ wherever it appears in our Hamiltonian.

More generally, we will have some action of a (local in space) group $g \in G$ of the form $\mathbf{U}_g$ on our $\mathcal{H}_{\text{big}}$. This means that $\mathcal{H}_{\text{big}}$ forms a (in general highly reducible) representation of G. The orbits of G are

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} \mathbf{U}_g |g_0\rangle$$

where $|G|$ is the order of the group G, and $|g_0\rangle$ is a reference state (like a highest-weight state in the theory of Lie groups). We may label the orbits by the reference

[32] Given a unitary operator on a Hilbert space and a state $|\psi_0\rangle$ in that space, what I mean by 'the orbit of $|\psi_0\rangle$ under the action of $\mathbf{U}$' is the set of images under repeated action of $\mathbf{U}$:

$$\text{orbit of } |\psi_0\rangle \text{ under action of } \mathbf{U} \equiv \{|\psi_0\rangle \, , \mathbf{U} |\psi_0\rangle \, , \mathbf{U}^2 |\psi_0\rangle \, ...\}$$

(if $\mathbf{U}$ represents a finite group, this contains a finite number of elements; call this order($\mathbf{U}$)). The associated eigenvector of $\mathbf{U}$ with eigenvalue 1 is

$$\frac{1}{\sqrt{\text{order}(\mathbf{U})}} \sum_{l=0}^{\text{order}(\mathbf{U})} \mathbf{U}^l |\psi_0\rangle \, .$$

It is just the average over the group. (For non-abelian groups, we must speak about the orbit of $|\psi_0\rangle$ under the action of the whole group.)

state. And more to the point, states that lie in the *same* orbit correspond to the *same* unit eigenvector, and can be regarded as equivalent. This is the origin of the usual way of thinking about gauge redundancy.

There is much more to say about the toric code in $d > 1$, for which I will teleologically refer to these notes. We have shown that this model reduces at energies $E \ll E_{\text{giant}}$ to the gauge theory description of the TFIM, which we have solved by the Jordan-Wigner transformation. The situation is a little bit different in 2d in that the model is not solvable away from the zero-correlation length limit.

Boundary conditions on fermions

How does this arise in terms of the fermions? Just as for domain walls, the sign of the fermion operator is also not well-defined. With periodic boundary conditions on the spins ($\mathbf{Z}_{j+N} = \mathbf{Z}_j$), we can also define the fermions by putting a *branch cut* at $j = N$, that is, we'll write the Jordan-Wigner formula as $\boldsymbol{\chi}_j = \mathbf{Z}_j \prod_{N \geq i > j} \mathbf{X}_i$, where the string of $\mathbf{X}_i$s stops at N. This is a choice. The inverse map is then

$$\mathbf{Z}_j = \left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right) \prod_{N \geq i > j} e^{\mathbf{i}\pi\mathbf{n}_i} \ . \tag{5.22}$$

The ferromagnetic term in the hamiltonian, for $j \neq N$, is

$$\mathbf{Z}_j\mathbf{Z}_{j+1} = \left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right) e^{\mathbf{i}\pi\mathbf{n}_{j+1}} \left(\mathbf{c}_{j+1}^\dagger + \mathbf{c}_{j+1}\right) = \left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right) \left(-\mathbf{c}_{j+1}^\dagger + \mathbf{c}_{j+1}\right)$$

where I used the identity (5.9). (This is as in our previous discussion.) But with periodic boundary conditions on $\mathbf{Z}$, the term at the branch cut is:

$$\mathbf{Z}_N\mathbf{Z}_1 = \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right) \left(\mathbf{c}_1^\dagger + \mathbf{c}_1\right) \prod_{N \geq j > 1} e^{\mathbf{i}\pi\mathbf{n}_j} \ .$$

This last red factor includes *all* of the sites except the first:

$$\prod_{N \geq j > 1} e^{\mathbf{i}\pi\mathbf{n}_j} = \prod_{j=1}^{N} e^{\mathbf{i}\pi\mathbf{n}_j} e^{\mathbf{i}\pi\mathbf{n}_1} = e^{\mathbf{i}\pi\mathbf{N}} e^{\mathbf{i}\pi\mathbf{n}_1} \equiv (-1)^{\mathbf{N}} e^{\mathbf{i}\pi\mathbf{n}_1} \ .$$

Here $\mathbf{N} \equiv \sum_j \mathbf{c}_j^\dagger \mathbf{c}_j$ operator which counts the total number of fermions, and $(-1)^{\mathbf{N}}$ is called the 'fermion parity'[33]. This is a symmetry generator: $[\mathbf{H}, (-1)^{\mathbf{N}}] = 0$, since $\mathbf{H}$ only contains terms with an even number of fermions. Returning to the outlying ferromagnetic coupling,

$$\begin{aligned} \mathbf{Z}_N\mathbf{Z}_1 &= \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right) \underbrace{\left(\mathbf{c}_1^\dagger + \mathbf{c}_1\right) e^{\mathbf{i}\pi\mathbf{n}_1}}_{=\mathbf{c}_1^\dagger - \mathbf{c}_1} (-1)^{\mathbf{N}} \\ &= \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right) \left(\mathbf{c}_1^\dagger - \mathbf{c}_1\right) (-1)^{\mathbf{N}}. \end{aligned} \tag{5.23}$$

This is the same as the other terms, but for a factor of $-(-1)^{\mathbf{N}}$. This means that we get periodic boundary conditions on $\mathbf{c}$, *i.e.* $\mathbf{c}_{N+1} = +\mathbf{c}_1$ if the total number of fermions is *odd* and antiperiodic boundary conditions, $\mathbf{c}_{N+1} = -\mathbf{c}_1$, in sectors where $\mathbf{N}$ is even. In summary[34] :

$$\boxed{\mathbf{c}_{N+1} = -\mathbf{c}_1(-1)^{\mathbf{N}}} \quad .$$

[33] please don't confuse the number operator $\mathbf{N}$ with the number of sites N – sorry about that

[34] This fact is perhaps simpler in terms of the majoranas:

$$\text{for } j < N, \ -\mathbf{i}\tilde{\boldsymbol{\chi}}_{j+1}\boldsymbol{\chi}_j = \mathbf{Z}_j\mathbf{Z}_{j+1}, \quad \text{but} \quad -\mathbf{i}\tilde{\boldsymbol{\chi}}_N\boldsymbol{\chi}_1 = \mathbf{Z}_N\mathbf{Z}_1\,(-\mathbf{S})\,.$$

So the full Hamiltonian is

$$\frac{1}{J}\mathbf{H}_{\text{TFIM}}^{\text{PBC on }\mathbf{z}} = -\sum_{j=1}^{N-1}\left(\mathbf{c}_j^\dagger + \mathbf{c}_j\right)\left(\mathbf{c}_{j+1}^\dagger - \mathbf{c}_{j+1}\right) - \left(\mathbf{c}_N^\dagger + \mathbf{c}_N\right)\left(\mathbf{c}_1^\dagger - \mathbf{c}_1\right)(-1)^{\mathbf{N}} - 2g\underbrace{\sum_j \mathbf{c}_j^\dagger \mathbf{c}_j}_{=\mathbf{N}}.$$

We can diagonalize this by fourier transformation

$$\mathbf{c}_j = \frac{1}{\sqrt{N}}\sum_k e^{\mathbf{i}kja}\mathbf{c}_k$$

where the allowed values of k depend on the boundary conditions on $\mathbf{c}$:

$$\begin{aligned} \text{PBC,}\quad (-1)^{\mathbf{N}} = -1: &\qquad k \in \frac{2\pi}{Na}\{1...N\} \\ \text{APBC,}\quad (-1)^{\mathbf{N}} = +1: &\qquad k \in \frac{2\pi}{Na}\left(\frac{1}{2} + \{1...N\}\right) \end{aligned} \tag{5.24}$$

And the Hamiltonian in each case is:

$$\mathbf{H}/J = \sum_k \left(\mathbf{c}_k^\dagger \mathbf{c}_k\left(2g - 2\cos ka\right) + \left(\mathbf{c}_k \mathbf{c}_{-k}\mathbf{i}\sin ka + h.c.\right)\right) \equiv \sum_k \mathbf{h}_k$$

The dynamics determine the boundary conditions: is it less energy to have an odd number of fermions or an even number? First notice that the dispersion has an inversion symmetry $k \to -k$, so each k value is degenerate in energy with $-k$, unless it is fixed by this map. The fixed points are $k = 0, \pi/a$ (recall that $ka = -\pi \simeq -\pi + 2\pi = \pi$). These only occur for PBC.

The twofold (in particular even) degeneracy for most k means that the occupation $\mathbf{N}_{k\neq -k}$ only changes by two, and we don't care about the associated $\mathbf{h}_k$. More algebraically, for PBC,

$$\mathbf{N} = \sum_j \mathbf{c}_j^\dagger \mathbf{c}_j = 2\sum_{k\neq 0,\pi} \mathbf{c}_k^\dagger \mathbf{c}_k + \mathbf{c}_0^\dagger \mathbf{c}_0 + \mathbf{c}_\pi^\dagger \mathbf{c}_\pi$$

(for APBC we can simply omit the last term and conclude that $\mathbf{N}$ is even, so $(-1)^{\mathbf{N}} = 1$.) So for determining $(-1)^{\mathbf{N}}$, we only care about $\mathbf{h}_{ka=0}$ and $\mathbf{h}_{ka=\pi}$.

$$\mathbf{h}_{ka=\pi} = \mathbf{c}_\pi^\dagger \mathbf{c}_\pi\left(2g + 2\right) > 0 \quad \Longrightarrow \quad \text{empty in the groundstate}, \forall g \geq 0$$

As usual, the zero-momentum mode is the hero of the story:

$$\mathbf{h}_{ka=0} = \mathbf{c}_0^\dagger \mathbf{c}_0\left(2g - 2\right)$$

which *changes sign* at $g = g_c = 1$:

- For $g > g_c$,, $\mathbf{h}_0 > 0$ is a positive energy cost for filling the zeromode. So the groundstate for $g > g_c$ has this odd mode empty and hence $(-1)^{\mathbf{N}} = 1$, and APBCs.

- For $g < g_c$,, $\mathbf{h}_0 < 0$ is an incentive for filling the zeromode, of which there is odd number (namely, 1), so the system can lower its energy by having PBC and filling the $k = 0$ mode. In the thermodynamic limit $L = Na \to \infty$, these two groundstates become degenerate.

Comments on the splitting between the groundstates

Comments about the L-dependence of the splitting between these states:

(1) We would like to identify them with (linear combinations of) the two ferromagnetic groundstates, which become the fully-polarized (in $\mathbf{Z}$) states $|+\rangle$ and $|-\rangle$ as $g \to 0$. As we've argued from many other points of view, these states are split by an amount which is exponentially small in L: $\Delta E \sim e^{-L/\xi}$.

(2) *At the critical point*, where the fermions are massless, the energy splitting between the groundstates with PBC and APBC (in the scaling limit, where $a \to 0$), is an oft-used result in string theory, and is $\frac{1}{16}\frac{\hbar c}{L}$ (PBC is higher energy). The L-dependence is determined by dimensional analysis, since there is no other scale. Away from the critical point I don't know the answer at the moment.

(3) Note that these two sectors with different $(-1)^{\mathbf{N}}$ do not mix via $\mathbf{H}_{\text{TFIM}}$, since $(-1)^{\mathbf{N}}$ is a symmetry generator; you can't create a single fermion using $\mathbf{H}_{\text{TFIM}}$. There is *no* tunneling between these vacua. So these two states must in fact be the cat states $\frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$ which are eigenvectors of the $\mathbb{Z}_2$ symmetry of the TFIM, which, after all, is

$$\mathbf{S} = \prod_{j=1}^{N} \mathbf{X}_j = \prod_{j=1}^{N} (-1)^{\mathbf{c}_j^\dagger \mathbf{c}_j} = (-1)^{\mathbf{N}}.$$

(Recall that the symmetric cat state $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ is the ground state in finite volume, so this must be the fermion state with antiperiodic boundary conditions.) Being forced to superpose states with different boundary conditions may be discomfiting, but it is part of the resolution of the puzzle of an odd number of domain walls on a circle.

[End of Lecture 17]

5.2 XY transition from superfluid to Mott insulator, and T-duality

In this subsection (and another one later) we're going to study ways to think about bosonic field theories with a $\mathsf{U}(1)$ symmetry, and dualities between them, in $D = 1+1$ and $D = 2+1$.

[This discussion is from Ashvin Vishwanath's lecture notes.] Consider the Bose-Hubbard model (in any dimension, but we'll specify to $D = 1+1$ at some point)

$$\mathbf{H}_{BH} = -\tilde{J}\sum_{\langle ij\rangle}\left(\mathbf{b}_i^\dagger \mathbf{b}_j + h.c.\right) + \frac{U}{2}\sum_i \mathbf{n}_i\left(\mathbf{n}_i - 1\right) - \mu\sum_i \mathbf{n}_i$$

where the $\mathbf{b}^\dagger$s and $\mathbf{b}$ are bosonic creation and annihilation operators at each site: $[\mathbf{b}_i, \mathbf{b}_j^\dagger] = \delta_{ij}$. $\mathbf{n}_i \equiv \mathbf{b}_i^\dagger \mathbf{b}_i$ counts the number of bosons at site i. The second Hubbard-U term is zero if $\mathbf{n}_j^b = 0, 1$, but exacts an energetic penalty $\Delta E = U$ if a single site j is occupied by two bosons. This model has a $\mathsf{U}(1)$ symmetry generated by $\mathbf{n}_j$, $\mathbf{U}(\theta) = e^{\mathbf{i}\theta\sum_j \mathbf{n}_j}$.

Notice that if we take the limit $U \to \infty$, there are exactly two states per site, and this reduces to a spin system. At each site, we can write $\mathbf{b} = S^+, \mathbf{b}^\dagger = S^-, \mathbf{b}^\dagger\mathbf{b} = \frac{1}{2}(1 - S^z)$. In fact it is the XY model:

$$\mathbf{H}_{XY} = -\frac{w}{2}\sum_{\langle ij\rangle}\left(\mathbf{X}_i\mathbf{X}_j + \mathbf{Y}_i\mathbf{Y}_j\right) + \frac{\mu}{2}\sum_j \mathbf{Z}_j.$$

In these variables, the $\mathsf{U}(1)$ boson-number symmetry is rotations about $\mathbf{Z}$: $\mathbf{U}(\theta) = \prod_j e^{\mathbf{i}\frac{\theta}{2}\mathbf{Z}_j}$. In $D = 1+1$, this model is in fact solvable by Jordan-Wigner, as you can see on the homework.

The Hilbert space that represents the boson algebra has a useful *number-phase* representation in terms of

$$[\mathbf{n}_i, \phi_j] = -\mathbf{i}\delta_{ij}, \quad \phi_i \equiv \phi_i + 2\pi, \quad \mathbf{n}_i \in \mathbb{Z}$$

(where the last statement pertains to the eigenvalues of the operator). The bosons are

$$\mathbf{b}_i = e^{-\mathbf{i}\phi_i}\sqrt{\mathbf{n}_i}, \quad \mathbf{b}_i^\dagger = \sqrt{\mathbf{n}_i}e^{+\mathbf{i}\phi_i}\ ;$$

these expressions have the same algebra as the original $\mathbf{b}$s. In terms of these operators, the hamiltonian is

$$\mathbf{H}_{BH} = -\tilde{J}\sum_{\langle ij\rangle}\left(\sqrt{\mathbf{n}_i}e^{\mathbf{i}(\phi_i - \phi_j)}\sqrt{\mathbf{n}_j} + h.c.\right) + \frac{U}{2}\sum_i \mathbf{n}_i\left(\mathbf{n}_i - 1\right) - \mu\sum_i \mathbf{n}_i.$$

If $\langle \mathbf{n}_i \rangle = n_0 \gg 1$, so that $\mathbf{n}_i = n_0 + \Delta\mathbf{n}_i, \Delta\mathbf{n}_i \ll n_0$ then $\mathbf{b}_i = e^{-\mathbf{i}\phi}\sqrt{\mathbf{n}_i} \simeq e^{-\mathbf{i}\phi_i}\sqrt{n_0}$ and

$$\mathbf{H}_{BH} \simeq -\underbrace{2\tilde{J}n_0}_{\equiv J}\sum_{\langle ij\rangle}\cos\left(\phi_i - \phi_j\right) + \frac{U}{2}\sum_i \left(\Delta\mathbf{n}_i\right)^2 \equiv \mathbf{H}_{\text{rotors}}$$

where we set $n_0 \equiv \mu/U \gg 1$. This is a *rotor model*.

This model has two phases:

$\boxed{U \gg J}$: then we must satisfy the U term first and the number is locked, $\Delta\mathbf{n} = 0$ in the groundstate. This is a Mott insulator, with a gap of order U. Since $\mathbf{n}$ and ϕ are conjugate variables, definite number means wildly fluctuating phase.

$\boxed{U \ll J}$: then we must satisfy the J term first and the phase is locked, $\phi =$ constant in the groundstate, or at least it will try. This is the superfluid (SF). That is, we can try to expand the cosine potential[35]

$$\mathbf{H}_{\text{rotors}} = U\sum_i \mathbf{n}_i^2 - J\sum_{\langle ij\rangle}\cos\left(\phi_i - \phi_j\right) \simeq U\sum_i \mathbf{n}_i^2 - J\sum_{\langle ij\rangle}\left(1 - \frac{1}{2}\left(\phi_i - \phi_j\right)^2 + ...\right) \tag{5.25}$$

which is a bunch of harmonic oscillators and can be solved by Fourier: $\phi_i = \frac{1}{\sqrt{N^d}}\sum_i e^{-\mathbf{i}k\cdot x_i}\phi_k$, so

$$\mathbf{H} \simeq \sum_k \left(U\pi_k\pi_{-k} + J\left(1 - \cos ka\right)\phi_k\phi_{-k}\right)$$

This has gapless *phonon* modes at $k = 0$, whose existence is predicted by Nambu-Goldstone. I have written the hamiltonian in 1d notation but nothing has required it so far. The low energy physics is described by the continuum lagrangian density

$$L_{\text{eff}} = \frac{\rho_s}{2}\left(\frac{\left(\partial_\tau\phi\right)^2}{c} + c\left(\vec{\nabla}\phi\right)^2\right) \tag{5.26}$$

with $\rho_s = \sqrt{J/U}, c = \sqrt{JU}$. ρ_s is called the superfluid stiffness. This is a free massless scalar theory. The demand of the $\mathsf{U}(1)$ symmetry $\phi \to \phi + \alpha$ forbids interactions which would be relevant; the only allowed interactions are derivative interactions (as you can see by keeping more terms in the Taylor expansion (5.25)) such as $(\partial\phi)^4$.

Now 1d comes in: In $d > 1$, there is long range order – the bosons condense and spontaneously break the phase rotation symmetry $\phi \to \phi + \alpha$; the variable ϕ is a Goldstone boson. In 1d there is no long-range order. The two phases are still distinct however, since one has a gap and the other does not. The correlators of the boson operator $b_i \sim e^{\mathbf{i}\phi_i}$ diagnose the difference. In the Mott phase they have exponential decay. In the "SF" they have

$$\left\langle e^{\mathbf{i}\phi(x)}e^{-\mathbf{i}\phi(y)}\right\rangle = \frac{c_0}{r^\eta}, \quad \eta = \frac{1}{2\pi\rho_s} = \frac{1}{2K}.$$

[35] From now on the background density n_0 will not play a role and I will write $\mathbf{n}_i$ for $\Delta\mathbf{n}_i$.

This is *algebraic long range order*. This is a sharp distinction between the two phases we've discussed, even though the IR fluctuations destroy the $\langle b \rangle$.

Massless scalars in $D = 1+1$ and T-duality-invariance of the spectrum. A lot of physics is hidden in the innocent-looking theory of the superfluid goldstone boson. Consider the following (real-time) continuum action for a free massless relativistic scalar field in 1+1 dimensions:

$$S[\phi] = \frac{T}{2} \int dt \int_0^L dx \left((\partial_0 \phi)^2 - (\partial_x \phi)^2 \right) = 2T \int dx dt \partial_+ \phi \partial_- \phi \ . \tag{5.27}$$

I have set the velocity of the bosons to $c = 1$ by rescaling t. Here $x^\pm \equiv t \pm x$ are lightcone coordinates; the derivatives are $\partial_\pm \equiv \frac{1}{2}(\partial_t \pm \partial_x)$. Space is a circle: the point labelled x is the same as the point labelled $x + L$. It will sometimes be useful to call actual space the 'base space', to distinguish it from the field space, aka the 'target space'. This denotation is common in the study of nonlinear sigma models, which are field theories of maps from the base space to the target space.

We will assume that the field space of ϕ itself is periodic:

$$\phi(x,t) \equiv \phi(x,t) + 2\pi, \quad \forall x, t \ .$$

So the *field* space is a circle S^1 with (angular) coordinate ϕ. It can be useful to think of the action (5.27) as describing the propagation of a string, since a field configuration describes an embedding of the real two dimensional space into the *target space*, which here is a circle. This is a simple special case of a nonlinear sigma model. The name T-duality comes from the literature on string theory. The worldsheet theory of a string propagating on a circle of radius $R = \sqrt{\rho_s}$ is governed by the Lagrangian (5.26). To see this, recall that the action of a 2d nonlinear sigma model with target space metric $g_{\mu\nu}\phi^\mu\phi^\nu$ is $\frac{1}{\alpha'}\int d^2\sigma g_{\mu\nu}\partial\phi^\mu\partial\phi^\nu$. Here $\frac{1}{\alpha'}$ is the tension (energy per unit length) of the string; work in units where this disappears from now on. Here we have only one target space dimension, with $g_{\phi\phi} = \rho_s$.

Notice that we could rescale $\phi \to \lambda\phi$ and change the radius; but this would change the periodicity of $\phi \equiv \phi + 2\pi$. The proper length of the period is $2\pi R$ and is invariant under a change of field variables. This proper length distinguishes different theories because the operators $: e^{\alpha\phi} :$ (and all good operators of definite scaling dimension in the theory of the free boson (unlike ϕ itself)) must be periodic; this determines the allowed values of α.

First a little bit of classical field theory. The equations of motion for ϕ are

$$0 = \frac{\delta S}{\delta \phi(x,t)} \propto \partial^\mu \partial_\mu \phi \propto \partial_+ \partial_- \phi$$

which is solved by

$$\phi(x,t) \equiv \phi_L(x^+) + \phi_R(x^-) \ .$$

In euclidean time, $\phi_{L,R}$ depend (anti-)holomorphically on the complex coordinate $z \equiv \frac{1}{2}(x + \mathbf{i}\tau)$ and the machinery of complex analysis becomes useful.

Symmetries: Since $S[\phi]$ only depends on ϕ through its derivatives, there is a simple symmetry $\phi \to \phi + \epsilon$. By the Nöther method the associated current is

$$j_\mu = T\partial_\mu \phi \ . \tag{5.28}$$

This symmetry is translations in the target space, and so I will sometimes call the associated conserved charge 'momentum'.

There is another symmetry which is less obvious. It comes about because of the topology of the target space. Since $\phi(x,t) \equiv \phi(x,t) + 2\pi m, m \in \mathbb{Z}$ describe the same point (it is a redundancy in our description, in fact a discrete gauge redundancy), we don't need $\phi(x+L,t) \stackrel{?}{=} \phi(x,t)$. To make the field configuration single-valued, it is enough to have

$$\phi(x+L,t) = \phi(x,t) + 2\pi m, \quad m \in \mathbb{Z}$$

The number m cannot change without the string breaking: it is a topological charge, a winding number:

$$m = \frac{1}{2\pi}\phi(x,t)|_{x=0}^{x=L} \stackrel{\text{FTC}}{=} \frac{1}{2\pi}\int_0^L dx \partial_x \phi \ . \tag{5.29}$$

The associated current whose charge density is $\frac{1}{\pi}\partial_x\phi$ (which integrates over space to the topological charge) is

$$\tilde{j}_\mu = \frac{1}{2\pi}(\partial_x\phi, -\partial_0\phi)_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\phi. \tag{5.30}$$

This is conserved because of the equality of the mixed partials: $\epsilon^{\mu\nu}\partial_\mu\partial_\nu = 0$.

Let's expand in normal modes: $\phi = \phi_L + \phi_R$ with

$$\begin{aligned}
\phi_L(t+x) &= q_L + \underbrace{(p+w)}_{\equiv \frac{1}{2T}p_L}(t+x) - \mathbf{i}\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0}\frac{\rho_n}{n}e^{\mathbf{i}n(t+x)\frac{2\pi}{L}}, \\
\phi_R(t-x) &= q_R + \underbrace{(p-w)}_{\equiv \frac{1}{2T}p_R}(t-x) - \mathbf{i}\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0}\frac{\tilde{\rho}_n}{n}e^{\mathbf{i}n(t-x)\frac{2\pi}{L}},
\end{aligned} \tag{5.31}$$

The factor of $\frac{1}{n}$ is a convention whose origin you will appreciate below, as are the other normalization factors. Real ϕ means $\rho_n^\dagger = \rho_{-n}$ (If we didn't put the $\mathbf{i}$ it would have been $-\rho_{-n}$).

Here $q \equiv \frac{1}{L}\int_0^L dx\phi(x,t) = q_L + q_R$ is the center-of-mass position of the string. The canonical momentum for ϕ is $\pi(x,t) = T\partial_0\phi(x,t) = T\left(\partial_+\phi_L + \partial_-\phi_R\right)$.

QM. Now we'll do quantum mechanics. Recall that a quantum mechanical particle on a circle has momentum quantized in units of integers over the period. Since ϕ is periodic, the wavefunction(al)s must be periodic in the center-of-mass coordinate q with period 2π, and this means that the total (target-space) momentum must be an integer

$$\mathbb{Z} \ni j = \pi_0 \equiv \int_0^L dx\pi(x,t) = T\int_0^L dx\partial_t\phi \overset{(5.31)}{=} LT2p$$

So our conserved charges are quantized according to

$$p = \frac{j}{2LT}, \qquad w \overset{(5.31)(5.29)}{=} \frac{\pi m}{L}\ , \qquad j,m \in \mathbb{Z}\ .$$

(Don't confuse the target-space momentum j with the 'worldsheet momentum' n!)

(Note that this theory is scale-free. We could use this freedom to choose units where $L = 2\pi$.)

Now I put the mode coefficients in boldface:

$$\begin{aligned}
\boldsymbol{\phi}_L(x^+) &= \mathbf{q}_L + \frac{1}{2T}\mathbf{p}_L x^+ - \mathbf{i}\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0}\frac{\boldsymbol{\rho}_n}{n}e^{\mathbf{i}\frac{2\pi}{L}nx^+}, \\
\boldsymbol{\phi}_R(x^-) &= \mathbf{q}_R + \frac{1}{2T}\mathbf{p}_R x^- - \mathbf{i}\sqrt{\frac{L}{4\pi T}}\sum_{n\neq 0}\frac{\tilde{\boldsymbol{\rho}}_n}{n}e^{\mathbf{i}\frac{2\pi}{L}nx^-},
\end{aligned} \tag{5.32}$$

The nonzero canonical equal-time commutators are

$$[\boldsymbol{\phi}(x), \boldsymbol{\pi}(x')] = \mathbf{i}\delta(x-x')$$

which determines the commutators of the modes (this was the motivation for the weird normalizations)

$$[\mathbf{q}_L, \mathbf{p}_L] = [\mathbf{q}_R, \mathbf{p}_R] = \mathbf{i}, \qquad [\boldsymbol{\rho}_n, \boldsymbol{\rho}_{n'}^\dagger] = n\delta_{n,n'}, \text{or}\ \ [\boldsymbol{\rho}_n, \boldsymbol{\rho}_{n'}] = n\delta_{n+n'}\ ,$$

and the same for the rightmovers with twiddles (and $[\boldsymbol{\rho}, \tilde{\boldsymbol{\rho}}] = 0$). This is one simple harmonic oscillator for each $n \geq 1$ (and each chirality); the funny normalization is conventional.

$$\begin{aligned}
\mathbf{H} &= \int dx\left(\boldsymbol{\pi}(x)\dot{\boldsymbol{\phi}}(x) - \mathcal{L}\right) = \frac{1}{2}\int dx\left(\frac{\boldsymbol{\pi}^2}{T} + T\left(\partial_x\boldsymbol{\phi}\right)^2\right) \\
&= L\underbrace{\frac{1}{4T}\left(\mathbf{p}_L^2 + \mathbf{p}_R^2\right)}_{\frac{\boldsymbol{\pi}_0^2}{2T} + \frac{T}{2}\mathbf{w}^2} + \pi\sum_{n=1}^{\infty}\left(\boldsymbol{\rho}_{-n}\boldsymbol{\rho}_n + \tilde{\boldsymbol{\rho}}_{-n}\tilde{\boldsymbol{\rho}}_n\right) + \mathfrak{a}
\end{aligned}$$

$$= \frac{1}{2L}\left(\frac{j^2}{T} + T\,(2\pi m)^2\right) + \pi\sum_{n=1}^{\infty} n\left(\mathbf{N}_n + \tilde{\mathbf{N}}_n\right) + \mathfrak{a} \tag{5.33}$$

Here $\mathfrak{a}$ is a (UV sensitive) constant which will not be important for us (it is very important in string theory), which is the price we pay for writing the hamiltonian as a sum of normal-ordered terms – the modes with negative indices are to the right and they annihilate the vacuum:

$$\boldsymbol{\rho}_n\,|0\rangle = 0, \quad \tilde{\boldsymbol{\rho}}_n\,|0\rangle = 0, \quad \text{for } n > 0 \ .$$

Energy eigenstates can be labelled by a target-momentum j and a winding m. Notice that there is an operator $\mathbf{w}$ whose eigenvalues are w, and it has a conjugate momentum $\mathbf{p}_L - \mathbf{p}_R$ which increments its value. So when I write $|0\rangle$ above, I really should label a vacuum of the oscillator modes with p, w.

$\mathbf{N}_n \equiv \frac{1}{n}\boldsymbol{\rho}_{-n}\boldsymbol{\rho}_n$ is the number operator; if we redefine $\mathbf{a}_n \equiv \sqrt{n}^{-1}\boldsymbol{\rho}_n (n > 0)$, we have $[\mathbf{a}_n, \mathbf{a}_m^\dagger] = \delta_{nm}$ and $\mathbf{N}_n = \mathbf{a}_n^\dagger \mathbf{a}_n$ is the ordinary thing.

Notice that the separate conservation of (5.28) and (5.30) means that there are separately-conserved left-moving and right-moving currents:

$$(j_L)^\mu = (j_L^z, j_L^{\bar{z}})^\mu \equiv (j_+, 0)^\mu$$

$$(j_R)^\mu = (j_R^z, j_R^{\bar{z}})^\mu \equiv (0, j_-)^\mu$$

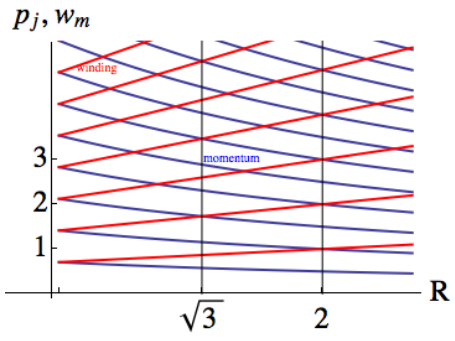
Here j_L only depends on the modes $\boldsymbol{\rho}_n$, and j_R only depends on the modes $\tilde{\boldsymbol{\rho}}_n$:

$$j_+ = \partial_+\phi = \partial_+\phi(x^+) = \mathbf{p} + \mathbf{w} + \sqrt{\frac{\pi}{LT}}\sum_{n\neq 0}\boldsymbol{\rho}_n e^{\mathbf{i}\frac{2\pi}{L}nx^+}$$

$$j_- = \partial_-\phi = \partial_-\phi(x^-) = \mathbf{p} - \mathbf{w} + \sqrt{\frac{\pi}{LT}}\sum_{n\neq 0}\tilde{\boldsymbol{\rho}}_n e^{\mathbf{i}\frac{2\pi}{L}nx^-}$$

Here's an **Observation (T-duality)**: At large T (think of this as a large radius of the target space), the momentum modes are closely-spaced in energy, and exciting the winding modes is costly, since the string has a tension, it costs energy-per-unit-length T to stretch it. But the spectrum (5.33) is invariant under the operation

$$m \leftrightarrow j, \quad T \leftrightarrow \frac{1}{(2\pi)^2 T}$$



which takes the radius of the circle to its inverse and exchanges the momentum and winding modes. This is called *T-duality*. The required duality map on the fields is

$$\phi_L + \phi_R \leftrightarrow \phi_L - \phi_R \ .$$

(The variable R in the plot is $R \equiv \sqrt{\pi T}$.)

T-duality says string theory on a large circle is the same as string theory on a small circle. On the homework you'll get to see a derivation of this statement in the continuum which allows some generalizations.

Vertex operators. It is worthwhile to pause for another moment and think about the operators that create the winding modes. They are like vortex creation operators. Since ϕ has logarithmic correlatations, you might think that exponentiating it is a good idea. First let's take advantage of the fact that the ϕ correlations split into left and right bits to write $\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})$:

$$\langle \phi_L(z)\phi_L(0)\rangle = -\frac{1}{\pi T}\log\frac{z}{a}, \quad \langle \phi_R(\bar{z})\phi_R(0)\rangle = -\frac{1}{\pi T}\log\frac{\bar{z}}{a}\ , \quad \langle \phi_L(z)\phi_R(0)\rangle = 0\ . \tag{5.34}$$

A set of operators with definite scaling dimension is:

$$\mathcal{V}_{\alpha,\beta}(z,\bar{z}) =: e^{\mathbf{i}(\alpha\phi_L(z)+\beta\phi_R(\bar{z}))} : \ .$$

This is a composite operator which we have defined by normal-ordering. The normal ordering prescription is: $\boxed{q,p,-,+}$, that is: positive-momentum modes (lowering operators) go on the right, and p counts as a lowering operator, so in particular using the expansion (please beware my factors here): $\phi_L(z) = \mathbf{q}_L + \mathbf{p}_L z + \mathbf{i}\sum_{n\neq 0}\frac{\boldsymbol{\rho}_n}{n}w^n$, we have

$$: e^{\mathbf{i}\alpha\phi_L(z)} :\equiv e^{\mathbf{i}\alpha\mathbf{q}_L}e^{\mathbf{i}\alpha\mathbf{p}_L z}e^{\mathbf{i}\alpha\sum_{n<0}\frac{\boldsymbol{\rho}_n}{n}w^n}e^{\mathbf{i}\alpha\sum_{n>0}\frac{\boldsymbol{\rho}_n}{n}w^n}$$

(I used the definition $w \equiv e^{2\pi i z/L}$.)

How should we think about this operator ? In the QM of a free particle, the operator $e^{\mathbf{i}p\mathbf{x}}$ inserts momentum p – it takes a momentum-space wavefunction $\psi(p_0) = \langle p_0|\psi\rangle$ and gives

$$\langle p_0|\, e^{\mathbf{i}p\mathbf{x}}\,|\psi\rangle = \psi(p_0+p).$$

It's the same thing here, with one more twist.

In order for $\mathcal{V}_{\alpha,\beta}$ to be well-defined under $\phi \to \phi + 2\pi$, we'd better have $p = \frac{\alpha+\beta}{2} \in \mathbb{Z}$ – momentum is quantized, just like for the particle (the center of mass is just a particle). Let's consider what the operator $\mathcal{V}_{\alpha,\beta}$ does to a winding and momentum eigenstate $|w,p\rangle$ (for simplicity, take one with no oscillator excitations, $\boldsymbol{\rho}_n\,|p,w\rangle = 0, n>0$):

$$\mathcal{V}_{\alpha\beta}(0)\,|w,p\rangle = e^{\mathbf{i}\left(\frac{\alpha+\beta}{2}\right)\mathbf{q}_0}e^{\mathbf{i}\left(\frac{\alpha-\beta}{2}\right)\tilde{\boldsymbol{\phi}}_0}e^{\mathbf{i}\alpha\sum_{n<0}\boldsymbol{\rho}_n}e^{\mathbf{i}\alpha\sum_{n>0}\boldsymbol{\rho}_n}\,|w,p\rangle = e^{\mathbf{i}\alpha\sum_{n<0}\boldsymbol{\rho}_n}\left|w+\frac{\alpha-\beta}{2}, p+\frac{\alpha+\beta}{2}\right\rangle \tag{5.35}$$

The monster in front here creates oscillator excitations. I wrote $\mathbf{q}_0 \equiv \mathbf{q}_L + \mathbf{q}_R$ and $\tilde{\boldsymbol{\phi}}_0 \equiv \mathbf{q}_L - \mathbf{q}_R$. The important thing is that the winding number has been incremented by $\alpha - \beta$; this means that $\alpha - \beta$ must be an integer, too. We conclude that

$$\alpha + \beta \in 2\mathbb{Z}, \quad \alpha - \beta \in 2\mathbb{Z} \tag{5.36}$$

so they can both be odd integers or they can both be even integers.

By doing the gaussian integral (or moving the annihilation operators to the right) their correlators are

$$\langle \mathcal{V}_{\alpha,\beta}(z,\bar{z}) \mathcal{V}_{\alpha',\beta'}(0,0)\rangle = \frac{D_0}{z^{\frac{\alpha^2}{\pi T}} \bar{z}^{\frac{\beta^2}{\pi T}}} \ . \tag{5.37}$$

The zeromode prefactor D_0 is:

$$D_0 = \left\langle e^{\mathbf{i}((\alpha+\alpha')\mathbf{q}_L + (\beta+\beta')\mathbf{q}_R)} \right\rangle_0 = \delta_{\alpha+\alpha'}\delta_{\beta+\beta'}.$$

This is charge conservation.

We conclude that the operator $\mathcal{V}_{\alpha,\beta}$ has scaling dimension $\Delta = h_L + h_R$ with

$$(h_L, h_R) = \frac{1}{2\pi T}(\alpha^2, \beta^2).$$

($h_L - h_R$ is the spin.) Notice the remarkable fact that the exponential of a dimension-zero operator manages to have nonzero scaling dimension. This requires that the multiplicative prefactor depend on the cutoff a to the appropriate power (and it is therefore nonuniversal). We could perform a multiplicative renormalization of our operators $\mathcal{V}$ to remove this cutoff dependence from the correlators.

The values of α, β allowed by single-valuedness of ϕ and its wavefunctional are best understood in terms of the integers j, m that we introduced above. We see (at least) three special values of the parameter T:

- **The $\mathsf{SU}(2)$ radius**: When $2\pi T = 1$, T-duality maps the theory to itself. The operators with $(\alpha, \beta) = (1, 1)$ are *marginal*. Also, the operators with $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$ have the scaling behavior of currents ($\langle j_-(z) j_-(0)\rangle \propto \frac{1}{z^2}$) and by holomorphicity are in fact conserved. The $\mathsf{U}(1) \times \mathsf{U}(1)$ symmetry is enhanced to $\mathsf{SU}(2) \times \mathsf{SU}(2)$. This model is the same as the $\mathsf{SU}(2)_1$ (pronounced $\mathsf{SU}(2)$ level one) WZW model. It can also be described as a NLSM whose target space is the $\mathsf{SU}(2)$ group manifold, with the addition of a WZW term; although the $\mathsf{SU}(2)$ manifold has positive curvature, so naively wants to shrink in the IR, the WZW term affects the running of the radius and leads to a fixed point at a radius determined by the level. See Witten's wonderful paper on the subject.

- **The free fermion radius**: when $2\pi T = 2$, $\mathcal{V}_{1,0}$ in (5.37) looks like $\langle \mathcal{V}_{1,0}(z)\mathcal{V}_{1,0}(0)\rangle = \frac{D}{z}$ which is the behavior of a left-moving free fermion, with action $\int dtdx\bar{\psi}\partial_+\psi$. In fact the scalar theory with this radius is (locally) equivalent to a massless Dirac fermion! This equivalence is an example of *bosonization*. In particular, the radius-changing deformation of the boson maps to a marginal four-fermion interaction: by studying free bosons we can learn about interacting fermions. (More precisely, just like in the Jordan-Wigner description of the TFIM, the scalar theory is equivalent to a Dirac fermion with the fermion number gauged.)

- **The supersymmetric radius**: when $2\pi T = \frac{2}{3}$, $\mathcal{V}_{1,0}$ has dimension $(\frac{3}{2}, 0)$ and represents a supersymmetry current.

After this detour, let's return to the drama of the bose-Hubbard model. Starting from large J/U, where we found a superfluid, what happens as U grows and makes the phase fluctuate more? Our continuum description in terms of harmonic oscillators hides (but does not ignore) the fact that $\phi \simeq \phi + 2\pi$. The system admits vortices, aka winding modes.

Lattice T-duality. To see their effects let us do T-duality on the lattice.

The dual variables live on the bonds, labelled by $\bar{i} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}....$

Introduce

$$\mathbf{m}_{\bar{i}} \equiv \frac{\phi_{i+1} - \phi_i}{2\pi}, \quad \Theta_{\bar{i}} \equiv \sum_{j<\bar{i}} 2\pi \mathbf{n}_j \tag{5.38}$$

which together imply

$$[\mathbf{m}_{\bar{i}}, \Theta_{\bar{j}}] = -\mathbf{i}\delta_{\bar{i}\bar{j}}.$$

To understand where these expressions come from, notice that the operator

$$e^{\mathbf{i}\Theta_{\bar{i}}} = e^{\mathbf{i}\sum_{j<\bar{i}} 2\pi \mathbf{n}_j}$$

rotates the phase of the boson on all sites to the left of $\bar{i}$ (by 2π). It inserts a vortex in between the sites i and $i+1$. The rotor hamiltonian is

$$\begin{aligned} \mathbf{H}_{\text{rotors}} &= \frac{U}{2}\sum_{\bar{i}} \left(\frac{\Theta_{\bar{i}+1} - \Theta_{\bar{i}}}{2\pi}\right)^2 - J\sum_{\bar{i}} \cos 2\pi \mathbf{m}_{\bar{i}} \\ &\overset{SF}{\simeq} \sum_{\bar{i}} \left(\frac{U}{2}\left(\frac{\Delta\Theta}{2\pi}\right)^2 + \frac{J}{2}(2\pi \mathbf{m}_{\bar{i}})^2\right) \end{aligned} \tag{5.39}$$

where in the second step, we assumed we were in the SF phase, so the phase fluctuations and hence $\mathbf{m}_{\bar{i}}$ are small. This looks like a chain of masses connected by springs again, but with the roles of kinetic and potential energies reversed – the second term should be regarded as a π^2 kinetic energy term. BUT: we must not forget that $\Theta \in 2\pi\mathbb{Z}$! It's oscillators with discretized positions. We can rewrite it in terms of continuous Θ at the expense of imposing the condition $\Theta \in 2\pi\mathbb{Z}$ energetically by adding a term $-\lambda \cos\Theta$[36]. The resulting model has the action

$$L_{\text{eff}} = \frac{1}{2(2\pi)^2\rho_s}\left(\partial_\mu \Theta\right)^2 - \lambda \cos\Theta. \tag{5.40}$$

Ignoring the λ term, this is the T-dual action, with ρ_s replaced by $\frac{1}{(2\pi)^2\rho_s}$. The coupling got inverted here because in the dual variables it's the J term that's like the π^2 inertia term, and the U term is like the restoring force. This $\Theta = \phi_L - \phi_R$ is therefore the T-dual variable, with ETCRs

$$[\phi(x), \Theta(y)] = 2\pi \mathsf{isign}(x-y). \tag{5.41}$$

This commutator follows directly from the definition of Θ (5.38). (5.41) means that the operator $\cos\Theta(x)$ jumps the SF phase variable ϕ by 2π – it inserts a 2π vortex, as we designed it to do. So λ is like a chemical potential for vortices.

This system has two regimes, depending on the scaling dimension of the vortex insertion operator:

• If λ is an irrelevant coupling, we can ignore it in the IR and we get a superfluid, with algebraic LRO.

• If the vortices are relevant, $\lambda \to \infty$ in the IR, and we pin the dual phase, $\Theta_{\bar{i}} = 0, \forall \bar{i}$. This is the Mott insulator, since $\Theta_{\bar{i}} = 0$ means $\mathbf{n}_i = 0$ – the number fluctuations are frozen.

When is λ relevant? Expanding around the free theory,

$$\left\langle e^{\mathbf{i}\Theta(x)} e^{-\mathbf{i}\Theta(0)} \right\rangle = \frac{c}{x^{2\pi\rho_s}}$$

this has scaling dimension $\Delta = \pi\rho_s$ which is relevant if $2 > \Delta = \pi\rho_s$. Since the bose correlators behave as $\left\langle b^\dagger b \right\rangle \sim x^{-\eta}$ with $\eta = \frac{1}{2\pi\rho_s}$, we see that only if $\eta < \frac{1}{4}$ do we

[36] This step seems scary at first sight, since we're adding degrees of freedom to our system, albeit gapped ones. $\Theta_{\bar{i}}$ is the number of bosons to the left of $\bar{i}$ (times 2π). An analogy that I find useful is to the fact that the number of atoms of air in the room is an integer. This constraint can have some important consequences, for example, were they to solidify. But in our coarse-grained description of the fluid phase, we use variables (the continuum number density) where the number of atoms (implicitly) varies continuously. The nice thing about this story (both for vortices and for air) is that the system tells us when we can't ignore this quantization constraint.

have a stable SF phase. (Recall that $\rho_s = \sqrt{J/U}$.) If $\eta > \frac{1}{4}$, the SF is unstable to proliferation of vortices and we end up in the Mott insulator, where the quantization of particle number matters. A lesson: we can think of the Mott insulator as a condensate of vortices. [End of Lecture 18]

Note: If we think about this euclidean field theory as a 2+0 dimensional stat-mech problem, the role of the varying ρ_s is played by temperature, and this transition we've found of the XY model, where by varying the radius the vortices become relevant, is the Kosterlitz-Thouless transition.

Most continuous phase transitions occur by tuning the coefficient of a relevant operator to zero (recall the general $\mathsf{O}(n)$ transition, where we have to tune $r \to r_c$ to get massless scalars). This is *not* what happens in the 2d XY model; rather, we are varying a *marginal* parameter and the dimensions of other operators depend on it and become relevant at some critical value of that marginal parameter. This leads to very weird scaling near the transition, of the form $e^{-\frac{a}{\sqrt{K-K_c}}}$ (for example, in the correlation length, the exponential arises from inverting expressions involving $G_R(z) = -\frac{1}{4\pi K}\log z$) – it is sometimes called an 'infinite order' phase transition, because all derivatives of such a function are continuous.

If you are eager to learn more about duality, I recommend these notes, the first section of which offers a useful parallel to the discussion above (part of which I actually followed in lecture).

5.3 (2+1)-d XY is dual to (2+1)d electrodynamics

5.3.1 Mean field theory

Earlier (during our discussion of boson coherent states) I made some claims about the phase diagram of the Bose-Hubbard model

$$H_{BH} = \sum_i \left(-\mu n_i + U n_i(n_i - 1)\right) + \sum_{ij} b_i^\dagger w_{ij} b_j$$

which I would like to clarify.

[Sachdev] Consider a variational approach to the BH model. We'll find the best product-state wavefunction $|\Psi_{\text{var}}\rangle = \otimes_i |\psi_i\rangle$, and minimize the BH energy $\langle\Psi_{\text{var}}| H_{BH} |\Psi_{\text{var}}\rangle$ over all ψ_i. We can parametrize the single-site states as the groundstates of the mean-field hamiltonian:

$$H_{\text{MF}} = \sum_i h_i = \sum_i \left(-\mu n_i + U n_i(n_i - 1) - \Psi^\star b_i - \Psi b_i^\dagger\right).$$

Here Ψ is an effective field which incorporates the effects of the neighboring sites. Notice that nonzero Ψ breaks the $\mathsf{U}(1)$ boson number conservation: particles can hop out of the site we are considering. This also means that nonzero Ψ will signal SSB.

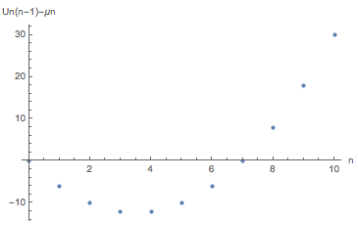
What does this simple approximation give up? For one, it assumes the ground-state preserves the lattice translation symmetry, which doesn't always happen. More painfully, it also gives up on any entanglement at all in the groundstate. Phases for which entanglement plays an important role will not be found this way.

We want to minimize over Ψ the quantity

$$\begin{aligned}\mathcal{E}_0 &\equiv \frac{1}{M}\left\langle\Psi_{\text{var}}\right| H_{BH}\left|\Psi_{\text{var}}\right\rangle = \frac{1}{M}\left(\left\langle\Psi_{\text{var}}\right|\left(\underbrace{H_{BH}-H_{MF}}_{=w\sum b^\dagger b+\Psi b+h.c.}+H_{MF}\right)\left|\Psi_{\text{var}}\right\rangle\right)\\ &= \frac{1}{M}E_{MF}(\Psi) - zw\left\langle b^\dagger\right\rangle\langle b\rangle + \langle b\rangle\,\Psi^\star + \left\langle b^\dagger\right\rangle\Psi. \qquad (5.42)\end{aligned}$$

Here z is the coordination number of the lattice (the number of neighbors of a site, which we assume is the same for every site), and $\langle..\rangle \equiv \langle\Psi_{\text{var}}|\,..\,|\Psi_{\text{var}}\rangle$.

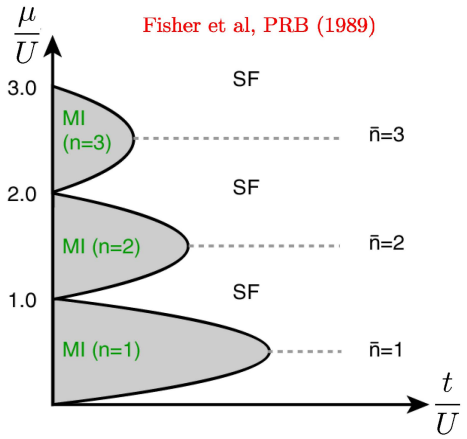
First consider $w = 0$, no hopping. Then $\Psi_B = 0$ (neighbors don't matter), and the single-site state is a number eigenstate $|\psi_i\rangle = |n_0(\mu/U)\rangle$, where $n_0(x) = 0$ for $x < 0$, and $n_0(x) = \lceil x\rceil$, (the ceiling of x, *i.e.* , the next integer larger than x), for $x > 0$. Precisely when μ/U is an integer, there is a twofold degeneracy per site.



This degeneracy is broken by a small hopping term. Away from the degenerate points, within a single Mott plateau, the hopping term does very little (even away from mean field theory). This is because there is an energy gap, and $[N, H_{BH}] = 0$, which means that a small perturbation has no other states to mix in which might have other eigenvalues of N. Therefore, within a whole open set, the particle number remains fixed. This means $\partial_\mu\langle N\rangle = 0$, the system is incompressible.

We can find the boundaries of this region by expanding $\mathcal{E}_0$ in Ψ, following Landau: $\mathcal{E}_0 = \mathcal{E}_0^0 + r|\Psi|^2 + \mathcal{O}(|\Psi|^4)$. We can compute the coefficients in perturbation theory, and this produces the following picture.

Mean field theory gives the famous picture at right, with lobes of different Mott insulator states with different (integer!) numbers of bosons per site. (The hopping parameter w is called t in the figure.)



(Fig credit: Roman Lutchyn)

5.3.2 Coherent state path integral

Actually we can do a bit better; some of our hard work will pay off. Consider the coherent state path integral for the Euclidean partition sum

$$Z = \int [d^2b] e^{-\int_0^{1/T} d\tau \mathcal{L}_b}$$

$$\text{with} \quad \mathcal{L}_b = \sum_i \left(b_i^\dagger \partial_\tau b_i - \mu b_i^\dagger b_i + U b_i^\dagger b_i^\dagger b_i b_i \right) - \sum_{ij} b_i^\dagger w_{ij} b_j$$

where we introduced the hopping matrix $w_{ij} = w$ if $\langle ij \rangle$ share a link, otherwise zero. Here the bs are numbers, coherent state eigenvalues. Here is another application of the Hubbard-Stratonovich transformation:

$$Z = \int [d^2b][d^2\Psi] e^{-\int_0^{1/T} d\tau \mathcal{L}'_b}$$

$$\text{with} \quad \mathcal{L}'_b = \sum_i \left(b_i^\dagger \partial_\tau b_i - \mu b_i^\dagger b_i + U b_i^\dagger b_i^\dagger b_i b_i - \Psi b_i^\dagger - \Psi^\star b_i \right) + \sum_{ij} \Psi_i w_{ij}^{-1} \Psi_j .$$

(Warning: if w has negative eigenvalues, so that the gaussian integral over Ψ is well-defined, we need to add a big constant to it, and subtract it from the single-particle terms.) Now integrate out the b fields. It's not gaussian, but notice that the resulting action for Ψ is the connected generating function $W[J]$: $\int [d^2b] e^{-S[b] + \int \Psi b + h.c.} = e^{-W[\Psi, \Psi^\star]}$. More specifically,

$$Z = \int [d^2\Psi] e^{-\frac{V}{T}\mathcal{F}_0 - \int_0^{1/T} d\tau \mathcal{L}_B}$$

$$\text{with} \quad \mathcal{L}_B = K_1 \Psi^\star \partial_\tau \Psi + K_2 |\partial_\tau \Psi|^2 + K_3 |\vec{\nabla} \Psi|^2 + \tilde{r} |\Psi|^2 + u |\Psi|^4 + \cdots$$

Here $V = Ma^d$ is the volume of space, and $\mathcal{F}_0$ is the mean-field free energy. The coefficients K etc are connected Green's functions of the bs. The choice of which terms I wrote was dictated by Landau, and the order in which I wrote them should have been determined by Wilson. The Mott-SF transition occurs when $\tilde{r}$ changes sign, that is, the condition $\tilde{r} = 0$ determines the location of the Mott-SF boundaries. You can see that generically we have $z = 2$ kinetic terms. Less obvious is that $\tilde{r}$ is proportional to the mean field coefficient r.

Here's the payoff. I claim that the coefficients in the action for Ψ are related by

$$K_1 = -\partial_\mu \tilde{r}. \tag{5.43}$$

This means that $K_1 = 0$ precisely when the boundary of the lobe has a vertical tangent. This means that right at those points (the ends of the dashed lines in the figure) the second-order kinetic term is the leading one, and we have $z = 1$.

Here's the proof of (5.43). $\mathcal{L}_B$ must have the same symmetries as $\mathcal{L}_b$. One such invariance is

$$b_i \to b_i e^{\mathbf{i}\phi(\tau)}, \ \Psi_i \to \Psi_i e^{\mathbf{i}\phi(\tau)}, \mu \to \mu + \mathbf{i}\partial_\tau \phi.$$

This is a funny transformation which acts on the couplings, so doesn't produce Noether currents. It is still useful though, because it implies

$$0 = \delta_\phi \left(K_1 \Psi^\star \partial_\tau \Psi + \tilde{r}|\Psi|^2 + ... \right) = K_1 |\Psi|^2 \mathbf{i}\partial_\tau \phi + \partial_\mu \tilde{r} \mathbf{i} \partial \phi |\Psi|^2 + ...$$

5.3.3 Duality

We have seen above (in §5.2) that the prevention of vortices is essential to superfluidity, which is the condensation of bosons. In $D = 1 + 1$, vortices are events in spacetime. In $D = 2 + 1$, vortices are actual particles, *i.e.* localizable objects, around which the superfluid phase variable winds by 2π (times an integer).

More explicitly, if the boson field which condenses is $b(x) = ve^{\mathbf{i}\phi}$, and we choose polar coordinates in space $x + \mathbf{i}y \equiv Re^{\mathbf{i}\varphi}$, then a vortex is a configuration of the order parameter field of the form $b(x) = f(R)e^{\mathbf{i}\varphi}$, where $f(R) \stackrel{R\to\infty}{\to} v$ far away: the phase of the order parameter winds around. Notice that the phase is ill-defined in the core of the vortex where $f(R) \stackrel{R\to 0}{\to} 0$. (This is familiar from our discussion of the Abelian Higgs model.)

To see the role of vortices in destroying superfluidity more clearly, consider superfluid flow in a 2d annulus geometry, with the same polar coordinates $x + \mathbf{i}y = Re^{\mathbf{i}\varphi}$. If the superfluid phase variable is in the configuration $\phi(R, \varphi) = n\varphi$, then the current is

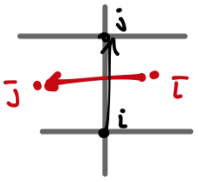
$$\vec{J}(R, \varphi) = \rho_s \vec{\nabla}\phi = \check{\varphi} \rho_s \frac{n}{2\pi R}.$$

The current only changes if the integer n changes. This happens if vortices enter from the outside; removing the current (changing n to zero) requires n vortices to tunnel all the way through the sample, which if they are gapped and the sample is macroscopic can take a cosmologically long time.

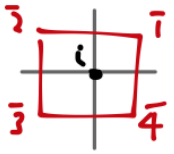
There is a dual statement to the preceding three paragraphs: a state where the bosons themselves are gapped and localized – that is, a Mott insulator – can be described starting from the SF phase by the condensation of vortices. To see this, let us consider again the (simpler-than-Bose-Hubbard) $2 + 1$d rotor model

$$\mathbf{H}_{\text{rotors}} = U \sum_i \mathbf{n}_i^2 - J \sum_{\langle ij \rangle} \cos\left(\phi_i - \phi_j\right)$$

and introduce dual variables. Introduce a dual lattice whose sites are (centered in) the faces of the original (direct) lattice; each link of the dual lattice crosses one link of the direct lattice.



• First let $e_{\bar{i}\bar{j}} \equiv \frac{\phi_i - \phi_j}{2\pi}$. Here we define $\bar{i}\bar{j}$ by the right hand rule: $ij \times \bar{i}\bar{j} = +\check{z}$ (ij denotes the unit vector pointing from i to j). This is a lattice version of $\vec{e} = \check{z} \times \vec{\nabla}\phi \frac{1}{2\pi}$. Defining lattice derivatives $\Delta_x \phi_i \equiv \phi_i - \phi_{i+\check{x}}$, the definition is $e_x = -\frac{\Delta_y \phi}{2\pi}, e_y = \frac{\Delta_x \phi}{2\pi}$. It is like an electric field vector.



• The conjugate variable to the electric field is $a_{\bar{i}\bar{j}}$, which must therefore be made from the conjugate variable of ϕ_i, namely $\mathbf{n}_i$: $[\mathbf{n}_i, \phi_j] = -\mathbf{i}\delta_{ij}$. Acting with $\mathbf{n}_i$ translates ϕ_i, which means that it shifts all the $e_{\bar{i}\bar{j}}$ from the surrounding plaquettes. More precisely:

$$2\pi \mathbf{n}_i = a_{\bar{1}\bar{2}} + a_{\bar{2}\bar{3}} + a_{\bar{3}\bar{4}} + a_{\bar{4}\bar{1}}.$$

This is a lattice, integer version of $n \sim \frac{1}{2\pi}\vec{\nabla} \times a \cdot \check{z}$. In terms of these variables,

$$\mathbf{H}_{\text{rotors}} = \frac{U}{2} \sum_i \left(\frac{\Delta \times a}{2\pi} \right)^2 - J \sum_{\langle \bar{i}\bar{j} \rangle} \cos\left(2\pi e_{\bar{i}\bar{j}}\right)$$

with the following *constraint*. If it were really true that $\vec{e} = \frac{1}{2\pi}\check{z} \times \vec{\nabla}\phi$, with single-valued ϕ, then $\vec{\nabla} \cdot \vec{e} = \vec{\nabla} \cdot \left(\check{z} \times \vec{\nabla}\phi\right) = 0$. But there are vortices in the world, where ϕ is not single valued. The number of vortices $n_v(R)$ in some region R with $\partial R = C$ is determined by the winding number of the phase around C:

$$2\pi n_v(R) = \oint_C d\vec{\ell} \cdot \vec{\nabla}\phi \overset{\text{Stokes}}{=} 2\pi \int_R d^2x \vec{\nabla} \cdot \vec{e}$$

(More explicitly, $2\pi \vec{\nabla} \cdot \vec{e} = \epsilon_{zij}\partial_i \partial_j \phi = [\partial_x, \partial_y]\phi$ clearly vanishes if ϕ is single-valued.) Since this is true for any region R, we have

$$\vec{\nabla} \cdot \vec{e} = 2\pi \delta^2(\text{vortices}).$$

Actually, the lattice version of the equation has more information (and is true) because it keeps track of the fact that the number of vortices is an integer:

$$\Delta_x e_x + \Delta_y e_y \equiv \vec{\Delta} \cdot \vec{e}(\bar{i}) = 2\pi n_v(\bar{i}), \qquad n_v(\bar{i}) \in \mathbb{Z}.$$

It will not escape your notice that this is Gauss' law, with the density of *vortices* playing the role of the charge density.

Phases of the 2d rotors. Since $\vec{e} \sim \vec{\nabla}\phi$ varies continuously, *i.e.* electric flux is not quantized, this is called *noncompact* electrodynamics. Again we will impose

the integer constraint $a \in 2\pi\mathbb{Z}$ energetically, *i.e.* let $a \in \mathbb{R}$ and add (something like) $\Delta\mathbf{H} \stackrel{?}{=} -t\cos a$ and see what happens when we make t finite. The expression in the previous sentence is not quite right, yet, however: This operator does not commute with our constraint $\vec{\Delta}\cdot\vec{e} - 2\pi n_v = 0$ – it jumps $\vec{e}$ but not n_v[37].

We can fix this by introducing explicitly the variable which creates vortices, $e^{-\mathbf{i}\chi}$, with:

$$[n_v(\bar{i}), \chi(\bar{j})] = -\mathbf{i}\delta_{\bar{i}\bar{j}} \ .$$

Certainly our Hilbert space contains states with different number of vortices, so we can introduce an operator which maps these sectors. Its locality might be an issue: certainly it is nonlocal with respect to the original variables, but we will see that we can treat it as a local operator (except for the fact that it carries gauge charge) in the dual description. Since $n_v \in \mathbb{Z}$, $\chi \simeq \chi + 2\pi$ lives on a circle. So:

$$\mathbf{H} \sim \sum_{\bar{i}} \left(\frac{U}{2} \left(\frac{\Delta \times a}{2\pi} \right)^2 + \frac{J}{2} (2\pi e)^2 - t\cos\left(\Delta\chi - a\right) \right)$$

still subject to the constraint $\vec{\Delta}\cdot\vec{e} = 2\pi n_v$.

Two regimes:

$\boxed{J \gg U, t}$: This suppresses e and its fluctuations, which means a fluctuates. The fluctuating a is governed by the gaussian hamiltonian

$$\mathbf{H} \sim \sum \left(\vec{e}^2 + \vec{b}^2 \right)$$

with $b \equiv \frac{\Delta \times a}{2\pi}$, which should look familiar. This *deconfined* phase has a gapless photon; a $2+1$d photon has a single polarization state. This is the goldstone mode, and this regime describes the superfluid phase (note that the parameters work out right in the original variables). The relation between the photon a and the original phase variable, in the continuum is

$$\epsilon_{\mu\nu\rho}\partial_\nu a_\rho = \partial_\mu \phi.$$

$\boxed{t \gg U, J}$: In this regime we must satisfy the cosine first. Like in $D = 1+1$, this can be described as the statement that vortices condense. Expanding around its minimum, the cosine term is

$$\mathbf{h} \ni t\left(a - \partial\chi\right)^2$$

[37] A set of words which has the same meaning as the above: $\cos a$ is not gauge invariant. Understanding these words requires us to think of the operator $G(\bar{i}) \equiv \vec{\Delta}\cdot\vec{e} - 2\pi n_v$ as the generator of a transformation,

$$\delta\mathcal{O} = \sum_{\bar{i}} s(\bar{i})[G(\bar{i}), \mathcal{O}].$$

It can be a useful picture.

– the photon gets a mass by eating the phase variable χ. There is an energy gap. This is the Mott phase.

If the vortices carry other quantum numbers, the (analog of the) Mott phase can be more interesting, as we'll see in section 5.5.

Compact electrodynamics in $D = 2 + 1$. Note that this free photon phase of $D = 2 + 1$ electrodynamics is not accessible if e is quantized (so-called compact electrodynamics) where monopole instantons proliferate and gap out the photon. This is the subject of §5.4.

5.3.4 Particle-vortex duality in the continuum

The above is easier to understand (but a bit less precise) in the continuum. Consider a quantum system of bosons in $D = 2+1$ with a $\mathsf{U}(1)$ particle-number symmetry (a real symmetry, not a gauge redundancy). Let's focus on a complex, non-relativistic bose field b with action

$$S[b] = \int \mathrm{d}t\mathrm{d}^2x \left(b^\dagger \left(\mathbf{i}\partial_t - \vec{\nabla}^2 - \mu \right) b - U(b^\dagger b)^2 \right) . \tag{5.44}$$

By Noether's theorem, the symmetry $b \to e^{\mathbf{i}\theta} b$ implies that the current

$$j_\mu = (j_t, \vec{j})_\mu = (b^\dagger b, \mathbf{i} b^\dagger \vec{\nabla} b + h.c.)_\mu$$

satisfies the continuity equation $\partial^\mu j_\mu = 0$.

This system has two phases of interest here. In the ordered/broken/superfluid phase, where the groundstate expectation value $\langle b \rangle = \sqrt{\rho_0}$ spontaneously breaks the U(1) symmetry, the goldstone boson θ in $b \equiv \sqrt{\rho_0} e^{\mathbf{i}\theta}$ is massless

$$S_{\text{eff}}[\theta] = \frac{\rho_0}{2} \int \left(\dot{\theta}^2 - \left(\vec{\nabla}\theta \right)^2 \right) \mathrm{d}^2x\mathrm{d}t, \quad j_\mu = \rho_0 \partial\theta \ .$$

In the disordered/unbroken/Mott insulator phase, $\langle b \rangle = 0$, and there is a mass gap. A dimensionless parameter which interpolates between these phases is $g = \mu/U$; large g encourages condensation of b.

We can 'solve' the continuity equation by writing

$$j^\mu = \epsilon^{\mu \cdots} \partial . a . \tag{5.45}$$

where $a.$ is a gauge potential. The time component of this equation says that the boson density is represented by the magnetic flux of a. The spatial components relate

the boson charge current to the electric flux of a. The continuity equation for j is automatic – it is the Bianchi identity for a – as long as a is single-valued. That is: as long as there is no magnetic charge present. A term for this condition which is commonly used in the cond-mat literature is: "a is non-compact." (More on the other case below.)

The relation (5.45) is the basic ingredient of the duality, but it is not a complete description: in particular, how do we describe the boson itself in the dual variables? In the disordered phase, adding a boson is a well-defined thing which costs a definite energy. The boson is described by a localized clump of magnetic flux of a. Such a configuration is energetically favored if a participates in a superconductor – i.e. if a is coupled to a condensate of a charged field. The Meissner effect will then ensure that its magnetic flux is bunched together. So this suggests that we should introduce into the dual description a scalar field, call it Φ, minimally coupled to the gauge field a:

$$S[b] \leftrightsquigarrow S_{\rm dual}[a, \Phi] \ .$$

And the disordered phase should be dual to a phase where $\langle \Phi \rangle \neq 0$, which gives a mass to the gauge field by the Anderson-Higgs mechanism.

Who is Φ? More precisely, what is the identity in terms of the original bosons of the particles it creates? When Φ is not condensed and its excitations are massive, the gauge field is massless. This the Coulomb phase of the Abelian Higgs model $S[a, \Phi]$; at low energies, it is just free electromagnetism in $D = 2 + 1$. These are the properties of the ordered phase of b. (This aspect of the duality is explained in Wen, §6.3.) The photon has one polarization state in $D = 2 + 1$ and is dual to the goldstone boson. This is the content of (5.45) in the ordered phase: $\epsilon^{\mu \cdots} \partial_. a_. = \rho_0 \partial_\mu \theta$ or $\star {\rm d} a = \rho_0 {\rm d}\theta$.

Condensing Φ gives a mass to the Goldstone boson whose masslessness is guaranteed by the broken $\mathsf{U}(1)$ symmetry. Therefore Φ is a disorder operator: its excitations are vortices in the bose condensate, which are gapped in the superfluid phase. The transition to the insulating phase can be described as a condensation of these vortices.

The vortices have relativistic kinetic terms, *i.e.* particle-hole symmetry. This is the statement that in the ordered phase of the time-reversal invariant bose system, a vortex and an antivortex have the same energy. An argument for this claim is the following. We may create vortices by rotating the sample, as was done in the figure at right. With time-reversal symmetry, rotating the sample one way will cost the same energy as rotating it the other way.

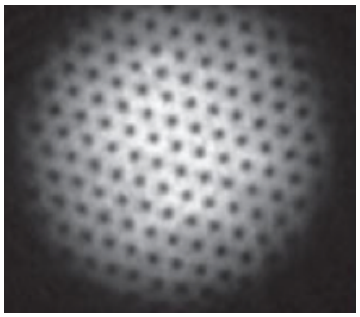


Fig: M. Zwierlein.

This means that the mass of the vortices $m_V^2 \Phi^\dagger \Phi$ is distinct from the vortex chemical potential $\mu_V \rho_V = \mu_V {\rm i} \Phi^\dagger \partial_t \Phi + h.c.$. The vortex mass[2]

maps under the duality to the boson chemical potential. Taking it from positive to negative causes the vortices to condense and disorder (restore) the $\mathsf{U}(1)$ symmetry.

To what does the vortex chemical potential map? It is a term which breaks time-reversal, and which encourages the presence of vortices in the superfluid order. It's an external magnetic field for the bosons. (This also the same as putting the bosons into a rotating frame.)

To summarize, a useful dual description is the Abelian Higgs model

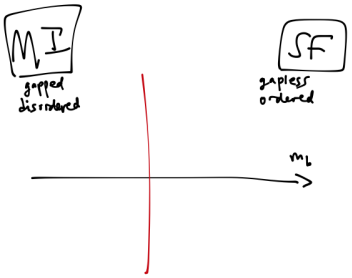
$$S[a,\Phi] = \int \mathrm{d}^2x\mathrm{d}t \left(\Phi^\dagger \left((\mathbf{i}\partial_t - \mathbf{i}A_t - \mu)^2 + \left(\vec{\nabla} + \vec{A}\right)^2 \right) \Phi - \frac{1}{e^2} f_{\mu\nu} f^{\mu\nu} - V(\Phi^\dagger\Phi) \right).$$

We can parametrize V as

$$V = \lambda \left(\Phi^\dagger\Phi - v \right)^2$$

– when $v < 0$, $\langle\Phi\rangle = 0$, Φ is massive and we are in the Coulomb phase. When $v > 0$ Φ condenses and we are in the Anderson-Higgs phase.

The description above is valid near the boundary of one of the MI phases. At the tips of the lobes are special points where the bosons b themselves have particle-hole symmetry (*i.e.* relativistic kinetic terms). For more on this diagram, see *e.g.* chapter 9 of Sachdev.



In the previous discussion I have been assuming that the vortices of b have unit charge under a and are featureless bosons, *i.e.* do not carry any non-trivial quantum numbers under any other symmetry. If *e.g.* the vortices have more-than-minimal charge under a, say charge q, then condensing them leaves behind a $\mathbb{Z}_q$ gauge theory and produces a state with topological order. If the vortices carry some charge under some other symmetry (like lattice translations or rotations) then condensing them breaks that symmetry. If the vortices are minimal-charge fermions, then they can only condense in pairs, again leaving behind an unbroken $\mathbb{Z}_2$ gauge theory.

5.4 Compact electrodynamics in $D = 2 + 1$

Since I emphasized above that we were speaking about non-compact electrodynamics, I should explain what is compact electrodynamics, why this makes a difference, and where it arises. First I'll introduce it in a lattice model of bosons. It can also emerge from spin systems; in this context, when a deconfined phase is realized, it is called a $\mathsf{U}(1)$ spin liquid.

Consider a quantum system on a two-dimensional lattice (say, square) with rotors $\Theta_l \equiv \Theta_l + 2\pi m$ on the links l. (Think of this as the phase of a boson or the direction of an easy-plane spin.) The conjugate variable $\mathbf{n}_l$ is an integer

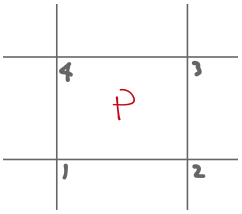
$$[\mathbf{n}_l, \Theta_{l'}] = -\mathbf{i}\delta_{l,l'}.$$

Here $\mathbf{n}_{ij} = \mathbf{n}_{ji}, \Theta_{ij} = \Theta_{ji}$ – we have not oriented our links (yet). We also impose the *Gauss' law constraint*

$$\mathbf{G}_s \equiv \sum_{l \in v(s)} \mathbf{n}_l = 0 \quad \forall \text{ sites } s,$$

where the notation $v(s)$ means the set of links incident upon the site s ('v' is for 'vicinity').

We'll demand that the Hamiltonian is 'gauge invariant', that is, that $[\mathbf{H}, \mathbf{G}_s] = 0 \forall s$. Any terms which depend only on $\mathbf{n}$ are OK. The natural single-valued object made from Θ is $e^{\mathbf{i}\Theta_l}$, but this is not gauge invariant. A combination which is gauge invariant is the *plaquette operator*, associated to a face p of the lattice:



$$\prod_{l \in \partial p} e^{(-1)^y \mathbf{i}\Theta_l} \equiv e^{\mathbf{i}(\Theta_{12} - \Theta_{23} + \Theta_{34} - \Theta_{41})}$$

– we put a minus sign on the horizontal links. ∂p denotes the links running around the boundary of p. So a good hamiltonian is

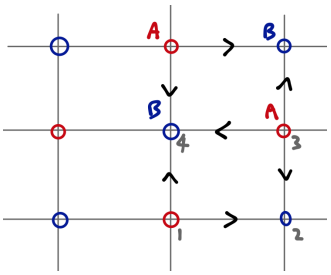
$$\mathbf{H} = \frac{U}{2} \sum_l \mathbf{n}_l^2 - K \sum_\square \cos\left(\sum_{l \in \partial \square} (-1)^y \Theta_l \right).$$

Local Hilbert space. The space of gauge-invariant states is not a tensor product over local Hilbert spaces. This sometimes causes some confusion, and seems like an obstruction to such a system emerging from condensed matter. Notice, however, that we can arrive at the gauge-theory hilbert space by imposing the Gauss' law constraint energetically (as in the toric code): Start with the following Hamiltonian acting on the full unconstrained rotor Hilbert space:

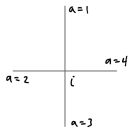
$$H_{\text{big}} = +\Gamma_\infty \sum_i \mathbf{G}_i + \mathbf{H}.$$

True to its name, the coefficient Γ_∞ is some huge energy scale which penalizes configurations which violate Gauss' law (if you like, such configurations describe some matter with rest mass Γ_∞). So, states with energy $\ll \Gamma_\infty$ all satisfy Gauss' law. Then further, we want $\mathbf{H}$ to act *within* this subspace, and not create excitations of enormous energies like Γ_∞. This requires $[\mathbf{G}_i, \mathbf{H}] = 0, \forall i$, which is exactly the condition that $\mathbf{H}$ is gauge invariant.

A useful change of variables gets rid of these annoying signs. Assume the lattice is bipartite: made of two sublattices A, B each of which only touches the other. Then draw arrows from A sites to B sites, and let



$$\begin{aligned} \mathbf{e}_{ij} &\equiv \eta_i \mathbf{n}_{ij} \\ \mathbf{a}_{ij} &\equiv \eta_i \Theta_{ij} \end{aligned}, \quad \eta_i \equiv \begin{cases} +1, & i \in A \\ -1, & i \in B \end{cases}.$$



Then the Gauss constraint now reads

$$0 = \mathbf{e}_{\bar{i}\bar{1}} + \mathbf{e}_{\bar{i}\bar{2}} + \mathbf{e}_{\bar{i}\bar{3}} + \mathbf{e}_{\bar{i}\bar{4}} \equiv \Delta \cdot \mathbf{e}(\bar{i}).$$

This is the lattice divergence operation. The plaquette term reads

$$\cos\left(\Theta_{12} - \Theta_{23} + \Theta_{34} - \Theta_{41}\right) = \cos\left(\mathbf{a}_{12} + \mathbf{a}_{23} + \mathbf{a}_{34} + \mathbf{a}_{41}\right) \equiv \cos\left(\Delta \times \mathbf{a}\right)$$

– the lattice curl (more precisely, it is $(\Delta \times \mathbf{a}) \cdot \check{z}$). In these variables,

$$\mathbf{H} = \frac{U}{2} \sum_l \mathbf{e}_l^2 - K \sum_{\square} \cos\left((\Delta \times \mathbf{a}) \cdot \check{n}_{\square}\right)$$

(in the last term we emphasize that this works in $D \geq 2+1$ if we remember to take the component of the curl normal to the face in question). This is (compact) lattice $\mathsf{U}(1)$ gauge theory, with no charges. The word 'compact' refers to the fact that the charge is quantized; the way we would add charge is by modifying the Gauss' law to

$$\underbrace{\Delta \cdot \mathbf{e}(\bar{i})}_{\in \mathbb{Z}} = \underbrace{\text{charge at } \bar{i}}_{\Longrightarrow \in \mathbb{Z}}$$

where the charge must be quantized because the LHS is an integer. (In the noncompact electrodynamics we found dual to the superfluid, it was the continuous angle variable which participated in the Gauss' law, and the discrete variable which was gauge variant.)

What is it that's compact in compact QED?

The operator appearing in Gauss' law

$$\mathbf{G}(x) \equiv \left(\vec{\nabla} \cdot \vec{\mathbf{e}}(x) - 4\pi \mathbf{n}(x)\right)$$

(here $\mathbf{n}(x)$ is the density of charge) is the generator of gauge transformations, in the sense that a gauge transformation acts on any operator $\mathcal{O}$ by

$$\mathcal{O} \mapsto e^{-\mathbf{i}\sum_x \alpha(x)\mathbf{G}(x)} \mathcal{O} e^{\mathbf{i}\sum_x \alpha(x)\mathbf{G}(x)} \tag{5.46}$$

This is a fact we've seen repeatedly above, and it is familiar from ordinary QED, where using the canonical commutation relations

$$[\mathbf{a}^i(x), \mathbf{e}^j(y)] = -\mathbf{i}\delta^{ij}\delta(x-y), \quad [\phi(x), \mathbf{n}(y)] = -\mathbf{i}\delta(x-y)$$

(ϕ is the phase of a charged field, $\Phi = \rho e^{\mathbf{i}\phi}$) in (5.46) reproduce the familiar gauge transformations

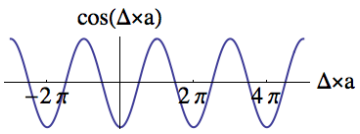
$$\vec{\mathbf{a}} \to \vec{\mathbf{a}} + \vec{\nabla}\alpha, \qquad \phi \to \phi + \alpha \quad .$$

SO: if all the objects appearing in Gauss' law are integers (which is the case if charge is quantized and electric flux is quantized), it means that the gauge parameter α itself only enters mod 2π, which means the gauge transformations live in $\mathsf{U}(1)$, as opposed to $\mathbb{R}$. So it's the gauge group that's compact.

This distinction is very important, because (in the absence of matter) this model does not have a deconfined phase! To see this result (due to Polyakov), first consider strong coupling:

$\boxed{U \gg K}$: The groundstate has $\mathbf{e}_{\bar{l}} = 0$, $\forall \bar{l}$. (Notice that this configuration satisfies the constraint.) There is a gap to excitations where some link has an *integer* $\mathbf{e} \neq 0$, of order U. (If $\mathbf{e}$ were continuous, there would not be a gap!) In this phase, electric flux is confined, *i.e.* costs energy and is generally unwanted.

$\boxed{U \ll K}$: The surprising thing is what happens when we make the gauge coupling weak.



Then we should first minimize the magnetic flux term: minimizing $-\cos(\Delta \times \mathbf{a})$ means $\Delta \times \mathbf{a} \in 2\pi\mathbb{Z}$. Near each minimum, the physics looks like Maxwell, $\mathbf{h} \sim \mathbf{e}^2 + \mathbf{b}^2 + \cdots$. BUT: it turns out to be a colossally bad idea to ignore the tunnelling *between* the minima. To see this, begin by solving the Gauss law constraint $\Delta \cdot \mathbf{e} = 0$ by introducing

$$\mathbf{e}_{\bar{1}\bar{2}} \equiv \frac{1}{2\pi}\left(\chi_2 - \chi_1\right) \tag{5.47}$$

(*i.e.* $\vec{\mathbf{e}} = \check{z} \cdot \Delta\chi \frac{1}{2\pi}$.) χ is a (discrete!) 'height variable'. Then the operator

$$e^{\mathbf{i}(\Delta\times\mathbf{a})(\bar{i})}$$

increases the value of $\mathbf{e}_{\bar{i}\bar{a}}$ for all neighboring sites $\bar{a}$, which means it jumps $\chi_{\bar{i}} \to \chi_{\bar{i}} + 2\pi$. So we should regard

$$(\Delta \times \mathbf{a})(\bar{i}) \equiv \Pi_\chi(\bar{i})$$

as the conjugate variable to χ, in the sense that

$$[\Pi_\chi(r), \chi(r')] = -\mathbf{i}\delta_{rr'}.$$

Notice that this is consistent with thinking of χ as the dual scalar related to the gauge field by our friend the (Hodge) duality relation

$$\partial_\mu \chi = \epsilon_{\mu\nu\rho}\partial_\nu a_\rho.$$

The spatial components i say $\partial_i \chi = \epsilon_{ij} f_{0j}$, which is the continuum version of (5.47). The time component says $\dot{\chi} = \epsilon_{ij} f_{ij} = \nabla \times a$, which indeed says that (if χ has quadratic kinetic terms), the field momentum of χ is the magnetic flux. So χ is the would-be transverse photon mode.

The hamiltonian is now

$$\mathbf{H} = \frac{U}{2}\sum_l (\Delta\chi)^2 - K\sum_r \cos \Pi_\chi(r)$$

with no constraint, but $\chi \in 2\pi\mathbb{Z}$. In the limit $U \gg K$, the spatial gradients of χ are forbidden – χ wants to be uniform. From the definition (5.47), uniform χ means there are no electric field lines, this is the confined phase. Deconfinement limit should be $K \gg U$, in which case it looks like we can Taylor expand the cosine $\cos \Pi_\chi \sim 1 - \frac{1}{2}\Pi_\chi^2$ about one of its minima, and get harmonic oscillators. But: tunneling between the neighboring vacua of $\Delta \times \mathbf{a}$ is accomplished by the *flux-insertion operator* (or *monopole operator*)

$$e^{\mathbf{i}\chi}, \quad \text{which satisfies } [e^{\mathbf{i}\chi(r)}, (\Delta \times \mathbf{a})(r')] = e^{\mathbf{i}\chi(r)}\delta_{rr'}$$

– that is, $e^{\mathbf{i}\chi}$ is a raising operator for $\Delta \times \mathbf{a}$. To analyze whether the Maxwell limit survives this, let's go to the continuum and study perturbations of the free hamiltonian

$$\mathbf{H}_0 = \int \left(\frac{U}{2}\left(\vec{\nabla}\chi\right)^2 + \frac{K}{2}\Pi_\chi^2\right)$$

by

$$\mathbf{H}_1 = -\int V_0 \cos\chi \ .$$

This operator introduces tunneling events by $\Pi_\chi \to \Phi_\chi \pm 2\pi$ with rate V_0. Alternatively, notice that again we can think of the addition of this term as energetically imposing the condition that $\chi \in 2\pi\mathbb{Z}$.

So: is V_0 irrelevant? Very much no. In fact

$$\langle \cos\chi(r) \cos\chi(0)\rangle_0 \sim \text{const} \tag{5.48}$$

has constant amplitude at large r! That means that the operator has dimension zero, and the perturbation in the action has $[S_1 = -\int V_0 \cos\chi d^2x d\tau] \sim L^3$, very relevant.

The result is that it pins the χ field (the would-be photon mode) to an integer, from which it can't escape. This result is due to Polyakov.

To see (5.48) begin with the gaussian identity

$$\left\langle e^{\mathbf{i}s\chi(x)} e^{\mathbf{i}s'\chi(0)} \right\rangle = e^{-\frac{ss'}{2}\langle\chi(x)\chi(0)\rangle} ,$$

with $s, s' = \pm$. The required object is

$$\begin{aligned}
\langle\chi(x)\chi(0)\rangle &= \frac{\mathbf{i}}{T}\int \mathrm{đ}^3 p \frac{e^{\mathbf{i}\vec{p}\cdot\vec{x}}}{p^2} = \mathbf{i}\frac{2\pi}{(2\pi)^3 T}\int_0^\infty dp \underbrace{\int_{-1}^{1} d\cos\theta e^{\mathbf{i}px\cos\theta}}_{=\frac{2\sin px}{px}} \\
&= \mathbf{i}\frac{2}{(2\pi)^2 T}\int_0^\infty dp \frac{\sin px}{px} \\
&= \mathbf{i}\frac{2}{2\pi T}\frac{1}{x}\frac{1}{2}\underbrace{\int_{-\infty}^{\infty} d\bar{p}\frac{\sin\bar{p}}{\bar{p}}}_{=\pi} \\
&= \frac{\mathbf{i}}{2Tx}.
\end{aligned} \tag{5.49}$$

(I have set the velocity of propagation to 1, and $T \equiv U/K$ is the coefficient in front of the Lagrangian, $S = T\int d^3x \partial_\mu\chi\partial^\mu\chi$.) So

$$\left\langle e^{\mathbf{i}s\chi(x)} e^{\mathbf{i}s'\chi(0)} \right\rangle = e^{-\mathbf{i}\frac{ss'}{4xT}} .$$

And

$$\langle\cos\chi(x)\cos\chi(0)\rangle = \cos\frac{1}{4Tx}$$

which does not decay at long distance, and in fact approaches a constant.

- The fact that the would-be-transverse-photon χ is massive means confinement of the gauge theory. To see that external charge is confined, think as usual about the big rectangular Wilson loop $\langle W(\square)\rangle = \left\langle e^{\mathbf{i}\oint_\square A}\right\rangle \overset{\text{euclidean}}{\sim} e^{-E(R)T}$ as an order parameter for confinement. In term of χ,

$$\oint_\square A = \int_\blacksquare F_{12} = \int_\blacksquare g\dot{\chi}$$

(I've absorbed a factor of the gauge coupling into χ to make the dimensions work nicely, $\epsilon_{\mu\nu\rho}\partial_\nu A_\rho = g\partial_\mu\chi$) and the expectation is

$$\langle W(\square)\rangle = Z^{-1}\int [d\chi] e^{-S_\chi + g\mathbf{i}\int_\blacksquare \dot{\chi}} \sim e^{-cg^2 m_\chi \cdot \text{area}(\blacksquare)}.$$

In the last step we did the gaussian integral from small χ fluctuations. This area-law behavior proportional to m_χ means that the mass for χ confines the gauge theory. This is the same (Polyakov) effect we saw in the previous section, where the monopole tunneling events produced the mass.

- Adding matter helps to produce a deconfined phase! In particular, the presence of enough massless charged fermions can render the monopole operator irrelevant. I recommend this paper by Tarun Grover for more on this.

- Think about the action of $e^{\mathrm{i}\chi(x,t)}$ from the point of view of $2+1$d spacetime: it inserts 2π magnetic flux at the spacetime point x,t. From that path integral viewpoint, this is an event localized in three dimensions which is a source of magnetic flux – a magnetic monopole. In Polyakov's paper, he uses a UV completion of the abelian gauge theory (not the lattice) in which the magnetic monopole is a smooth solution of field equations (the 't Hooft-Polyakov monopole), and these solutions are instanton events. The $\cos\chi$ potential we have found above arises from, that point of view, by the same kind of dilute instanton gas sum that we did in the $D=1+1$ Abelian Higgs model.

5.5 Deconfined Quantum Criticality

[The original papers are this and this; this treatment follows Ami Katz' BU Physics 811 notes.] Consider a square lattice with quantum spins (spin half) at the sites, governed by the Hamiltonian

$$H_{JQ} \equiv J\sum_{\langle ij\rangle} \vec{S}_i\cdot\vec{S}_j + Q\sum_{[ijkl]}\left(\vec{S}_i\cdot\vec{S}_j-\frac{1}{4}\right)\left(\vec{S}_k\cdot\vec{S}_l-\frac{1}{4}\right).$$

Here $\langle ij\rangle$ denotes pairs of sites which share a link, and $[ijkl]$ denotes groups of four sites at the corners of a plaquette. This JQ-model is a somewhat artificial model designed to bring out the following competition which also exists in more realistic models:

$\boxed{J\gg Q}$: the groundstate is a Neel antiferromagnet (AFM), with local order parameter $\vec{n}=\sum_i(-1)^{x_i+y_i}\vec{S}_i$, whose expectation value breaks the spin symmetry $\mathsf{SU}(2)\to\mathsf{U}(1)$. Hence, the low-energy physics is controlled by the (two) Nambu-Goldstone modes. This is well-described by the field theory we studied in §3.3.

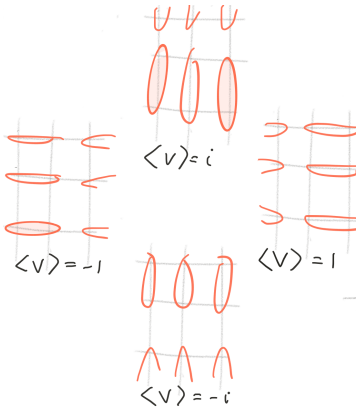
$\boxed{Q\gg J}$: The Q-term is designed to favor configurations where the four spins around each square form a pair of singlets. A single Q-term has a two-fold degenerate groundstate, which look like $|{=}\rangle$ and $|||\rangle$. The sum of all of them has four groundstates, which look like ... These are called *valence-bond solid* (VBS) states. The VBS order

parameter on the square lattice is

$$V = \sum_i \left((-1)^{x_i} \vec{S}_i \cdot \vec{S}_{i+x} + \mathbf{i}(-1)^{y_i} \vec{S}_i \cdot \vec{S}_{i+y} \right) \in \mathbb{Z}_4.$$

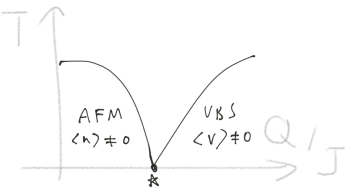
In the four solid states, it takes the values $1, \mathbf{i}, -1, -\mathbf{i}$. Notice that they are related by multiplication by $\mathbf{i} = e^{\mathbf{i}\pi/2}$. V is a singlet of the spin $\mathsf{SU}(2)$, but the VBS states do break spacetime symmetries: a lattice rotation acts by $R_{\pi/2} : V \to -\mathbf{i}V$ (the Neel order $\vec{n}$ is invariant), while a translation by a single lattice site acts by

$$T_{x,y} : \vec{n} \to -\vec{n}, \; T_x : V \to -V^\dagger, T_y : V \to V^\dagger. \tag{5.50}$$

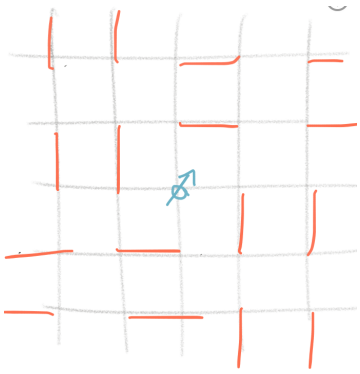


The VBS phase is gapped (it only breaks discrete symmetries, so no goldstones).

Claim: There seems to be a continuous transition between these two phases as a function of Q/J. (If it is first order, the latent heat is very small.) Here's why this is weird and fascinating: naively, the order parameters break totally different symmetries, and so need have nothing to do with each other. Landau then predicts that generically there should be a region where both are nonzero or where both are zero. Why should the transitions coincide? What are the degrees of freedom at $\star$?



To get a big hint, notice that the VBS order parameter is like a discrete rotor: if we had a triangular lattice it would be in $\mathbb{Z}_6$ and would come closer to approximating a circle-valued field. In any case, we can consider vortex configurations, where the phase of V rotates (discretely, between the four quadrants) as we go around a point in space. Such a vortex looks like the picture at right.



Notice that inside the core of the vortex, there is necessarily a spin which is not paired with another spin: The vortex carries spin: it transforms as a doublet under the spin $\mathsf{SU}(2)$. Why do we care about such vortices? I've been trying to persuade you for the past two sections that the way to think about destruction of (especially $\mathsf{U}(1)$) ordered phases is by proliferating vortex defects. Now think about proliferating this kind of VBS vortex. Since it carries spin, it necessarily must break the $\mathsf{SU}(2)$

symmetry, as the Neel phase does. This is why the transitions happen at the same point.

To make this more quantitative, let's think about it from the AFM side: how do we make V from the degrees of freedom of the low energy theory? It's not made from n since it's a spin singlet which isn't 1 (spin singlets made from n are even under a lattice translation). What about the $\mathbb{CP}^1$ version, aka the Abelian Higgs model, aka scalar QED (now in $D = 2 + 1$)?

$$L = -\frac{1}{4g^2}F^2 + |Dz|^2 - m^2|z|^2 - \frac{\lambda}{4}|z|^4$$

where $z = \begin{pmatrix} z_\uparrow \\ z_\downarrow \end{pmatrix}$, and $D_\mu z = (\partial_\mu - \mathbf{i}A_\mu)z$ as usual. Let's think about the phases of this model.

$\boxed{m^2 < 0}$: Here z condenses and breaks $\mathsf{SU}(2) \to \mathsf{U}(1)$, and A_μ is higgsed. A gauge invariant order parameter is $\vec{n} = z^\dagger \vec{\sigma} z$, and there are two goldstones associated with its rotations. This is the AFM. The cautionary tale I told you about this phase in $D = 1 + 1$ doesn't happen because now the vortices are *particles* rather than instanton events. More on these particles below.

$\boxed{m^2 > 0}$: Naively, in this phase, z are uncondensed and massive, leaving at low energies only $L_{\text{low-E}} \stackrel{?}{=} -\frac{1}{4g^2}F^2$, Maxwell theory in $D = 2 + 1$. This looks innocent but it will occupy us for quite a few pages starting now. This model has a conserved current (conserved by the Bianchi identity)

$$J_F^\mu \equiv \epsilon^{\mu\nu\rho}F_{\nu\rho} = \frac{1}{8\pi}n^a\partial_\mu n^b \partial_\nu n^c \epsilon_{abc}.$$

In terms of the gauge variables, the thing that's conserved is the magnetic flux; in terms of the spins, it's the skyrmion number (as you showed on the homework) We can follow these around more effectively by introducing the dual scalar field by a by-now-familiar duality relation:

$$J_F^\mu \equiv \epsilon^{\mu\nu\rho}F_{\nu\rho} \equiv g\partial^\mu \chi. \tag{5.51}$$

You can think of the last equation here as a *solution* of the conservation law $\partial_\mu J_F^\mu = 0$. The symmetry acts on χ by shifts: $\chi \to \chi + \text{constant}$. In terms of χ, the Maxwell action is

$$L_{\text{low-E}} \stackrel{?}{=} -\frac{1}{4g^2}F^2 = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi.$$

But this is a massless scalar, a gapless theory. And what is the $\chi \to \chi + c$ symmetry in terms of the spin system? I claim that it's the rotation of the phase of the VBS order parameter, which is explicitly broken by the squareness of the square lattice. An

improvement would then be

$$L_{\text{low-E}} = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V_k\cos\left(k\chi/g\right)$$

where $V_k \sim \frac{1}{a^2}$ (a is the lattice spacing) comes from the breaking the rotation invariance by the lattice down to $\mathbb{Z}_k$ ($k = 4$ for the square lattice).

To see that shifts of χ are VBS rotations, let's reproduce the lattice symmetries in the Abelian Higgs model. Here's the action of lattice translations $T \equiv T_x$ or T_y (take a deep breath.): $T : n^a \to -n^a$ but $n^a = z^\dagger\sigma^a z$, so on z we must have $T : z \to \mathbf{i}\sigma^2 z^\star$. The gauge current is $j_\mu = \mathbf{i}z^\dagger\partial_\mu z + h.c. \to -j_\mu$ which means we must have $A_\mu \to -A_\mu$ and $F_{\mu\nu} \to -F_{\mu\nu}$. Therefore by (5.51) we must have $T : \partial\chi \to -\partial\chi$ which means that

$$T_{x,y} : \chi \to -\chi + g\alpha_{x,y}$$

where $\alpha_{x,y}$ are some so-far-undetermined numbers, and g is there on dimensional grounds. Therefore, by choosing $T_{x,y}\chi \to -\chi \pm g\pi/2, R_{\pi/2} : \chi \to \chi - g\pi/2$ we can reproduce the transformation (5.50) by identifying

$$V = ce^{\mathbf{i}\chi/g}$$

(up to an undetermined overall complex number). Notice for future reference the canonical commutation relation between the flux current density ($J^0_F = g\dot{\chi} = \frac{g}{\mathbf{i}}\frac{\delta}{\delta\chi}$) and V:

$$[J^0_F(x), V(0)] = V(0)\delta^2(x). \tag{5.52}$$

It creates flux.

So χ is like the phase of the bosonic operator V which is condensed in the VBS phase; lattice effects break the $\mathsf{U}(1)$ symmetry down to some discrete subgroup ($\mathbb{Z}_4$ for the square lattice, $\mathbb{Z}_6$ for triangular, $\mathbb{Z}_3$ for honeycomb), with a potential of the form $\mathcal{V}(V^k) = m_\chi^3\cos(4\chi/g) + \cdots$, where $k = 4, 6, 3...$ depends on the lattice, which has k minima, corresponding to the k possible VBS states. By (5.52), such a potential has charge k under J_F.

Consider this phase from the point of view of the gauge theory now. Notice that χ is the same (up to a factor) dual variable we introduced in our discussion of compact QED, and the Wilson loop will again produce an area law if χ is massive, as with the Polyakov effect.

In order for this story to make sense, we need that $M, g^2 \ll \frac{1}{a^2}$, so that χ is actually a low-energy degree of freedom. The idea is that the critical point from tuning J/Q to the critical value is reached by taking $m_\chi \to 0$. What is the nature of this critical theory? It has emergent deconfined gauge fields, even though the phases on either side

of the critical point do not (they are confined $m > 0$ and Higgsed $m < 0$ respectively). Hence the name deconfined quantum criticality.

The conjecture (which would explain the phase diagram above) is that this gauge theory is a critical theory (in fact a conformal field theory) with only one relevant operator (the one which tunes us through the phase transition, the mass for χ) which is a singlet under all the symmetries. Recall that $e^{\mathbf{i}k\chi}$ has charge k under the J_F symmetry, and the square lattice preserves a $\mathbb{Z}_4 \subset \mathsf{U}(1)$ subgroup, so only allows the 4-vortex-insertion operator $e^{\mathbf{i}4\chi}$. What is the dimension of this operator at the critical point? The conjecture is that it has dimension larger than 3.

Insanely brief sketch of a check at large N. Actually, this can be checked very explicitly in a large-N version of the model, with N component z fields, so that the spin is $\phi^A = z^\dagger T^A z, A = 1..N^2-1$. This has $\mathsf{SU}(N)$ symmetry. When $m^2 < 0$, it is broken to $\mathsf{SU}(N-1)$, with $2(N-1)$ goldstone bosons. (Actually there is a generalization of the lattice model which realizes this – just make the spins into $N \times N$ matrices.)

Introducing an H-S field σ to decouple the $|z|^4$ interaction, we can make the z integrals gaussian, and find (this calculation is just like our earlier analysis in §4.3)

$$S[A,\sigma] = \int đp \left(\frac{1}{4} F_{\mu\nu}(p) \left(\frac{1}{g_{UV}^2} + \frac{c_1 N}{\mathbf{i}p} \log \frac{2m + \mathbf{i}p}{2m - \mathbf{i}p} \right) F^{\mu\nu}(-p) + \sigma(p) \left(-\frac{1}{\lambda} + \frac{c_2 N}{\mathbf{i}p} \log \frac{2m + \mathbf{i}p}{2m - \mathbf{i}p} \right) \sigma(-p) \right)$$

In the IR limit, $m \ll p \ll g_{UV}^2 N, \lambda N$, this is a scale-invariant theory with $\langle FF \rangle \sim p, \langle \sigma\sigma \rangle \sim p$ so that both F and σ have dimension near 2. (Actually the dimension of F is fixed at 2 by flux conservation.) z doesn't get any anomalous dimension at leading order in N.

This is all consistent with the claim so far. What is the dimension of $V_4 = e^{\mathbf{i}4\chi}$? To answer this question, we use a powerful tool of conformal field theory called *radial quantization*. Consider the theory on a cylinder, $S^2 \times \mathbb{R}$, where the last factor we can interpret as time. In a conformal field theory there is a one-to-one map between local operators and states of the theory on $S^d \times \mathbb{R}$. The state corresponding to an operator $\mathcal{O}$ is just $\mathcal{O}(0)\,|0\rangle$. The energy of the state on the sphere is the scaling dimension of the operator. (For an explanation of this, I refer to §4 of these notes.)

The state created by acting with $V_k(0)$ on the vacuum maps by this transformation to an initial state with flux k spread over the sphere (think of it as the 2-sphere surrounding the origin in spacetime): this state has charge k under $Q_F = \int_{S^2} J_F^0 = \int_{S^2} F_{12}$. The dimension of V_k is the energy of the lowest-energy state with $Q_F = k$. We can compute this by euclidean-time path integral:

$$Z_k = \mathrm{tr}_{Q_F=k} e^{-TH_{\mathrm{cyl}}} \overset{T\to 0}{\to} e^{-T\Delta_k}.$$

This is

$$Z_k = \int [dA]\delta\left(\int F - k\right)\int [dzdz^\dagger] e^{-S[z,A]} \equiv e^{-F_k}$$

which at large-N we can do by saddle point. The dominant configuration of the gauge field is the charge-k magnetic monopole $\underline{A}_\varphi = \frac{k}{2}(1-\cos\varphi)$, and we must compute

$$\int [d^2z] e^{z^\dagger\left(-D_{\underline{A}}^\dagger D_{\underline{A}} + m^2\right)z} = \det\left(-D_{\underline{A}}^\dagger D_{\underline{A}} + m^2\right)^{-N/2} = e^{-\frac{N}{2}\mathrm{tr}\log\left(-D_{\underline{A}}^\dagger D_{\underline{A}} + m^2\right)}$$

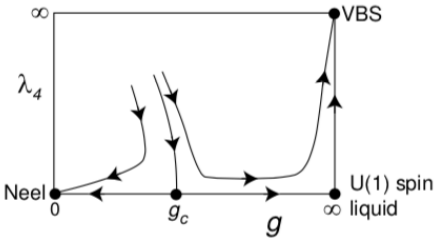
The free energy is then a sum over eigenstates of this operator

$$\left(-\partial_\tau^2 - \vec{D}_{\underline{A}}^2\right) f_\ell e^{\mathrm{i}\omega\tau} = \left(\omega^2 + \lambda_\ell(k)\right) f_\ell e^{\mathrm{i}\omega\tau}$$

$$F_k = NT\int đ\omega \sum_\ell (2\ell+1)\log(\omega^2 + \lambda_\ell(k) + m^2).$$

The difference $F_k - F_0$ is UV finite and gives $\Delta_k = Nc_k,\ c_1 \sim .12, c_4 \sim .82$. Unitarity requires $\Delta_1 \geq \frac{1}{2}$ (= the free scalar dimension), so don't trust this for $N < 4$.

So the idea can be summarized by the flow diagram at right, where the horizontal axis is $m_z^2 \sim g - g_c$. For $m_z^2 < 0$, we are in the Higgs phase of the gauge theory, where the spin symmetry is spontaneously broken – the Neel phase. For $m_z^2 > 0$, if we managed to tune $V_k = 0$, we would end up at a state with a massless photon. But this state is unstable to the proliferation of monopoles (created by the operator whose coefficient in the action is V_k). And on the lattice, the coefficient V_k of the monopole fugacity is inevitably nonzero, so we end up instead in the gapped VBS phase. But right at the critical point, there's a deconfined $\mathsf{U}(1)$ gauge theory!



[Fig is from this nice summary].

Pure field theory description. We've been discussing a theory with $\mathsf{U}(1)_{\mathrm{VBS}} \times \mathsf{SU}(2)_{\mathrm{spin}}$ symmetry. Lattice details aside, how can we encode the way these two symmetries are mixed up which forces the order parameter of one to be the disorder operator for the other? To answer this, briefly consider enlarging the symmetry to $\mathsf{SO}(5) \supset \mathsf{U}(1)_{\mathrm{VBS}} \times \mathsf{SU}(2)_{\mathrm{spin}}$, and organize $(\mathrm{Re}\,V, \mathrm{Im}\,V, n^1, n^2, n^3) \equiv n^a$ into a 5-component mega-voltron-spin vector. We saw that in $D = 0+1$, we could make a WZW term with a 3-component spin

$$\mathcal{W}_0[(n^1, n^2, n^3)] = \int_{B_2} \epsilon^{abc} n^a dn^b \wedge dn^c.$$

Its point in life was to impose the spin commutation relations at spin s when the coefficient is $2s$. In $D = 1 + 1$, we can make a WZW term with a 4-component spin, which can have $\mathsf{SO}(4)$ symmetry

$$\mathcal{W}_1[(n^1, n^2, n^3, n^4)] = \int_{B_3} \epsilon^{abcd} n^a dn^b \wedge dn^c \wedge dn^d.$$

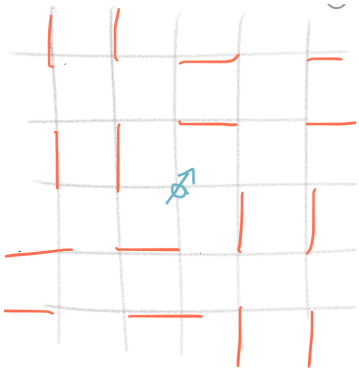
[38] Once we've got this far, how can you resist considering

$$\mathcal{W}_2[(n^1, n^2, n^3, n^4, n^5)] = \int_{B_4} \epsilon^{abcde} n^a dn^b \wedge dn^c \wedge dn^d \wedge dn^e.$$

What does this do? Break the $\mathsf{SO}(5) \to \mathsf{U}(1) \times \mathsf{SU}(2)$ and consider a vortex configuration of V at $x^2 = x^3 = 0$. Suppose our action contains the term $k\mathcal{W}_2[n]$ with $k = 1$. Evaluate this in the presence of the vortex:

$$k\mathcal{W}_2[(n^1, n^2, n^3, n^4, n^5)|_{\text{vortex of } n^1 + \mathbf{i}n^2 \text{ at } x^2 = x^3 = 0}] = \frac{k}{16\pi^2} \int_{B_2 | x^2 = x^3 = 0} \epsilon^{abc} n^a dn^b \wedge dn^c = k\mathcal{W}_0[(n^1, n^2, n^3)].$$

This says the remaining three components satisfy the spin-half commutation relations: there is a spin in the core of the vortex, just as in the lattice picture at right.



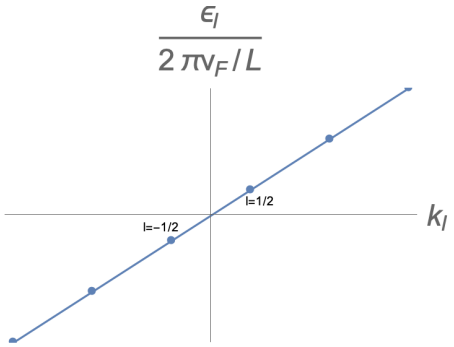
[38] In fact the $D = 1 + 1$ version of this is extremely interesting. A few brief comments: (1) involves a *real* VBS order parameter n^4.) (2) The $D = 1 + 1$ term has the same number of derivatives (in the EOM) as the kinetic term $\partial n^a \partial n^a$. This means they can compete at a fixed point. The resulting CFTs are called WZW models. (3) The above is in fact a description of the spin-half chain, which previously we've described by an $\mathsf{O}(3)$ sigma model at $\theta = \pi$.

5.6 Bosonization

In addition to T-duality, the theory of a massless compact scalar field ϕ in $D = 1+1$ enjoys yet another dual description, namely a description in terms of a Dirac fermion Ψ.

5.6.1 Bosonization, part 1: counting

[Sachdev, chapter 20] Consider a massless Dirac fermion in $D = 1+1$ with spinor components $\Psi = (\mathbf{L}, \mathbf{R})$, where the name indicates the direction of propagation. Let's quantize this system on a circle of length L with APBC: $\mathbf{L}(x+L) = -\mathbf{L}(x), \mathbf{R}(x+L) = -\mathbf{R}(x)$. The mode expansion is:



$$\mathbf{R}(x) = \frac{1}{\sqrt{L}} \sum_{l \in \mathbb{Z}+\frac{1}{2}} \mathbf{R}_l e^{\frac{2\pi l x \mathbf{i}}{L}}$$

and a similar expression for $\mathbf{L}$. We'll focus on $\mathbf{R}$ for a while. The modes satisfy

$$\{\mathbf{R}_l, \mathbf{R}_{l'}^\dagger\} = \delta_{ll'}.$$

and the Hamiltonian (minus the vacuum energy) is

$$\mathbf{H}_R = \frac{2\pi v_F}{L} \sum_{l \in \mathbb{Z}+\frac{1}{2}} l \underbrace{\mathbf{R}_l^\dagger \mathbf{R}_l}_{\equiv \mathbf{n}_l^R} - E_0$$

so $\mathbf{H}_R |\text{gs}\rangle = 0$. (The full $\mathbf{H}$ is $\mathbf{H} = \mathbf{H}_R + \mathbf{H}_L$.) The groundstate has all modes with $l > 0$ empty and all modes with $l < 0$ filled. Notice that the Hamiltonian is symmetric under the interchange $l \to -l$: *removing* a particle with negative momentum $-|l|$ (or energy) *adds* momentum $|l|$. The charge operator is

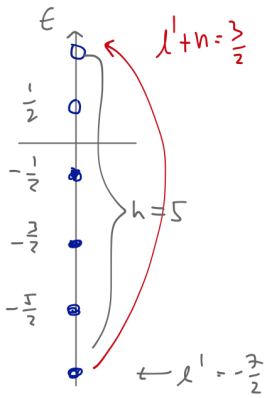
$$\mathbf{Q}_R = \sum_l : \mathbf{R}_l^\dagger \mathbf{R}_l : \in \mathbb{Z}\ , \quad \mathbf{Q}_R |gs\rangle = 0, \quad [\mathbf{Q}_R, \mathbf{H}] = 0.$$

Let's count states. The thermal partition sum for the rightmover is

$$\begin{aligned} Z_R(T) &= \text{tr}_{\mathcal{H}_R} e^{-\mathbf{H}_R/T} \\ &= \prod_{l \in \mathbb{Z}+\frac{1}{2}} \sum_{\mathbf{n}_l^R = 0,1} e^{-\frac{1}{T}\frac{2\pi v_F}{L}|l|\mathbf{n}_l^R} \equiv \prod_{l \in \mathbb{Z}+\frac{1}{2}} \sum_{\mathbf{n}_l^R = 0,1} q^{|l|\mathbf{n}_l^R} \qquad (q \equiv e^{-\frac{2\pi v_F}{TL}}) \\ &= \prod_{l \in \mathbb{Z}+\frac{1}{2}} \left(1 + q^{|l|}\right) \end{aligned}$$

$$= \prod_{n=1}^{\infty} \left(1 + q^{n-\frac{1}{2}}\right)^2 \ . \tag{5.53}$$

This is an infinite-product representation of an elliptic theta function.



Another way to arrive at this function is to keep track of the particle-hole excitations. If we take a particle out of the Fermi sea from a filled level l' and put it in an empty level $l'+n$, we create an excitation with momentum $l'+n-(l')=n$. n is an integer. The change in energy is

$$\Delta\epsilon = v_F \Delta k = \frac{2\pi v_F}{L} n.$$

Notice that it is independent of l'; this is because of the linear dispersion near the Fermi surface. The two fermions (the particle and the hole) move at the same speed, and in the same direction (because we are in 1d and they are both right-moving!) so it might be a good idea to think of them as a single particle. The operator which makes this happen is $\mathbf{R}^\dagger_{l'+n}\mathbf{R}_{l'}$. It is a bosonic operator. More precisely, the object $\boldsymbol{\rho}^\dagger_n \equiv \sum_{l'} \mathbf{R}^\dagger_{l'+n}\mathbf{R}_{l'}$ is a bosonic operator which accomplishes the above operation and can act many times without giving zero. It raises the momentum by n (in units of $\frac{2\pi}{L}$) and the energy by $\frac{2\pi n}{L} v_F$. Notice that it is a Fourier mode of the (rightmoving) density operator $\rho_R(x) = \mathbf{R}^\dagger(x)\mathbf{R}(x)$.

An arbitrary fermion state is

$$|F\rangle = \prod_{l>0} \mathbf{R}^\dagger_l \prod_{l'<0} \mathbf{R}_{l'} \, |gs\rangle \, .$$

If it has charge Q_R, we can rearrange this as

$$|F\rangle = (\text{particle-hole excitations}) \, |Q_R\rangle$$

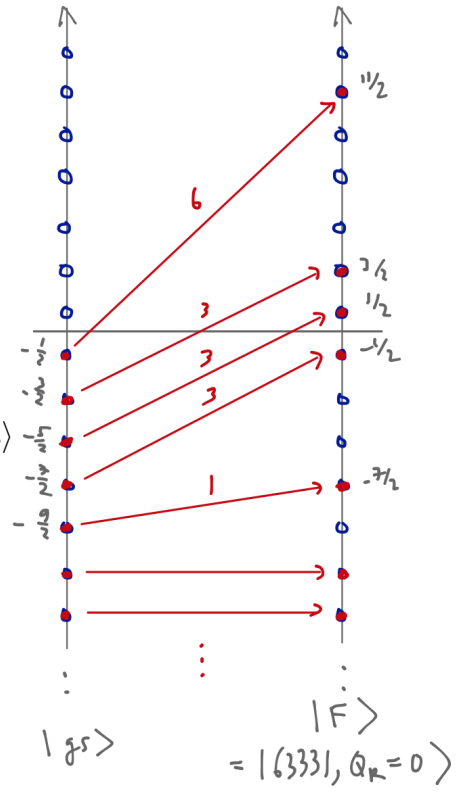
where $|Q_R\rangle$ is the lowest-energy state with charge Q_R. This state has only ls of the same sign, and only the first Q_R of them. For $Q_R > 0$ it is

$$|Q_R\rangle = \mathbf{R}^\dagger_{Q_R-\frac{1}{2}} \cdots \mathbf{R}^\dagger_{\frac{3}{2}} \mathbf{R}^\dagger_{\frac{1}{2}} \, |\text{gs}\rangle \, .$$

(For $Q_R < 0$, remove the daggers and reverse the signs.) It has energy (for either sign of Q_R)

$$E_0(Q_R) = \frac{2\pi v_F}{L} \sum_{l=\frac{1}{2}}^{|Q_R|-\frac{1}{2}} l = \frac{\pi v_F}{L} Q_R^2 \, .$$

For simplicity, let's consider particle-hole excitations above the groundstate, $|\text{gs}\rangle = |Q_R = 0\rangle$, as in the figure at right. (To understand what happens for $Q_R \neq 0$, just shift all the labels by Q_R.) The state in the figure is



$$|63331\rangle =$$

$$\left(\mathbf{R}^\dagger_{-\frac{7}{2}}\mathbf{R}_{-\frac{9}{2}}\right)\left(\mathbf{R}^\dagger_{-\frac{1}{2}}\mathbf{R}_{-\frac{7}{2}}\right)\left(\mathbf{R}^\dagger_{\frac{1}{2}}\mathbf{R}_{-\frac{5}{2}}\right)\left(\mathbf{R}^\dagger_{\frac{3}{2}}\mathbf{R}_{-\frac{3}{2}}\right)\left(\mathbf{R}^\dagger_{\frac{11}{2}}\mathbf{R}_{-\frac{1}{2}}\right)|\text{gs}\rangle$$

It is important that we act first (first means its at the right of the queue) with the boson operator making the largest jump – this guarantees that we never try to annihilate a fermion that is not there!

So we may label the excitation above $|Q_R\rangle$ by a set of numbers r_n indicating how many steps up the fermion in box $n = -l - \frac{1}{2}$ was pushed:

$$r_1 \geq r_2 \geq r_3 ... \geq r_{n_\star} = 0.$$

The corresponding state is:

$$\boldsymbol{\rho}^\dagger_{r_{n_\star}} \cdots \boldsymbol{\rho}^\dagger_{r_2}\boldsymbol{\rho}^\dagger_{r_1}|Q_R\rangle$$

This information can be represented by a *Young diagram*: For example, we represent the state in the figure, which has $\{r_n\} = \{6, 3, 3, 3, 1\}$, by:

$$\left|\ydiagram{5,1}, Q_R = 0\right\rangle = \boldsymbol{\rho}^\dagger_1\boldsymbol{\rho}^\dagger_3\boldsymbol{\rho}^\dagger_3\boldsymbol{\rho}^\dagger_3\boldsymbol{\rho}^\dagger_6|Q_R = 0\rangle$$

Notice that some of the r_n may be equal; the state still makes sense because these operators are bosonic.

We can see how the Young diagram encodes the state in terms of the fermions if we do anticommutator algebra to cancel $\mathbf{R}_l^\dagger$ and $\mathbf{R}_l$ whenever they both appear:

$$\left|\quad\right\rangle =$$

$$\mathbf{R}^\dagger_{-\frac{7}{2}}\mathbf{R}_{-\frac{9}{2}}\cdot\mathbf{R}^\dagger_{-\frac{1}{2}}\mathbf{R}_{-\frac{7}{2}}\cdot\mathbf{R}^\dagger_{\frac{1}{2}}\mathbf{R}_{-\frac{5}{2}}\cdot\mathbf{R}^\dagger_{\frac{3}{2}}\mathbf{R}_{-\frac{3}{2}}\cdot\mathbf{R}^\dagger_{\frac{11}{2}}\mathbf{R}_{-\frac{1}{2}}\,|\mathrm{gs}\rangle$$
$$=\left(\mathbf{R}^\dagger_{\frac{11}{2}}\mathbf{R}_{\frac{3}{2}}{}^\dagger\mathbf{R}_{\frac{1}{2}}{}^\dagger\right)\mathbf{R}_{-\frac{3}{2}}\mathbf{R}_{-\frac{5}{2}}\mathbf{R}_{-\frac{9}{2}}\,|\mathrm{gs}\rangle$$

Look at the momentum indices on the surviving annihilation and creation operators. You can see these very numbers are the $(\pm)$ the numbers of boxes to the right and below the diagonal of the diagram respectively.

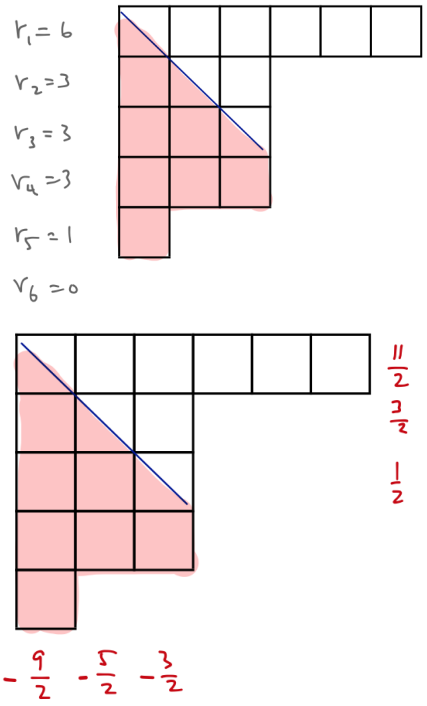


Figure 2: The two ways of looking at the Young diagram which determine the boson state and the fermion state, respectively.

What is the Hamiltonian in terms these boson operators?

$$\mathbf{H}^B=\frac{\pi v_F}{L}\mathbf{Q}_R^2+\frac{2\pi v_F}{L}\sum_{n=l+\frac{1}{2}=\frac{1}{2}}^{\infty}n\boldsymbol{\rho}_n^\dagger\boldsymbol{\rho}_n.$$

This is a *chiral* boson, since $n>0$. Its partition sum is

$$\begin{aligned}Z^R_{\text{boson}}(T)&=\mathrm{tr}_{\mathcal{H}_B}e^{-\mathbf{H}^B/T}\\&=\left(\sum_{Q_R=-\infty}^{\infty}q^{Q_R^2/2}\right)\prod_{n=1}^{\infty}\underbrace{\sum_{m_n=\boldsymbol{\rho}_n^\dagger\boldsymbol{\rho}_n=0,1,..\infty}q^{nm_n}}_{\frac{1}{1-q^n}}\end{aligned}\tag{5.54}$$

This is an infinite-sum representation of the same elliptic theta function. (For more about the theta functions, if you must, see *e.g.* Polchinski vol. 1, page 215.)

5.6.2 Bosonization dictionary

[Fradkin 2d ed, page 133; Sachdev 2d ed, page 420] Let's set $T=\frac{1}{\pi}$ for a bit. The left and right $\mathsf{U}(1)$ currents in terms of bosons and fermions are

$$\Psi_L^\dagger\Psi_L=\frac{1}{2\pi}\partial_z\phi,\quad \Psi_R^\dagger\Psi_R=\frac{1}{2\pi}\bar\partial_z\phi.$$

The fermions themselves are:

$$\Psi_L(z)^\dagger =: e^{\mathbf{i}\phi_L(z)} :, \quad \Psi_R(\bar{z})^\dagger =: e^{\mathbf{i}\phi_R(\bar{z})} : \ .$$

How might a human have figured this out? (One human who did was Mandelstam.)

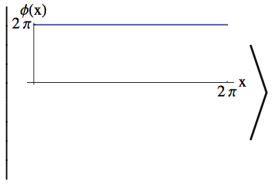
The relation for the charges says that the total fermion charge is the winding number:

$$\Psi_L^\dagger \Psi_L + \Psi_R^\dagger \Psi_R = \frac{1}{2\pi}\partial_x \phi. \tag{5.55}$$

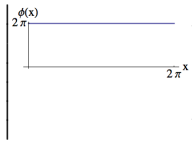
Let's consider the refined statement:

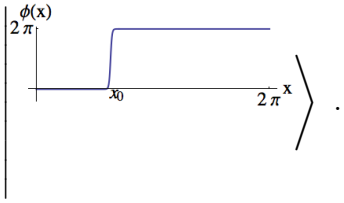
$$\Psi_L^\dagger \Psi_L = \frac{1}{2\pi}\partial_x \phi_L = j_L. \tag{5.56}$$

This means that if we create a left-moving fermion at $x = x_0$, the value of $\phi_L(x)$ should *jump* for all $x < x_0$, by a fixed amount so that if we integrate the BHS of (5.55) we get 1. (That amount is 2π.) For example, according to (5.56) a state with no fermions looks like:

$|\text{no fermions}\rangle =$ 

Acting on this state with one fermion at $x = x_0$ should do the following:

$\Psi_L^\dagger(x_0)$ 

$=$ 

.

Such a jump is produced by the conjugate field momentum (recall the translation operator in QM is $e^{-\mathbf{i}p^a}\psi(x) = \psi(x+a)$):

$$\Psi_L(x_0) \sim e^{\mathbf{i}2\pi \int_{-\infty}^{x_0} dx \boldsymbol{\pi}_L(x)} \ .$$

Now who is the canonical momentum $\boldsymbol{\pi}_L(x)$ in terms of ϕ_L, ϕ_R? It's the thing with a canonical commutator with ϕ_L. One way to figure this out is to go back to the mode expansions and show that

$$[\partial_x \boldsymbol{\phi}_L(x), \boldsymbol{\phi}_L(y)] = \frac{2\mathbf{i}}{T}\delta(x-y) \tag{5.57}$$

which shows that $\boldsymbol{\pi}_L(x) = \frac{T}{2}\partial_x \boldsymbol{\phi}_L(x)$ (and similarly $\boldsymbol{\pi}_R = -\frac{T}{2}\partial_x \boldsymbol{\phi}_R(x)$). (Notice that this equation is roughly the imaginary part of the x derivative of (5.34).) So (when $T = \frac{1}{\pi}$, which is when this operator is single-valued!) the fermion operator is

$$\Psi_L(x_0) \sim e^{\mathbf{i}\int_{-\infty}^{x_0} dx \ \partial_x \boldsymbol{\phi}_L} = e^{\mathbf{i}\boldsymbol{\phi}_L} \ .$$

We already saw that this kind of object creates winding in (5.35). So again the fermion is a domain-wall creation operator; notice that because the field ϕ is periodic, it can be possible to have a single such domain wall on a circle. We'll say a bit more about the periodicity of ϕ_L below.

In order for this to be a fermion operator, we require $\{\Psi_L(x), \Psi_L(x')\} = 0, x \neq x'$. Is it true? Yes in fact, as we can see by BCH, in the form $e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{B}}e^{\mathbf{A}}e^{-[\mathbf{A},\mathbf{B}]}$ if $[\mathbf{A}, \mathbf{B}]$ is a c-number. A useful intermediate fact comes from the x-integral of (5.57) (from $-\infty$ to x) which says

$$[\boldsymbol{\phi}_L(x), \boldsymbol{\phi}_L(y)] = \mathbf{i}\pi\mathsf{sign}(x-y)$$

(the additive constant of integration on the RHS is fixed by demanding antisymmetry in $x \to y$ which is manifest on the LHS). Therefore:

$$: e^{\mathbf{i}\phi_L(x)} :: e^{\mathbf{i}\phi_L(0)} := e^{-\mathbf{i}\pi\mathsf{sign}(x)} : e^{\mathbf{i}\phi_L(0)} :: e^{\mathbf{i}\phi_L(x)} := - : e^{\mathbf{i}\phi_L(0)} :: e^{\mathbf{i}\phi_L(x)} : .$$

More generally, the same calculation shows that

$$: e^{\mathbf{i}n\phi_L(x)} :: e^{\mathbf{i}m\phi_L(0)} := (-1)^{nm} : e^{\mathbf{i}m\phi_L(0)} :: e^{\mathbf{i}n\phi_L(x)} :$$

So $e^{\mathbf{i}n\phi_L}$ is bosonic for even n and fermionic for odd n.

You can also see the fermi statistics from the OPEs[39]:

$$: e^{\mathbf{i}\phi_L(z_1)} :: e^{-\mathbf{i}\phi_L(z_2)} :\sim \frac{1}{z_1 - z_2} = -\frac{1}{z_2 - z_1}.$$

$$: e^{\mathbf{i}\phi_L(z_1)} :: e^{\mathbf{i}\phi_L(z_2)} :\sim (z_1 - z_2) : e^{\mathbf{i}\phi_L(z_1)+\mathbf{i}\phi_L(z_2)} :$$

Notice that this object

$$e^{\mathbf{i}\frac{1}{\pi}\int_{-\infty}^{x_0} dx \partial_x \boldsymbol{\phi}_L} = e^{\mathbf{i}\int_{-\infty}^{x_0} dx j_L(x)}$$

[39] In these notes I have not discussed the operator product expansion carefully. Here is the executive summary. The idea is that if we have two local operators near each other in spacetime inserted in the path integral, then from the point of view of the other operators they look like a single local operator. Which one? In general, it is a sum over all of them.

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_a C_{12}^a(x_1 - x_2)\mathcal{O}_a(x_1).$$

This expansion is very powerful in a CFT where we can label the operators by their scaling dimensions. Dimensional analysis says that only a finite number of terms can be singular as a function of the separation of the two operators.

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_a \frac{c_{12}^a}{(x_1 - x_2)^{\Delta_1+\Delta_2-\Delta_a}}\mathcal{O}_a(x_1).$$

In the OPE with a $\sim$ we only write the singular terms.

has a lot in common with the Jordan-Wigner string, which counts the number of particles to the left.

Periodicity of chiral bosons. Since $\phi \simeq \phi + 2\pi m, m \in \mathbb{Z}$ and $\phi = \phi_L + \phi_R$ it is tempting to suggest that ϕ_L has period π. Such a period would mean that $e^{-\mathbf{i}\phi_L} \sim \Psi_L$ is not single-valued. But it is not quite correct. To see the correct statement, notice that the T-dual field is $\tilde{\phi} = \phi_L - \phi_R$. With our normalization, this field also has periodicity 2π. The winding modes of the dual fields are momentum modes of the original one, so

$$\begin{matrix} \phi_L + \phi_R \simeq \phi_L + \phi_R + 2\pi m \\ \phi_L - \phi_R \simeq \phi_L + \phi_R + 2\pi j \end{matrix} \implies \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} \simeq \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} + \pi \begin{pmatrix} m+j \\ m-j \end{pmatrix} ,$$

where m and j are independent integers. But then $m \pm j$ are *not* independent integers: when $m+j$ is odd or even, so is $m-j$. This is the same fact as my earlier statement about the labels α, β on $\mathcal{V}_{\alpha,\beta}$ at Eq. (5.36).

Note that the T-dual field is often called $\vartheta \equiv \phi_L - \phi_R$ in discussions of Luttinger liquids (*e.g.* in Fradkin's and Sachdev's books).

Possibly Depressing comment. So now you are starting to see that this duality business is actually often a sad story: we thought we could solve *two* systems (free bosons and free fermions) but since they are really the same system in disguise, it turns out we can only solve one!

Multiple fermions. Notice that if we had N complex fermions ψ^a (for example a could be a spin index), we could bosonize them by N scalar fields at the free fermion radius:

$$\psi^a(z) \overset{?}{\sim} e^{\mathbf{i}\phi^a(z)}, \quad a = 1..N \ . \tag{5.58}$$

This formula *almost* works, but it has the problem that the operators on the RHS with different values of a will still commute (since the associated scalars don't have any singularities in their OPE, *i.e.* they commute). This must be fixed by so-called *Klein factors* (or 2-cocycles) which introduce the extra necessary signs. The necessary object is constructed by ordering the species of fermions $a = 1..N$ and defining:

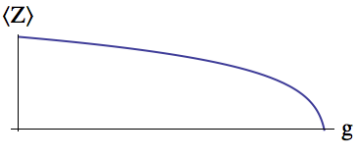
$$\mathfrak{c}_a \equiv (-1)^{\sum_{b<a} \mathbf{N}_b}$$

where $\mathbf{N}_b$ is the total fermion number of species b. The correct version of (5.58)

$$\psi^a(z) = \mathfrak{c}_a e^{\mathbf{i}\phi^a(z)}, \quad a = 1..N \qquad \text{(no sum on } a\text{)}. \tag{5.59}$$

Spin fields in (two copies of) the Ising model. Now I will partially fulfill my promise to use bosonization to determine the scaling dimension of the spin operator at the Ising critical point.

Recall that I claimed without any justification that in the TFIM, the behavior of the magnetization $\langle Z\rangle$ for $g \lesssim g_c$ was



$$\langle Z\rangle \sim (g_c - g)^{1/8}$$

(as in the figure). This 1/8 is the scaling dimension of the order parameter operator in the critical Ising CFT, the operator onto which the spin Z_j matches.

To begin, let's consider just the holomorphic part of a complex fermion $\psi(z) = \frac{1}{\sqrt{2}}(\chi_1(z) + \mathbf{i}\chi_2(z))$. This is not exactly the Ising critical theory, which is instead a non-chiral majorana mode, $\chi_L(z), \chi_R(\bar{z})$, but this is what we know how to bosonize. The bosonization map is

$$\psi(z) \sim e^{\mathbf{i}\phi(z)}.$$

Recall that the fermions in our spin chain arose as domain wall operators. They satisfy

$$\mathbf{Z}_l \mathbf{c}_j = -\mathbf{c}_j \mathbf{Z}_l, \quad l > j.$$

We can interpret this formula to say that the spin $\mathbf{Z}_j$ creates a branch cut for the fermion field. The CFT version of this statement is:

$$\psi(z)\sigma(0) \sim z^{-\frac{1}{2}}\mu(0) \quad . \tag{5.60}$$

Here σ, μ are *spin fields* (or more generally, *twist fields*) whose presence at $z = 0$ creates a branch cut. The actual location of the branch cut is not physical, but the location of the branch *points* (here $0, \infty$) is meaningful. Can we find operators that accomplish this?

Yes: they are

$$\sigma(w) = e^{-\frac{\mathbf{i}}{2}\phi(w)}, \quad \mu(w) = e^{+\frac{\mathbf{i}}{2}\phi(w)}. \tag{5.61}$$

To check this claim, their OPEs with the fermion operator are:

$$\begin{aligned}
\psi(z)\sigma(0) &=: e^{\mathbf{i}\phi(z)} :: e^{-\frac{\mathbf{i}}{2}\phi(0)} :\sim: e^{+\frac{\mathbf{i}}{2}\phi(0)} : z^{-\frac{1}{2}} + \text{regular} \\
\psi(z)\mu(0) &=: e^{\mathbf{i}\phi(z)} :: e^{+\frac{\mathbf{i}}{2}\phi(0)} :\sim: e^{+3\frac{\mathbf{i}}{2}\phi(0)} : z^{+\frac{1}{2}} + \text{regular}
\end{aligned} \tag{5.62}$$

As a check on my ability to do wick contractions, consider the OPE between (and hence the two-point function of) two of the spin fields, which is a special case of (5.37):

$$e^{\alpha \mathbf{i}\phi(z)} e^{-\alpha \mathbf{i}\phi(0)} \sim \frac{1}{z^{\alpha^2}} \ .$$

For $\alpha = \pm\frac{1}{2}$, this gives

$$\Delta_\sigma = \Delta_\mu = \frac{1}{2}\alpha^2|_{\alpha=\frac{1}{2}} = \frac{1}{8}.$$

The dimension of $e^{\mathbf{i}\frac{3}{2}\phi}$ is $\frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8}$ so the dimensional analysis in (5.62) works.

The answer $\Delta_\sigma = \frac{1}{8}$ is the dimension of the spin field in the theory of a *complex* chiral fermion. This is two copies of the left-moving part of the Ising CFT. The spin field in the Ising model which makes a branch cut for *both* χ_L and χ_R has the same dimension (the one that makes a branch cut in just χ_L has dimension $\frac{1}{16}$). The basic fact I am using here is that the spin-field dimension is additive in the number of fermion fields which feel the branch cut, that is, in the number of fermion fields whose boundary conditions are twisted.

To see this, let's make spin fields for N complex fermions ψ^a, each bosonized as we just did by N scalar fields

$$\psi^a(z) \sim e^{\mathbf{i}\phi^a(z)}, \quad a = 1..N$$

The operators

$$\sigma_s(z) \equiv e^{\mathbf{i}s^a\phi^a(z)}, \quad \{s^a = \pm\frac{1}{2}\}$$

create branch cuts for all N fermions at once:

$$\psi^a(z)\sigma_s(0) \sim z^{s^a}\sigma_{s'}(0) + \text{regular}$$

where $(s')^a = s^a + 1, (s')^{b\neq a} = s^b$. The dimension of these operators is the sum of the dimensions:

$$\Delta_s = \frac{1}{2}\left(s_1^2 + s_2^2 + \cdots + s_N^2\right) = \frac{N}{8}.$$

Now take $N = \frac{1}{2}$ to get the answer for a single majorana fermion. If you are not satisfied by this argument, there is a more direct way to compute the dimension of the Ising model spin field, using the defining equation (5.60), which requires a bit more CFT technology (it's explained on p. 74 of Ginsparg's CFT notes). We'll come to that soon.

Derivation of spin field. Notice that the object (5.61) would not be considered single-valued on the original boson hilbert space! However, the operator that creates a branch cut for *both* ψ_L and ψ_R at the same time is $e^{\frac{\mathbf{i}}{2}(\phi_L(z)\pm\phi_R(\bar{z}))}$ and this *is* an allowed operator. And in fact, here is a *derivation* of its relation to the spin operator. Recall the JW formula

$$\boldsymbol{\sigma}_j^+ = (-1)^{\sum_{i<j} n_i}\mathbf{c}_j^\dagger.$$

(Previously we put the JW string to the right; here I defer to the bosonization convention we used above.) Here (in the basis we used earlier, where $\mathbf{Z}$ is the Ising order parameter), $\boldsymbol{\sigma}^+ \equiv \mathbf{Z} + \mathbf{iY}$, so $\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^- = 2\mathbf{Z}$. In the continuum, in terms of the fermions, this is

$$\boldsymbol{\sigma}^+(x) = e^{\mathbf{i}\pi\int_{-\infty}^x dx' j_0(x')}\Psi(x)^\dagger \ .$$

Now, we expand our fermion operator near the fermi surface:

$$\mathbf{\Psi}(x,\tau) = e^{\mathbf{i}k_F x}\Psi_R(x,\tau) + e^{-\mathbf{i}k_F x}\Psi_L(x,\tau) \tag{5.63}$$

Let's do just the R piece first (then we'll put it all together):

$$\begin{aligned} & e^{\mathbf{i}\pi\int_{-\infty}^{x} dx' j_0(x')}\Psi_R(x)^\dagger \\ \overset{(5.55)}{=} & e^{\mathbf{i}\pi\int_{-\infty}^{x} dy\ \frac{\partial_y\phi_L+\partial_y\phi_R}{2\pi}} e^{-\mathbf{i}\phi_R(x)} \\ = & e^{\mathbf{i}\frac{\phi_L(x)+\phi_R(x)}{2}} e^{-\mathbf{i}\phi_R(x)} \\ \sim & e^{\mathbf{i}\frac{\phi_L(x)-\phi_R(x)}{2}} \end{aligned} \tag{5.64}$$

OK, that was practice; the full expression is:

$$\begin{aligned} \boldsymbol{\sigma}^+(x) = & e^{\mathbf{i}\pi\int_{-\infty}^{x} dx' j_0(x')}\Psi(x)^\dagger \\ \overset{(5.55),(5.63)}{=} & e^{\mathbf{i}\pi\int_{-\infty}^{x} dy\ \frac{\partial_y\phi_L+\partial_y\phi_R}{2\pi}} \left(e^{-\mathbf{i}k_F x}e^{-\mathbf{i}\phi_R(x)} + e^{+\mathbf{i}k_F x}e^{-\mathbf{i}\phi_L(x)}\right) \\ \sim & e^{\mathbf{i}\frac{\phi_L(x)-\phi_R(x)}{2}}e^{-\mathbf{i}k_F x} + e^{-\mathbf{i}\frac{\phi_L(x)-\phi_R(x)}{2}}e^{+\mathbf{i}k_F x} \end{aligned} \tag{5.65}$$

The role of the anomaly in bosonization. Consider the canonical commutator of the bose mode operators ($\partial_z\phi \propto \sum_n \boldsymbol{\rho}_n z^{-n-1}$):

$$[\boldsymbol{\rho}_n, \boldsymbol{\rho}_{-n'}] = \delta_{n,n'}n$$

in terms of the fermions, $\boldsymbol{\rho}_n = \sum_l \mathbf{L}_l^\dagger \mathbf{L}_{l+n}$, we have:

$$\begin{aligned} [\boldsymbol{\rho}_n, \boldsymbol{\rho}_{-n'}] &= \sum_{l,'l} [\mathbf{L}_l^\dagger \mathbf{L}_{l+n}, \mathbf{L}_{l'}^\dagger \mathbf{L}_{l'-n'}] \\ &= \sum_{l'} \left(\mathbf{L}_{l'-n}^\dagger \mathbf{L}_{l'-n'} - \mathbf{L}_{l'}^\dagger \mathbf{L}_{l'+n-n'}\right) \overset{?}{=} 0 \end{aligned} \tag{5.66}$$

This expression has the classic form of an anomaly: it looks like this vanishes since we can redefine the dummy index in the second term by $n' \to n' + n$ and make the two terms look the same with opposite sign. *However*, in the presence of a UV cutoff, *i.e.* in the regulated theory, that shift will shift the cutoff and will not be innocuous since the summand is not zero for arbitrarily high energies.[40] If, for example, we regulate the short distance behavior using normal-ordered operators, we have

$$[\boldsymbol{\rho}_n, \boldsymbol{\rho}_{-n'}] = \sum_{l',l} [: \mathbf{L}_l^\dagger \mathbf{L}_{l+n} :, : \mathbf{L}_{l'}^\dagger \mathbf{L}_{l'-n'} :]$$

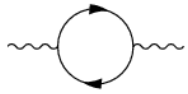
[40] To see how the high-energy stuff comes in consider the analogous infinite-volume expression

$$0 = \int_{-\infty}^{\infty} dp(f(p) - f(p)) \overset{?}{=} \int_{-\infty}^{\infty} dp(f(p+s) - f(p)) = \int dp(sf'(p) + \mathcal{O}(s^2)) = s(f(\infty) - f(-\infty)).$$

$$= \delta_{nn'} \sum_l \left(\left\langle \mathbf{L}^\dagger_{l-n} \mathbf{L}_{l-n} \right\rangle - \left\langle \mathbf{L}^\dagger_n \mathbf{L}_n \right\rangle \right) = \delta_{nn'} n. \tag{5.67}$$

This calculation is related to the anomaly because it says that the OPE of the chiral current j_R with itself has a singularity with a definite coefficient. Specifically

$$j(z)j(0) \sim \frac{k}{z^2}$$

with $k = 1$ is called the *level* of the (here $\mathsf{U}(1)$) current algebra. This singularity implies the anomalous nonconservation when coupling to a background gauge field. In terms of the fermions, recall that both of these effects come from the 'diangle' diagram: 

Twist fields and boundary conditions. Above I used the phrase "fermion fields whose boundary conditions are twisted" by the spin fields. With this phrase I betrayed that I am secretly using the following picture, which is called *radial quantization*. Imagine that the radial coordinate of the plane is the time coordinate (it is euclidean so this interpretation is not problematic). Then equal-time surfaces are circles centered at the origin. (In a CFT this is equivalent to the ordinary notion of time evolution by a conformal transformation, namely the exponential map that takes the cylinder to the plane.) In this case, the boundary conditions on the spatial coordinate are determined by going around the origin; if there's a branch point at the origin, we must cross the branch cut in doing so, and the field comes back to itself up to the discontinuity across the branch cut, which here is a minus sign. So the spin fields can be regarded as "boundary-condition-changing operators": they take the fermions from the sector with PBC (and hence a fourier expansion with integer powers of z: $\psi_{PBC}(z) = \sum_{n\in\mathbb{Z}} z^n \psi_n$; this is called the *Ramond sector*) to APBC (half-integer mode expansion $\psi_{APBC}(z) = \sum_{n\in\mathbb{Z}} z^{n+\frac{1}{2}} \psi_{n+\frac{1}{2}}$; this is called the *Neveu-Schwarz sector*) and vice versa.

Understanding in terms of group theory. I must comment on the very essential group theory underlying the previous discussion of spin fields for multiple fermions. The theory of N complex free chiral fermions has a manifest $\mathsf{U}(N)$ symmetry rotating the complex fermions into each other: $\psi^a \to \mathsf{U}^a_b \psi^b$, $\mathsf{U}\mathsf{U}^\dagger = 1$. The free fermion action actually enjoys a larger symmetry which ignores the grouping into complex fermions: $\psi^a = \frac{1}{\sqrt{2}}(\chi^{2a-1} + \mathbf{i}\chi^{2a})$; since $S \propto \int \delta_{AB} \chi^A \bar{\partial} \chi^B$, these $2N$ chiral majorana fermions transform in the fundamental of an $\mathsf{SO}(2N)$ symmetry, $\chi^A \to \mathsf{O}^A_B \chi^B, \mathsf{O}^\mathsf{T}\mathsf{O} = 1$.

This $\mathsf{SO}(2N)$ acts on the hilbert space. In terms of the fermions, the generators are just fermion bilinears $j^{AB} = \chi^A \chi^B$, the currents (recall that the adjoint representation (the one in which the generators transform) of $\mathsf{SO}(2N)$ is made from antisymmetrizing two fundamentals). There are $2N(2N-1)/2$ of these.

In terms of the bosons, the generators come in two types (in the theory of Lie groups this is called Cartan-Weyl basis). N of them are just the ordinary boson currents $\partial\phi^a$. This is called the *Cartan subalgebra* (the number of them is $N = \text{rank}(\mathsf{SO}(2N))$ – they all commute with each other. The non-commuting (charged) generators are vertex operators $e^{\mathbf{i}\phi^a w_a}$ where in order for this to be a conserved current, it must have dimension 1, which requires $\sum_{a=1}^{N} w_a w_a = 2$. (in Lie group terminology, w_a is a *root vector*). Recall that the entries of w must be integers. This condition is solved by vectors of the form $\{w_a\} = \{(0...0, \pm 1, 0 \cdots, 0, \pm 1, ...)\}$, of which there are $N(N-1)/2 \times 4$. Altogether we have recovered the same number of generators $N + 2N(N-1) = N(2N-1), \checkmark$.

The states in the Neveu-Schwarz (APBC) sector are of the form

$$\psi^A_{-\frac{1}{2}} \psi^B_{-\frac{3}{2}} \cdots |0\rangle \,, \qquad \psi^A_{r>0} |0\rangle = 0, \forall r$$

and therefore transform in various antisymmetric tensor representations of this $\mathsf{SO}(2N)$.

What is going on with these spin field objects $e^{\mathbf{i}s^a\phi^a} = e^{\mathbf{i}\left(\pm\frac{1}{2}\phi^1 \pm \frac{1}{2}\phi^2 \cdots \pm\frac{1}{2}\phi^N\right)}$? These operators map the Neveu-Schwarz sector to the Ramond sector. For a hint about what they are doing, notice that the fermion zeromodes in the Ramond (PBC) sector satisfy

$$\{\psi^a_0, \psi^b_0\} = 2\delta^{ab} \implies \{\chi^A_0, \chi^B_0\} = \delta^{AB}$$

which is the algebra of Dirac gamma matrices for $\mathsf{SO}(2N)$, *i.e.* Clifford algebra. This algebra must be represented on the Ramond groundstates, which are therefore not unique, *i.e.* they are degenerate. The point in life of Clifford algebra is to construct spinor representations. So the answer is that these vectors $\{s^a\} = \{\pm\frac{1}{2}, \pm\frac{1}{2}, \cdots, \pm\frac{1}{2}\}$ are *weight vectors* for the spinor representation(s) of $\mathsf{SO}(2N)$, in which the Ramond groundstates transform. The spin fields create the Ramond groundstates from the NS groundstate:

$$\sigma_s(0) |0\rangle_{NS} = |s\rangle \,.$$

A very interesting application of this group theory arises in the demonstration (by Fidkowski and Kitaev) that eight free majorana chains in the topological phase (with the dangling majoranas at the ends) can be adiabatically connected to the trivial phase (only) via interacting Hamiltonians.

5.6.3 Application 1: Briefly, what is a Luttinger liquid?

[Fradkin p. 152] Let us consider a model which is actually made of fermions (which may have spin) at finite chemical potential. The actual density of the microscopic fermions

Ψ is

$$\boldsymbol{\rho}_{\rm actual}(x) = \sum_\sigma \Psi^\dagger_\sigma(x)\Psi_\sigma(x) = \underbrace{\rho_0}_{=N_{\rm spin}\frac{k_F}{\pi}} + \underbrace{\boldsymbol{\rho}_L}_{=\psi^\dagger_L\psi_L} + \boldsymbol{\rho}_R + \sum_\sigma \left(e^{2k_F\mathbf{i}x}\psi^\dagger_R\psi_L + h.c.\right)...$$

where I am reminding you that $\Psi(x) = e^{\mathbf{i}k_Fx}\psi_R(x) + e^{-\mathbf{i}k_Fx}\psi_L(x)$. The .... are even-higher momentum contributions which we will ignore. The quantity $2k_F$ is important in any system with a fermi surface as it is the *diameter* of the fermi surface, and (bosonic) particle-hole excitations near (just below) this momentum can be cheap despite its large value.

Consider the hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_{\rm pot}$ where the free Galilean-invariant hamiltonian is

$$\mathbf{H}_0 = \sum_\sigma \int đp\ pv_F \left(\psi^\dagger_{\sigma R}\psi_{\sigma R} - \psi^\dagger_{\sigma L}\psi_{\sigma L}\right) \ .$$

Subject this system to an (at-the-moment) external potential

$$\mathbf{H}_{\rm pot} = \int dx\ eV(x)\boldsymbol{\rho}(x) = -e\int đp \int đq \tilde{V}(q)\Psi^\dagger(p+q)\Psi(p) \ .$$

Let us consider the special wavenumber $V(x) = V_0\cos(2k_Fx)$, so

$$\mathbf{H}_{\rm pot} = -eV_0\int dp\psi^\dagger_R(p)\psi_L(p) + h.c. = -eV_0\int dx\psi^\dagger_R(x)\psi_L(x) + h.c.$$

In the relativistic language, this is a mass term:

$$\mathbf{H} = \int dx\psi^\dagger\left(\sigma^3v_F\mathbf{i}\partial_x + eV_0\sigma^1\right)\psi = \int dx\left(\bar\psi v_F\gamma^x\partial_x\psi + eV_0\bar\psi\psi\right)$$

(here I used $\gamma^0 = \sigma^1, \gamma^x = \sigma^1\sigma^3, \bar\psi \equiv \psi^\dagger\gamma^0$).

Where does this potential come from? It can be generated by (*e.g.* Coulomb) interactions between the fermions themselves

$$\mathbf{H}_{\rm int} = \int dxeV\boldsymbol{\rho} + \frac{e}{2}\int\int VKV \ .$$

If we treat this interaction in mean field theory, we get back (for some choice of K more on which more below) the previous background potential, $\mathbf{H}_{\rm int} \sim \mathbf{H}_{\rm pot}$. On the other hand, if we integrate out V, we find a density-density interaction

$$\mathbf{H}_{\rm int} = e\int\int \boldsymbol{\rho}K^{-1}\boldsymbol{\rho}$$

which for the right kernel, K, might be the Coulomb interaction. In a metal, the coulomb interaction is screened and therefore short-ranged. Expanding in this range, we find

$$\mathbf{H}_{\text{int}} = \int dx \left(\underbrace{2g_2 \boldsymbol{\rho}_R(x)\boldsymbol{\rho}_L(x)}_{\text{back-scattering, 1d}} + \underbrace{g_4\left(\boldsymbol{\rho}_R^2 + \boldsymbol{\rho}_L^2\right)}_{\text{forward scattering, ubiquitous}} + g_u \lim_{y\to x}\left(\psi_R^\dagger(x)\psi_R^\dagger(y)\psi_L(x)\psi_L(y) + h.c.\right)\right)$$

The gs can be written in terms of various (fourier) components of the interaction potential K^{-1}. In the last term, you should think of $y = x+a$, one lattice spacing. The last term describes a process whereby two L/R fermions turn into two R/L fermions, which clearly violates momentum conservation by $\pm 2k_F$; this can happen in a lattice model if the two sides of the fermi surface differ by a reciprocal lattice vector:

$$k_F \equiv -k_F + \frac{2\pi}{a}.$$

This is called *umklapp* scattering. When does it happen? k_F is determined by the density of electrons. The maximum density is when all states are filled and is $N_s\frac{2\pi}{a}$, and then there is no fermi surface. Umklapp happens at *half-filling*, when $k_F = \frac{\pi}{a}$. Otherwise g_u can be ignored. Notice that this effect explicitly breaks the chiral symmetry $\psi_{L/R} \to e^{\mathbf{i}\alpha_{L/R}}\psi_{L/R}$ which independently rotates the phases of the left-movers and right-movers down to a $\mathbb{Z}_2$ symmetry, which is just fermion number (which is hard to violate).

Bosonize. Now we bosonize. Using $\phi = \phi_L + \phi_R, \Theta = \phi_L - \phi_R$ and our dictionary above, we find $\mathbf{H}_0 \equiv \int dx \mathbf{h}_0(x)$, with

$$\mathbf{h}_0(x) = \frac{v_F}{2}\left(\Pi^2 + (\partial_x\phi)^2\right) = \frac{v_F}{2}\left((\partial_x\Theta)^2 + (\partial_x\phi)^2\right)$$

where in the second step we used the canonical commutators we found previously. Using $\boldsymbol{\rho}_L = \frac{1}{2\pi}\partial_x\phi_L$ etc this is

$$\mathbf{h}_0(x) = \pi v_F\left((\boldsymbol{\rho}_L + \boldsymbol{\rho}_R)^2 + (\boldsymbol{\rho}_L - \boldsymbol{\rho}_R)^2\right) = 2\pi v_F\left(\boldsymbol{\rho}_L^2 + \boldsymbol{\rho}_R^2\right).$$

The forward scattering hamiltonian density is

$$\mathbf{h}_{\text{forward}}(x) = g_4\left(\boldsymbol{\rho}_R^2 + \boldsymbol{\rho}_L^2\right)$$

of exactly the same form as the free term (!) and therefore just renormalizes the velocity, the coefficient of the spatial gradient term. The back-scattering term is less innocuous

$$\mathbf{h}_{\text{back}}(x) = 2g_2\boldsymbol{\rho}_R\boldsymbol{\rho}_L = \frac{g_2}{\#}\left((\partial_x\phi)^2 - \Pi^2\right)$$

Umklapp is a little trickier: to define the bosonized form we must absorb the OPE singularity

$$: e^{-\mathbf{i}\phi_R(x)} :: e^{-\mathbf{i}\phi_R(y)} :\sim (x-y)^{\frac{1}{2\pi T}} : e^{-2\mathbf{i}\phi_R(x)} : +...$$

into the coupling constant:

$$\mathbf{h}_u = \tilde{g}_u \cos 2\phi.$$

(It is useful to set $\tilde{g}_u = g_u a^{-\frac{1}{2\pi T}}$ where a is a short-distance cutoff.) With $\tilde{g}_u \neq 0$, the remaining chiral symmetry acts by $\phi \to \phi + \pi$.

The full hamiltonian density (without umklapp, so away from half-filling) is

$$\mathbf{h} = \frac{v}{2}\left(\frac{1}{K}\left(\partial_x \Theta\right)^2 + K\left(\partial_x \phi\right)^2\right)$$

where v and K can be written in terms of the gs and v_F[41]. Notice that T-duality is clear in this expression as $\mathsf{T} : K \leftrightarrow \frac{1}{K}, \phi \leftrightarrow \Theta$.

A few comments:

- The backscattering term g_2 affects both the velocity and the 'stiffness' K (this is the parameter we called the tension T earlier). The signs are such that a repulsive interaction $g_2 > 0$ increases K.
- In the absence of umklapp, the chiral symmetry acts on the boson as $\phi \to \phi + \alpha$, $\alpha \in [0, 2\pi)$. Think about how this acts on the fermions: since $\boldsymbol{\rho} = ... + \frac{1}{2\pi}\partial\phi$, the density transforms as

  $$\boldsymbol{\rho}(x) \to \rho_0 + \boldsymbol{\rho}_R + \boldsymbol{\rho}_L + \left(e^{\mathbf{i}2(k_F x - \alpha)}\psi_R^\dagger(x)\psi_L(x) + h.c.\right) = \boldsymbol{\rho}\left(x - \frac{\alpha}{k_F}\right)$$

  – this is a rigid translation of the density profile! It makes sense then that lattice effects would break this symmetry.
- We have ignored the fact that the dispersion will not be exactly linear arbitrarily far from the Fermi surface. Such terms can also be bosonized and become things like $(\partial_x\phi)^3$ which change the story dramatically; in particular they will lead to damping of the 'sound mode' described by ϕ even away from half-filling.

[41] The relation is

$$v = \sqrt{\left(v_F + g_4/\pi\right)^2 - \left(g_2/\pi\right)^2}, \quad K = \sqrt{\frac{v_F + g_4/\pi + g_2/\pi}{v_F + g_4/\pi - g_2/\pi}}.$$

- If we study spinful fermions, there are even more 4-fermion terms to consider. A payoff of studying the case with spin is that it provides an example of *spin-charge separation*: by bosonizing like $\phi_c \equiv \phi_\downarrow + \phi_\uparrow, \phi_s \equiv \phi_\downarrow - \phi_\uparrow$ we can see that the charge sector (ϕ_c) and the spin sector (ϕ_s) decouple and can have independent vs and Ks – the spin excitations of the electron travel at a different speed than its charge! There is a lot more to say about this and I recommend Fradkin's treatment in chapter 6.

How is a Luttinger liquid different from a Fermi liquid?

Continuing with the model with actual microscopic fermions, we have encountered a gapless phase in $D = 1 + 1$ described by free fermions, which we also know how to build in $D > 1 + 1$. The excitations are characterized by long-lived fermionic quasiparticles with the same $\mathsf{U}(1)$ charge as the constituent fermions $\mathbf{c}_j$ – they are Landau quasiparticles. This has the consequence that there are sharp peaks in the fermion green's functions,

$$\int e^{\mathbf{i}\omega t - \mathbf{i}kja} \left\langle \mathbf{c}_j^\dagger(t)\mathbf{c}_0(0) \right\rangle = G(\omega, k) \sim \frac{Z}{\omega - v_F(k - k_F)}$$

(the RHS is the contribution from the right-mover $\Psi_R(t - x)$). There is (therefore) a *jump* in the fermion occupation number at $k = k_F$:

$$n(k) \equiv \mathbf{c}^\dagger(k)\mathbf{c}(k) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \underbrace{\mathrm{Im}G(\omega, k)}_{\sim Z\pi\delta(\omega - v_F(k-k_F))} = Z\theta(k - k_F).$$

[Fradkin §6.8] These last two characteristic features of Fermi liquid theory can be destroyed by interactions in 1+1 dimensions, as we can see using the boson description. As a simple first step, notice that as we change T away from $\frac{1}{\pi}$, the dimension of the operator which at $T = \frac{1}{\pi}$ creates the right-moving fermion $e^{\mathbf{i}\phi_R}$ changes away from $\frac{1}{2}$:

$$\left\langle e^{\mathbf{i}\phi_R(x,t)} e^{-\mathbf{i}\phi_R(0,0)} \right\rangle \sim \frac{1}{|t - x|^{\frac{1}{\pi T}}} \ .$$

It is tempting to conclude that the fermion correlation function $\Psi(x) = e^{\mathbf{i}k_F x}\Psi_R(x) +$ ... is

$$\left\langle \Psi(x,t)^\dagger \Psi(0,0) \right\rangle \overset{?}{\sim} \frac{\sin(k_F x)}{|t - x|^{\frac{1}{\pi T}}}$$

The fourier transform of this has a broad power-law peak (a branch point) instead of a pole; the spectral density has no delta function, but rather goes like

$$\mathrm{Im}G(\omega, k) \overset{?}{\sim} |\omega - v_F(k - k_F)|^{\frac{1}{\pi T}+1}\theta(\omega - v_F(k - k_F)),$$

and hence there is no jump at $k = k_F$.

This is not quite the correct story however, since the operator $e^{\mathbf{i}\phi_R}$ is not quite the operator which creates an electron, except at $T = \frac{1}{\pi}$ (for example, it is not a fermionic object!). Instead what happens is that the interactions mix the R and L operators; the operator which creates the rightmoving electron is actually the linear(-in-the-exponent) combination:

$$\Psi_R = e^{\mathbf{i}\alpha_+\phi_R + \mathbf{i}\alpha_-\phi_L}, \qquad \text{with } \alpha_\pm = \frac{(K \pm 1)^2}{4K}$$

(recall that $K = \frac{1}{\pi T}$); it has total dimension $\Delta = \frac{1}{4}\left(K + \frac{1}{K}\right)$. The statement about the pole being replaced by a branch point is correct.

(Pieces of) the fermion Green's function are (relatively) directly measured in angle-resolved photoemission spectroscopy (ARPES) experiments. Other combinations of them can be measured by scanning-tunneling microscopy (STM).

[Fradkin, 2d ed. p. 170] There is a lot more to say about this subject. In particular, one can understand in great detail various ordering instabilities of interacting fermions in one dimension using the technology we have developed.

5.6.4 Application 2: sine-Gordon and Thirring

Let us reconsider the scale invariant theory described by the free scalar again.

$$\mathcal{L} = \frac{T}{2}\left(\partial_\mu \phi\right)^2 + g_n : \cos n\phi :$$

I allow the possibility of a cosine term. For the case of the Luttinger liquid we found that umklapp produced $n = 2$. There are several important points to make here:

(-1) In the absence of lattice effects or a background gauge field, there is a $\mathsf{U}(1)\times\mathsf{U}(1)$ symmetry acting on ϕ_L and ϕ_R independently.

(0) The theory is scale invariant for any T: there is a *line of fixed points* parametrized by T. That is, T is the coefficient of an *exactly marginal operator*. This is weird and the only examples in $D > 1 + 1$ (that I know) require supersymmetry.

(1) by changing T we can study interacting fermion systems: $T - \frac{1}{\pi}$ is the coefficient of the 4-fermion interaction.

(2) the fermion mass corresponds to a $\cos\phi$ term.

(3) These $\cos n\phi$ terms have a dimension that varies with T. This means that as we vary T they can go from irrelevant to relevant. If they are relevant they produce a mass for everybody (as you can see by Taylor expanding about the minimum of the

potential). This happens when

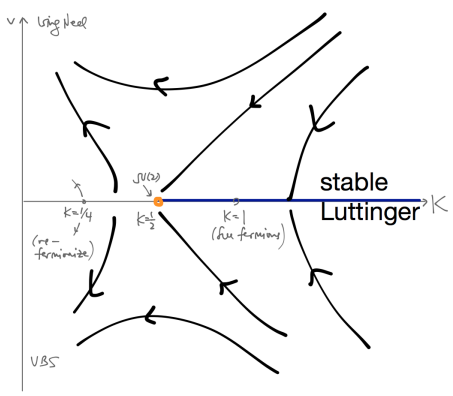
$$\Delta_n = \frac{n^2}{2\pi T} = K\frac{n^2}{2} \leq 2$$

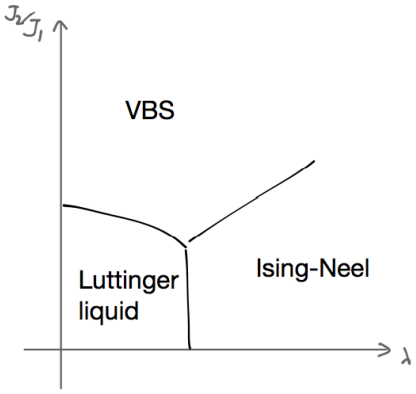
that is, for $T > T_c = \frac{n^2}{4\pi}$ (or $K < K_c = \frac{n^2}{2}$)

Brief comments on phase diagram. [Sachdev Ch. 20 – beware that $\phi_{\text{Sachdev}} = 2\phi_{\text{here}}$] The RG equations (for $n = 1$) are

$$\frac{dv}{d\ell} = (2 - 4K)v, \quad \frac{dK}{d\ell} = -\delta v^2$$

where $\delta > 0$ is some non-universal quantity. This leads to the following RG flow diagram in the space of these couplings (from Sachdev chapter 20):

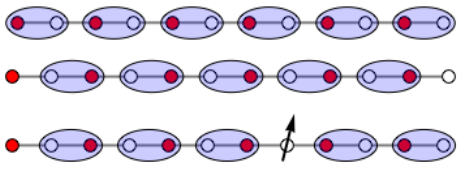




The figure at right is the phase diagram we can then infer for the model

$$\mathbf{H} = -J_1 \sum_{\langle ij \rangle} (\mathbf{X}_i\mathbf{X}_j + \mathbf{Y}_i\mathbf{Y}_j + \lambda\mathbf{Z}_i\mathbf{Z}_j) - J_2 \sum_{\langle\langle ij \rangle\rangle} (\mathbf{X}_i\mathbf{X}_j + \mathbf{Y}_i\mathbf{Y}_j + \mathbf{Z}_i\mathbf{Z}_j) \ .$$

where $\langle\langle ij \rangle\rangle$ denotes next-nearest neighbors. The simpler model we've been discussing moves along the λ axis.



• When $v \to -\infty$, we can find the ground-states by making the $-|v| \cos 2\phi$ happy first. The minima occur at $\phi = (n + \frac{1}{2})\pi$, $n \in \mathbb{Z}$. They can be distinguished by the value of $\sin\phi_i = (-1)^n = \langle(-1)^i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}\rangle$. This is described by singlets on alternating bonds. This is called a *valence bond solid* state. The low energy excitations in this state are interesting – they are not just the gapped ϕ excitations rolling in the minima of the cosine potential. Rather, the domain walls between the two groundstates (created in the CFT by $e^{\mathbf{i}\frac{1}{2}\phi}$!) have lower energy. We can see this explicitly at the special

point $K = \frac{1}{4}$ (which is $< K_c$) by 'refermionizing'. The previous original fermions where created by $\psi_L^\dagger = e^{\mathbf{i}\phi_R} = e^{\mathbf{i}\frac{1}{2}(\phi+\Theta)}$. Introduce new fermions

$$\tilde{\psi}_L^\dagger \equiv e^{\mathbf{i}\frac{1}{2}\left(2\phi+\frac{\Theta}{2}\right)}, \tilde{\psi}_R^\dagger \equiv e^{\mathbf{i}\frac{1}{2}\left(2\phi-\frac{\Theta}{2}\right)}.$$

At $K = \frac{1}{4}$ these are free fermions, and the $v\cos 2\phi$ term is just a mass term $v\tilde{\psi}_L^\dagger\tilde{\psi}_R + h.c.$. Interestingly, there is a regime where the quantum of the boson can *decay* into a pair of these (fermionic) domain walls.

• The phase at $v \to +\infty$ is similar. Minimizing $+|v|\cos 2\phi$ gives $\phi = n\pi$ which are distinguished by $\cos\phi = (-1)^n = \langle(-1)^i \mathbf{Z}_i\rangle$. This is 'Ising Neel' order, that is: staggered Ising spins. (Neel order refers to staggered magnetization in an arbitrary direction, spontaneously breaking $\mathsf{SU}(2)$ spin rotation symmetry.)

The two phases at large $|v|$ are related by the relabelling $v \to -v, \phi \to \phi + \pi/2$. In terms of the bosons or fermions both of these large $|v|$ states are charge density waves, where the density of particles breaks the translation symmetry down to a smaller discrete subgroup.

In the high-energy literature, this is called duality between the Thirring model and the sine-Gordon model.

[End of Lecture 19]

5.7 AdS/CFT

Quantum gravity is different. For three quarters now we've been talking about field theories, which you'll recall I defined as systems with extensive degrees of freedom. In a system with extensive degrees of freedom:

$$(\text{total \# of possible states}) \text{ is proportional to } (\text{\# possible states of each site})^{(\text{\# of sites})} \sim 2^V$$

The number of sites V (the volume of space in units of the lattice spacing) is something bigger than Avogadro's number 10^{23}, so these are some catastrophically big numbers. To make the numbers a bit more manageable let's take the logarithm of both sides:

$$\log(\text{\# of states}) \sim V \log 2.$$

This quantity appears in the thermodynamics of the system. Recall that the *entropy* is the log of the number of states consistent with the information we have about the state of the system:

$$S(F) = \log(\text{\# of states with macroscopic feature } F) \leq S_{\max} \sim V \log 2.$$

The biggest the entropy can be is the log of the total number of states of the system. The conclusion is that in a QFT, the maximum entropy goes like the volume of space.

The main point I want to convey in this final subsection is that quantum gravity is not like this. Here is a rough four-step argument for this crazy claim (due to 't Hooft and Susskind in the early 1990s). The key ingredient is:

Black Hole Thermodynamics.

1. A black hole is an object from which there is (classically) no escape, because the escape velocity is larger than the speed of light. The region from which there is no escape is bounded by the black hole's *event horizon*. Their existence is a direct consequence of (the universally attractive nature of) gravity.

2. If you throw together enough stuff in a small-enough region, you will make a black hole.

3. Consistent laws of thermodynamics in the presence of gravity require that we assign a black hole an entropy. The idea is: if we didn't assign an entropy to a black hole, we could violate the second law of thermodynamics and build perpetual-motion machines. We could do this just by throwing all our trash into a black hole. Since the high-entropy trash would just be gone, doing so would reduce the entropy of the world.

The entropy we must assign to a black hole in Einstein gravity is

$$S_{\rm BH} = \frac{\text{area of horizon}}{4l_P^2},$$

$l_P \equiv \sqrt{\frac{G_N\hbar^2}{c^3}}$ is the *Planck length*, the intrinsic length scale of the gravitational interactions.

With this realization (due to Bekenstein and Hawking), there is a consistent set of laws of thermodynamics including black holes. The second law is generalized to the statement that $S + S_{\rm BH}$ doesn't decrease (with overwhelming probability).

4. Now, let's ask: what is the maximum entropy in a gravitating system? I claim that

$$S_{\rm max}(\text{stuff in a region of space with surface area } A \text{ in a gravitating system}) = S_{\rm BH}(\text{biggest}) = \frac{A}{4l_P^2},$$

where $S_{\rm BH}$(biggest) is the entropy of the biggest black hole that fits in the region. The idea is: suppose this were not the case, *i.e.* we have a collection of stuff in the region in question with $E_{\rm stuff} < M_{\rm BH}$ and $S_{\rm stuff} > S_{\rm BH}$. Now throw some trash into the region. This increases the energy at fixed volume, so increases the density. If we do it enough, the system will gravitationally collapse, and form a black hole. But then the resulting entropy would be $S_{\rm final} \leq S_{\rm BH} \leq S_{\rm initial}$. In this way we would could violate the second law of thermodynamics, and build perpetual motion machines and save the world from the infestation of humans. So that means we probably can't do this.

But now recall that the maximum entropy is

$$S_{\rm max} \propto \#\text{ of dofs.}$$

We've just shown that for a system with black holes

$$S_{\rm max} \propto \#\text{ of dofs} \propto \frac{\text{area}}{l_P^2} \ll \frac{\text{volume}}{l_P^3}$$

where the last inequality is true for regions large enough compared to the Planck length.

This means that gravity has the same number of degrees of freedom as an ordinary system (field theory) living on the *boundary* of space. This is called the *Holographic Principle*.

This is a general principle about gravity, and makes us really want to know: *who* is this field theory living on the boundary of space? In general we don't know the answer.

There is one set of examples where we do know, which is for gravity with a negative cosmological constant. (In our world we have a very teeny positive cosmological constant.) In this case, solving Einstein's equations shows that space has a naturally-occurring boundary, which is frozen in a sense. Near this boundary, the geometry is anti-de Sitter space. Gravity in asymptotically-anti-de Sitter space (AdS) *is* a special kind of field theory (called a conformal field theory (CFT)) living on the one-lower-dimensional boundary. This AdS/CFT correspondence was discovered using string theory, and we know lots of examples and can check it in lots of ways. It is an explicit realization of the holographic principle.

I mentioned that black holes have an entropy. This means they must also have a temperature. But a hot body in outer space will radiate. This means that black holes evaporate, and it seems that all that is left is some thermal radiation labelled by the temperature. Where does the information about what made the black hole go when this happens? This is the black hole information problem.

There can be black holes in AdS, which (if they're small enough) behave just like black holes in flat space, and will evaporate. But that whole process is described by the dual CFT, which is an ordinary quantum system with unitary time evolution, from which information cannot disappear. So the AdS/CFT correspondence tells us the resolution of the black hole information problem (the information does not go away), but doesn't tell us exactly how it is resolved.

A word about *entanglement*: in an ordinary system (a field theory), the Hilbert space is a *tensor product* of the Hilbert spaces of the individual degrees of freedom. For example, the Hilbert space of one spin is $\mathcal{H}_2 = \text{span}\{|0\rangle, |1\rangle\}$, the vector space spanned by these two vectors, *i.e.*all linear combinations of the form $a_0 |0\rangle + a_1 |1\rangle$. If I have two spins, the Hilbert space is

$$\mathcal{H}_2 \otimes \mathcal{H}_2 = \text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}.$$

The dimension of the Hilbert space for N spins is 2^N, and this is the number we were counting above.

A state of two spins of the form can be a product state like $|00\rangle$ or $|11\rangle$, or it can be some superposition like $|00\rangle + |11\rangle$. The latter state is said to be entangled. The *entanglement entropy* is a way to quantify this[42].

[42] If $\mathcal{H}_A = \text{span}\{|a\rangle, a = 1..N_A\}$ and $\mathcal{H}_B = \text{span}\{|b\rangle, b = 1..N_B\}$, then a general state of $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$|w\rangle = \sum_{a=1}^{N_A} \sum_{b=1}^{N_B} w_{ab} |a\rangle \otimes |b\rangle .$$

So all the information is in the matrix w_{ab}. The entanglement entropy of A (or of B) is $S = -\sum \lambda^2 \log \lambda^2$ where $\{\lambda\}$ are the eigenvalues of the matrix w.

If we have degrees of freedom spread over space, we can ask about the entanglement of some region with its complementary region.

But notice that the very notion of entanglement between regions of space required the degrees of freedom to be extensive, as in a field theory. So we don't really know what we mean by the entanglement of a region in a theory of gravity.

The temperature of a black hole. A black hole in $D = 3 + 1$ looks like

$$ds^2_{\rm Sch} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ . \tag{5.68}$$

$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere. The event horizon is located at a zero of the emblackening factor $f(r)$. For a generic (non-extremal) black hole, $f(r) \overset{r\sim r_H}{\sim} 2\kappa(r - r_H)$ has a single zero at the horizon (this defines the *surface gravity* κ), and the metric near the horizon can be rewritten in the form:

$$\begin{aligned} ds^2_{\rm Sch} &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \\ &\approx \kappa^2 R^2 dt^2 + dR^2 + r(R)^2 d\Omega^2 \\ &= -R^2 d\eta^2 + dR^2 + \dots \\ &\approx -dT^2 + dZ^2 + dX^2 + dY^2 \ . \end{aligned}$$

For definiteness I used the form of the Schwarzschild black hole in $D = 4$: $f = 1 - r_H/r$, $r_H = 2GM$, $R(r) = \sqrt{r(r - 2GM)} + \dots$, $\kappa = 1/4GM$, $\eta = \kappa t$, $T = R\sinh\eta$ and $Z = R\cosh\eta$. Notice that after the transformation, we obtain the Minkowski space $\mathbb{R}^{3,1}$. Rindler space is Minkowski space in the coordinate frame of a uniformly accelerating observer, whose worldline sits at fixed R. η is called the Rindler time; you can see from the last step that it is the rapidity from the point of view of the final Minkowski space. What we get is shown in Fig. 3. Notice that the two lines defined by $\{r = 2GM, \eta = \infty\}$ and $\{r = 2GM, \eta = -\infty\}$ divide the space into four regions { I, II, III, IV }.

The simplest way to see that a black hole has a temperature is to think about the euclidean geometry. Let $\theta = -\mathbf{i}\eta$. Then the metric is

$$ds^2 = R^2 d\theta^2 + dR^2 + ...$$

which is just flat space in polar coordinates. But this space is only actually flat if $\theta \equiv \theta + 2\pi$. Otherwise $R = 0$ is the tip of a cone, at which there is localized curvature. Such curvature does not solve the vacuum Einstein equations. So the black hole requires euclidean time to be periodic with $\tau = -\mathbf{i}t \equiv \tau + 2\pi\kappa$. The periodicity of euclidean time is $\beta = 1/T$ (recall that the thermal partition function is $\mathrm{tr}e^{-\beta H}$, and $e^{-\beta H}$ generates a translation in euclidean time by an amount β), so $T = \frac{1}{2\pi\kappa}$ is the temperature of the black hole.

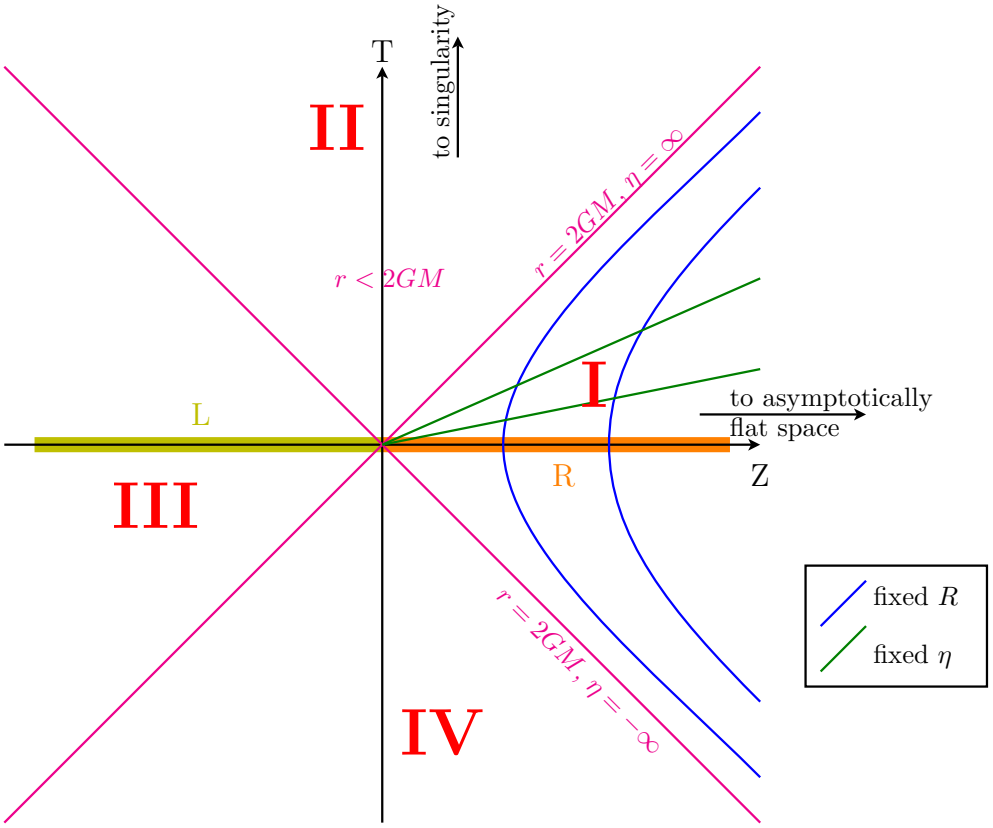


Figure 3: The geometry of the near-horizon region of a generic black hole is Rindler space. [Figure by Wing-Ko Ho]

We can understand this in more detail as follows. Region I is the region outside the horizon of the black hole. The key observation is that region I is self-contained. That is, region II and III can't communicate with it, while information from region IV passes through the line $\{r = 2GM, \eta = -\infty\}$ and hence corresponds to *initial data*. Therefore, at $T = 0$, the degrees of freedom at $Z < 0$ (marked by L on the figure) have no effect on region I. Thus we should trace over them when computing stuff in region I.

Now, Claim: Let $|\text{g.s.}\rangle$ be the ground state of any quantum field theory in Minkowski space, and let $T_{\mu\nu}$ be its stress-energy tensor. Its reduced density matrix on region I is

$$\rho_R = \text{tr}_L |\text{g.s.}\rangle\langle\text{g.s.}| = \frac{1}{\mathcal{Z}} e^{-2\pi H_R}$$

where H_R is the Rindler Hamiltonian:[43]

$$H_R = \int_{\text{constant } T} d^d x T_{\eta T}$$

[43]Note that $[H_R] = 1$ since the Rindler time is dimensionless $[\eta] = 1$.

and $\mathcal{Z} = \mathrm{tr}_R\, e^{-2\pi H_R}$.

Notice that the density matrix we obtained is not a pure state but is entangled with the L d.o.f. (hence $S = -\mathrm{tr}\rho_R \ln \rho_R \neq 0$). Much more specifically, it is a thermal density matrix with temperature $T_{\text{Rindler}} = 1/2\pi$.

Proof of Claim [Unruh, Bisognano-Wichmann 1975-6]. We will write the argument for a scalar field ϕ, but the argument is general. So consider a scalar field ϕ in Rindler space. A complete set of commuting d.o.f's at $T = 0$ is

$$\phi(x, y, z) = \begin{cases} \phi_R(x, y, z) & \text{for } z > 0 \\ \phi_L(x, y, z) & \text{for } z < 0 \end{cases}$$

ϕ_R and ϕ_L commute because they are at spacelike separated points. Any wavefunctional of the field ϕ can then be written as $\Psi = \Psi[\phi_L, \phi_R] = \langle \phi_L \phi_R | \Psi \rangle$.

The ground state is just the Minkowski space vacuum. The Feynman-Kac formula gives us a path integral representation of its wavefunctional:

$$\Psi_{\text{g.s.}}[\phi_L, \phi_R] = \frac{1}{\sqrt{\mathcal{Z}}} \int_{x^0>0, \phi(\vec{x}, x^0=0)=(\phi_L, \phi_R)} [d\phi] e^{-S_{\text{Eucl}}[\phi]}$$

This equation is written using constant-$x^0 = -iT$ time slices. We can gain some insight by instead slicing up the path integral by constant euclidean Rindler time $\theta (= -i\eta)$ (see Fig. 4(a) for illustration); euclidean Rindler time is just the angular coordinate in the x^0, Z plane. This yields:[44]

$$\Psi_{\text{g.s.}}[\phi_L, \phi_R] = \frac{1}{\sqrt{\mathcal{Z}}} \int_{\substack{0<\theta<\pi \\ \phi(\theta=0)=\phi_R \\ \phi(\theta=\pi)=\phi_L \\ \partial_R \phi(R=0)=0}} [d\phi] e^{-S_{\text{Eucl}}[\phi]} = \frac{1}{\sqrt{\mathcal{Z}}} \langle \phi_L | \, e^{-\pi H_R} \, | \phi_R \rangle \tag{5.69}$$

The last condition $\partial_R \phi(R = 0) = 0$ is simply a requirement that the function ϕ is regular at the origin. We identify the RHS of (5.69) $\langle \phi_L | \, e^{-\pi H_R} \, | \phi_R \rangle$ as the transition amplitude.

Hence the matrix elements of $\rho_R = \mathrm{tr}_L |\mathrm{gs}\rangle\langle \mathrm{gs}|$ are

$$\begin{aligned} \langle \phi_R | \, \rho_R \, | \phi'_R \rangle &= \int [d\phi_L] \Psi^*[\phi_L, \phi_R] \Psi[\phi_L, \phi'_R] \\ &= \left(\frac{1}{\sqrt{\mathcal{Z}}} \right)^2 \int [d\phi_L] \, \langle \phi_R | \, e^{-\pi H_R} \, | \phi_L \rangle \, \langle \phi_L | \, e^{-\pi H_R} \, | \phi'_R \rangle \\ &= \frac{1}{\mathcal{Z}} \, \langle \phi_R | \, e^{-2\pi H_R} \, | \phi'_R \rangle \end{aligned}$$

■

[44] We are transforming the variable that the function takes, rather than the function itself, and it is the function that's being integrated. Thus there is no Jacobian involved here.

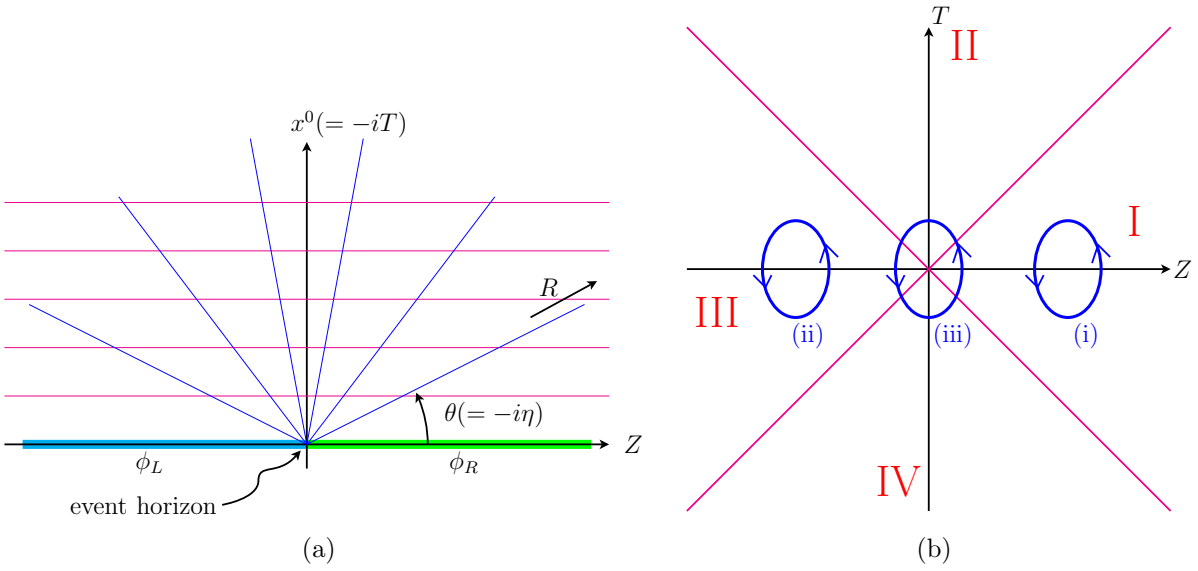


Figure 4: (a) Change in slicing sequence. The magenta lines are the old x^0 slices while the blue lines are the new θ slices. (b) Fluctuation in different regions of the Rindler space. [Figures by Wing-Ko Ho]

Some results and comments:

1. Forgetting for a moment about the motivation from black holes, this shows that a constantly accelerated observer[45] in $\mathbb{R}^{n,1}$ sees a *Unruh radiation* with temperature $T_{\text{proper}} = \dfrac{1}{2\pi R} = \dfrac{a}{2\pi}$. (The factor of R comes from the fact that the proper time is $\tau = R\eta$, so $\partial_\tau = \frac{1}{R}\partial_\eta = \frac{1}{R}H_R$.)

2. By the equivalence principle, the same happens outside a black hole. The corresponding Schwarzschild time is $t = \eta/\kappa$, which implies that $T_{\text{BH}} = T_{\text{Rindler}} = \dfrac{\kappa}{2\pi}$. This determines the temperature of the black hole in terms of the horizon radius, or equivalently in terms of its mass, and fixes the proportionality constant in the relation between entropy and area.

3. Where does the thermal radiation come from? Very heuristically, there are three types of fluctuations (see Fig. 4(b)):
   (i) is just ordinary pair production fluctuation, with lifetime $\propto \hbar/E$
   (ii) is something that we need not care about, if we live in region I

[45] Here a is the constant acceleration, defined by $a^2 \equiv g^{\mu\nu}a_\mu a_\nu$, where $a^\mu = \partial_\tau^2 X^\mu$ is the proper acceleration vector, and τ is the proper time along the worldline. A uniformly accelerating observer follows a trajectory $T = R\sinh\eta, Z = R\cosh\eta$ with fixed R, and the proper time satisfies $d\tau^2 \equiv -ds^2 = R^2 d\eta^2$ along the worldline. This equation shows that $a = \frac{1}{R}$. Thanks to Bowen Shi for pointing out an error here.

(iii) corresponds to stuff that enters at $\eta = -\infty, R = 0$ and falls back at $\eta = +\infty, R = 0$. In Schwarzschild time this particle stays forever. Hence this particle is real.

4. What we have calculated is the density matrix of a quantum field in a neighborhood outside $r \sim r_H$. Not all energy will get to ∞ because the black hole is like a potential well. The rate of particle emission includes what is called a *greybody factor* to account for this. The more complicated calculation done by Hawking includes this factor, but has the huge drawback of only applying to free fields.

5. *Information paradox.* A black hole can form from a dictionary or worse a pure quantum state, and then (apparently) evaporate into thermal radiation. The thermal density matrix is determined by one number only.
   Q1. Is this a unitarity evolution?
   Q2. Entropy results from coarse-graining. What are the microstate?
   Q3. How is the information stored in the black hole?
   via AdS/CFT it is possible to answer Q1 and Q2.

Holographic duality. [This is a highly condensed version of these notes, which in turn are a condensed version of this course. The logic is from here.]
Bold Assertion:
(a) Some ordinary quantum field theories (QFTs) are secretly quantum theories of gravity.
(b) Sometimes the gravity theory is classical, and therefore we can use it to compute interesting observables of the QFT.

Here are three facts which make the Assertion seem less unreasonable.

1) First we must define what we mean by a quantum gravity (QG). As a working definition, let's say that a QG is a quantum theory with a dynamical metric. In enough dimensions, linearizing equations of motion (EoM) for a metric usually reveals a propagating mode of the metric, some spin-2 massless particle which we can call a 'graviton'.

So at the least the assertion must mean that there is some spin-two graviton particle that is somehow a composite object made of QFT degrees of freedom. This statement seems to run afoul of the Weinberg-Witten no-go theorem, which says:
Theorem [Weinberg-Witten]**:** A QFT with a Poincaré covariant conserved stress tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ which carry momentum (*i.e.* with $P^\mu = \int d^D x T^{0\mu} \neq 0$).

You may worry that the assumption of Poincaré invariance plays an important role in the proof, but the set of QFTs to which the Bold Assertion applies includes

relativistic theories.

Like any good no-go theorem, it is best considered a sign pointing away from wrong directions. The loophole in this case is blindingly obvious in retrospect: the graviton needn't live in the same spacetime as the QFT.

2) is the Holographic Principle described above. This suggests that the gravity theory should live in a spacetime with an extra dimension.

3) A beautiful hint as to the possible identity of the extra dimensions is this. Wilson taught us that a QFT is best thought of as being sliced up by length (or energy) scale, as a family of trajectories of the renormalization group (RG). A remarkable fact about this is that the RG equations for the behavior of the coupling constants as a function of RG scale u are *local* in scale:

$$u\partial_u g = \beta(g(u)) \ .$$

The beta function is determined by the coupling constant evaluated at the energy scale u, and we don't need to know its behavior in the deep UV or IR to figure out how it's changing. This fact is basically a consequence of locality in ordinary spacetime. This opens the possibility that we can associate the extra dimension suggested by the Holographic idea with energy scale. This notion of locality in the extra dimension actually turns out to be much *weaker* than what we will find in AdS/CFT, but it is a good hint.

To summarize, we have three hints for interpreting the Bold Assertion:

1. The Weinberg-Witten theorem suggests that the graviton lives on a different space than the QFT in question.

2. The holographic principle says that the theory of gravity should have a number of degrees of freedom that grows more slowly than the volume. This suggests that the quantum gravity should live in more dimensions than the QFT.

3. The structure of the Renormalization Group suggests that we can identify one of these extra dimensions as the RG-scale.

Clearly the field theory in question needs to be strongly coupled. Otherwise, we can compute and we can see that there is no large extra dimension sticking out.

Next we will make a simplifying assumption in an effort to find concrete examples. The simplest case of an RG flow is when $\beta = 0$ and the system is self-similar. In a Lorentz invariant theory (which we also assume for simplicity), this means that the following scale transformation $x^\mu \to \lambda x^\mu$ ($\mu = 0, 1, 2, ...d-1$) is a symmetry. If the

extra dimension coordinate u is to be thought of as an energy scale, then dimensional analysis says that u will scale under the scale transformation as $u \to \frac{u}{\lambda}$. The most general $(d+1)$-dimensional metric (one extra dimension) with this symmetry and Poincaré invariance is of the following form:

$$ds^2 = \left(\frac{\tilde{u}}{\tilde{L}}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\tilde{u}^2}{\tilde{u}^2} L^2 \quad .$$

We can bring it into a more familiar form by a change of coordinates, $\tilde{u} = \frac{\tilde{L}}{L} u$:

$$ds^2 = \left(\frac{u}{L}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} L^2 \quad .$$

This is AdS_{d+1}[46]. It is a family of copies of Minkowski space, parametrized by u, whose size varies with u (see Fig. 5). The parameter L is called the 'AdS radius' and it has dimensions of length. Although this is a dimensionful parameter, a scale transformation $x^\mu \to \lambda x^\mu$ can be absorbed by rescaling the radial coordinate $u \to u/\lambda$ (by design); we will see below more explicitly how this is consistent with scale invariance of the dual theory. It is convenient to do one more change of coordinates, to $z \equiv \frac{L^2}{u}$, in which the metric takes the form

$$ds^2 = \left(\frac{L}{z}\right)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right) \quad . \tag{5.70}$$

These coordinates are better because fewer symbols are required to write the metric. z will map to the length scale in the dual theory.

So it seems that a d-dimensional[47] conformal field theory (CFT) should be related to a theory of gravity on AdS_{d+1}. This metric (5.70) solves the equations of motion of the following action (and many others)[48]

$$S_{\text{bulk}}[g, \dots] = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left(-2\Lambda + \mathcal{R} + \dots\right) \quad . \tag{5.71}$$

Here, $\sqrt{g} \equiv \sqrt{|\det g|}$ makes the integral coordinate-invariant, and $\mathcal{R}$ is the Ricci scalar curvature. The cosmological constant Λ is related by the equations of motion

$$0 = \frac{\delta S_{\text{bulk}}}{\delta g^{AB}} \quad \Longrightarrow \quad R_{AB} + \frac{d}{L^2} g_{AB} = 0 \tag{5.72}$$

[46] It turns out that this metric also has conformal invariance. So scale and Poincaré symmetry implies conformal invariance, at least when there is a gravity dual. This is believed to be true more generally (see Polchinski's 1987 paper), but there is no proof for $D > 1+1$. Without Poincaré invariance, scale invariance definitely does *not* imply conformal invariance; indeed there are scale-invariant metrics without Poincaré symmetry, which do not have have special conformal symmetry.

[47] Note that I am forced to violate my convention for d and D in either the bulk or the boundary of this correspondence. Below I use d to denote the number of spatial dimensions of the bulk, or the number of spacetime dimensions of the boundary.

[48] For verifying statements like this, it can be helpful to use Mathematica or some such thing.

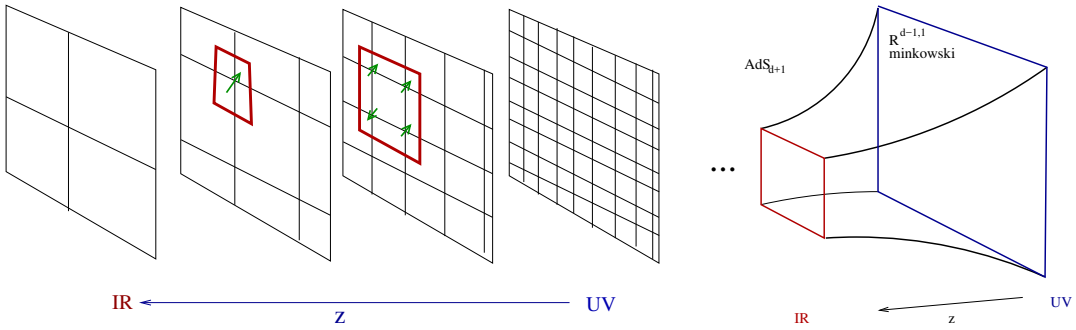


Figure 5: The extra ('radial') dimension of the bulk is the resolution scale of the field theory. The left figure indicates a series of block spin transformations labelled by a parameter z. The right figure is a cartoon of AdS space, which organizes the field theory information in the same way. In this sense, the bulk picture is a hologram: excitations with different wavelengths get put in different places in the bulk image.

to the value of the AdS radius: $-2\Lambda = \frac{d(d-1)}{L^2}$. This form of the action (5.71) is what we would guess using Wilsonian naturalness (*i.e.*, the 'Landau-Ginzburg-Wilson paradigm' or the EFT strategy): we include all the terms that respect the symmetries and redundancies (in this case, this is general coordinate invariance), organized by decreasing relevantness, *i.e.* by the number of derivatives. The Einstein-Hilbert term (the one with the Ricci scalar) is an irrelevant operator: $\mathcal{R} \sim \partial^2 g + (\partial g)^2$ has dimensions of length^{-2}, so G_N here is a length^{d-1}, the Planck length: $G_N \equiv \ell_{pl}^{d-1} \equiv M_{pl}^{1-d}$ (in units where $\hbar = c = 1$). The gravity theory is classical if $L \gg \ell_{pl}$. In this spirit, the ... on the RHS denote more irrelevant terms involving more powers of the curvature. Also hidden in the ... are other bulk fields that vanish in the dual of the CFT vacuum (*i.e.* in the AdS solution).

This form of the action (5.71) is indeed what comes from string theory at low energies and when the curvature (here, $\mathcal{R} \sim \frac{1}{L^2}$) is small (compared to the *string tension*, $\frac{1}{\alpha'} \equiv \frac{1}{\ell_s^2}$; this is the energy scale that determines the masses of excited vibrational modes of the string), at least in cases where we are able to tell. The main role of string theory in this business (at the moment) is to provide consistent ways of filling in the dots.

But wait: in a theory of gravity, the space-time metric is a dynamical variable, and we only get to specify the boundary behavior. The *AdS* metric above has a boundary at $z = 0$. This is a bit subtle. Keeping x^μ fixed and moving in the z direction from a finite value of z to $z = 0$ is actually infinite distance. However, massless particles in *AdS* (such as the graviton discussed above) travel along null geodesics; these reach the boundary in finite time. This means that in order to specify the future evolution of the system from some initial data, we have also to specify boundary conditions at

$z = 0$. These boundary conditions will play a crucial role in the discussion below.

So we should amend our statement to say that a d-dimensional conformal field theory is related to a theory of gravity on spaces which are *asymptotically* AdS_{d+1}. Note that this case of negative cosmological constant (CC) turns out to be much easier to understand holographically than the naively-simpler (asymptotically-flat) case of zero CC, which has a null boundary. Let's not even talk about the case of positive CC (asymptotically de Sitter), which has a boundary in the future.

Different CFTs will correspond to such theories of gravity with different field content and different bulk actions, *e.g.* different values of the coupling constants in $S_{\rm bulk}$. The example which is understood best (and was found first) is the case of the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four dimensions. This is dual to maximal supergravity in AdS_5 (which arises by dimensional reduction of ten-dimensional IIB supergravity on $AdS_5 \times S^5$). In that case, we know the precise values of many of the coefficients in the bulk action. This will not be very relevant for our discussion below. An important conceptual point is that the values of the bulk parameters which are realizable will in general be discrete[49]. This discreteness is hidden by the classical limit.

We will focus on the case of relativistic CFT, but let me emphasize here that the name 'AdS/CFT' is a very poor one: the correspondence is much more general. It can describe deformations of UV fixed points by relevant operators, and it has been extended to cases which are not even relativistic CFTs in the UV: examples include fixed points with dynamical critical exponent $z \neq 1$, Galilean-invariant theories and theories which do more exotic things in the UV like the 'duality cascade'

Counting of degrees of freedom. [Susskind-Witten] We can already make a check of the conjecture that a gravity theory in AdS_{d+1} might be dual to a QFT in d dimensions. The holographic principle tells us that the area of the boundary in Planck units is the number of degrees of freedom (dof), *i.e.* the maximum entropy:

$$\frac{\text{Area of boundary}}{4\text{G}_\text{N}} \stackrel{?}{=} \text{number of dof of QFT} \equiv N_d \quad .$$

Is this true? Yes: both sides are equal to infinity. We need to regulate our counting.

Let's regulate the field theory first. There are both UV and IR divergences. We put the thing on a lattice, introducing a short-distance cut-off ϵ (*e.g.*, the lattice spacing) and we put it in a cubical box of linear size R. The total number of degrees of freedom is the number of cells $\left(\frac{R}{\epsilon}\right)^{d-1}$, times the number of degrees of freedom per lattice site, which we will call 'N^2'. The behavior suggested by the name we have given this number is found in well-understood examples. It is, however, clear (for example

[49] An example of this is the relationship (5.76) between the Newton constant in the bulk and the *number* of species in the field theory, which we will find next.

from the structure of known AdS vacua of string theory) that other behaviors N^b are possible, and that's why I made it a funny color and put it in quotes. So $N_d = \frac{R^{d-1}}{\epsilon^{d-1}} N^2$.

The picture we have of AdS_{d+1} is a collection of copies of d-dimensional Minkowski space of varying size; the boundary is the locus $z \to 0$ where they get really big. The area of the boundary is

$$A = \int_{\mathbb{R}^{d-1},\ z/\to 0,\ \text{fixed } t} \sqrt{g} d^{d-1}x = \int_{\mathbb{R}^{d-1},\ z\to 0} d^{d-1}x \frac{L^{d-1}}{z^{d-1}} \quad . \tag{5.73}$$

As in the field theory counting, this is infinite for two reasons: from the integral over x and from the fact that z is going to zero. To regulate this integral, first put $x \simeq x + R$ in a box again, and second evaluate not at $z = 0$ but rather cut it off at $z = \epsilon$. This idea is that the boundary of AdS is associated with the UV behavior of the field theory, and that cutting off the geometry at $z = \epsilon$ is a UV cutoff (not identical to the lattice cutoff, but close enough for our present purposes). Given this,

$$A = \int_0^R d^{d-1}x \frac{L^{d-1}}{z^{d-1}}|_{z=\epsilon} = \left(\frac{RL}{\epsilon}\right)^{d-1} \quad . \tag{5.74}$$

The holographic principle then says that the maximum entropy in the bulk is

$$\frac{A}{4G_N} \sim \frac{L^{d-1}}{4G_N}\left(\frac{R}{\epsilon}\right)^{d-1} . \tag{5.75}$$

We see that the scaling with the system size agrees – the both-hand-side goes like R^{d-1}. So AdS/CFT is indeed an implementation of the holographic principle. We can learn more from this calcluation: In order for the prefactors of R^{d-1} to agree, we need to relate the AdS radius in Planck units $\frac{L^{d-1}}{G_N} \sim (LM_{pl})^{d-1}$ to the number of degrees of freedom per site of the field theory:

$$\boxed{\frac{L^{d-1}}{G_N} = N^2} \tag{5.76}$$

up to numerical prefactors.

An important conclusion from this calculation is that the gravity theory is classical $L \gg \ell_P$ precisely when the number of degrees of freedom at each point of space in the QFT, N^2, is large.

Preview of the AdS/CFT correspondence. Here's the ideology:

fields in AdS	$\longleftrightarrow$	local operators of CFT
spin		spin
mass		scaling dimension Δ

In particular, for a scalar field in AdS, the formula relating the mass of the scalar field to the scaling dimension of the corresponding operator in the CFT is $m^2 L_{AdS}^2 = \Delta(\Delta - d)$.

One immediate lesson from this formula is that a simple bulk theory with a small number of light fields is dual to a CFT with a hierarchy in its spectrum of operator dimensions. In particular, there need to be a small number of operators with small (*e.g.* of order N^0) dimensions. (If you are aware of explicit examples of such theories, please let me know.) This is to be distinguished from the thus-far-intractable case where some whole tower of massive string modes in the bulk are needed.

Now let's consider some observables of a QFT (we'll assume Euclidean spacetime for now), namely vacuum correlation functions of local operators in the CFT:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle \quad .$$

We can write down a generating functional $Z[J]$ for these correlators by perturbing the action of the QFT:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_A J_A(x)\mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x)$$
$$Z[J] = \left\langle e^{-\int \mathcal{L}_J} \right\rangle_{CFT} \quad .$$

where $J_A(x)$ are arbitrary functions (sources) and $\{\mathcal{O}_A(x)\}$ is some basis of local operators. The n-point function is then given by:

$$\left\langle \prod_n \mathcal{O}_n(x_n) \right\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z\Big|_{J=0} \quad .$$

Since $\mathcal{L}_J$ is a *UV* perturbation (because it is a perturbation of the *bare* Lagrangian by *local* operators), in AdS it corresponds to a perturbation near the boundary, $z \to 0$. (Recall from the counting of degrees of freedom in section 5.7 that QFT with UV cutoff $E < 1/\epsilon \longleftrightarrow$ AdS cutoff $z > \epsilon$.) The perturbation J of the CFT action is encoded in the boundary condition on bulk fields.

The idea (often referred to as GKPW) for computing $Z[J]$ is then, schematically:

$$Z[J] \equiv \left\langle e^{-\int \mathcal{L}_J} \right\rangle_{CFT} = \underbrace{Z_{\rm QG}[\text{b.c. depends on } J]}_{=???} \underset{N\gg 1}{\sim} e^{-S_{\rm grav}}\Big|_{\text{EOM, b.c. depend on } J} \quad . \tag{5.77}$$

The middle object is the partition function of quantum gravity. We don't have a very useful idea of what this is, except in perturbation theory and via this very equality. In a limit where this gravity theory becomes classical, however, we know quite well what we're doing, and we can do the path integral by saddle point, as indicated on the RHS of (5.77).

An important point here is that even though we are claiming that the QFT path integral is dominated by a classical saddle point, this does not mean that the field theory degrees of freedom are free. How this works depends on what kind of large-N limit we take to make the gravity theory classical. In the kind of vector-like large-N limit that we've studied so far this quarter, the large-N expectation values can be computed by saddle point. In contrast, in a theory where the degrees of freedom are $N \times N$ matrices, such as a Yang-Mills theory with gauge group $\mathsf{SU}(N)$, there are infinitely-many leading-order diagrams.

One nice conclusion that we can state is that black hole thermodynamics, for black holes in AdS, is the ordinary thermodynamics of the dual CFT. In particular, the canonical ensemble partition function is obtained by periodic identification of euclidean time:

$$e^{-\beta F} = \mathrm{tr}_{\mathrm{CFT}} e^{-\beta H_{\mathrm{CFT}}} = Z_{\mathrm{QG}}[\mathrm{bdy} S^1_\beta \times \mathbb{R}^d] \simeq e^{-S_{\mathrm{grav}}}|_{\mathrm{BH}}. \tag{5.78}$$

The solution to the bulk eom with these boundary conditions is the euclidean black hole. Its action then determines the free energy, and the entropy is

$$S = -\partial_T F = \frac{\mathrm{Area}}{4G_N} = S_{\mathrm{BH}}. \tag{5.79}$$

This calculation was first done by Gibbons and Hawking, without the knowledge of whose free energy was being computed[50].

[End of Lecture 20]

[50] If you want to actually do the calculation there is a wrinkle you must know about: there are some boundary terms in the action. First, is a term proportional to the extrinsic curvature of the boundary, required to make the equations of motion consistent with the boundary conditions we are imposing. Second, we must add some counterterms to get a finite answer. You shouldn't be surprised by this, since the boundary is associated with the UV of a QFT. Adding all possible local terms and fixing their coefficients to cancel the divergences arising in the on-shell action will give you the right answer.