University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 10

Due 11:59pm Monday, June 6, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Jordan-Wigner.**

   Solve the following spin chains using the mapping to Majorana fermions.

   (a) **XY-model.**

   $$\mathbf{H} = -J \sum_j \left(\mathbf{Z}_j \mathbf{Z}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} \right)$$

   This model has a U(1) symmetry which rotates $\mathbf{Z}$ into $\mathbf{Y}$, *i.e.* acting by $U = e^{\mathbf{i}\alpha\mathbf{X}}$. How does it act on the fermions?

   (b) **Solve an interacting fermion system.**

   $$\mathbf{H}_{\text{int}} = -J \sum_j \left(\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} \right) \tag{1}$$

   This model is in fact related by a basis rotation ($\mathbf{U} = \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Y}_j}$) to the one in part 1a.

   But if you directly use the mapping we introduced in class in these variables, you'll find quartic terms in the fermions.

   The basis transformation above therefore solves this interacting fermion system.

   How does the $\mathsf{U}(1)$ symmetry of (1) act on these fermion variables?

   (c) **A spin chain with a non-onsite Ising symmetry.**

   Consider the Hamiltonian

   $$\mathbf{H} = -J \sum_j \left(\mathbf{X}_j + \lambda \mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1} \right)$$

   i. [Slightly more optional] Show that when $\lambda = -1$ this model is invariant under the action of

   $$\mathbf{S}_1 \equiv \prod_j \mathbf{X}_j \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Z}_j \mathbf{Z}_{j+1}}. \tag{2}$$

This symmetry is "not-onsite" in that its action on the spin at site j depends on the state of the neighboring sites.

ii. Solve this model by Jordan-Wigner. Show that the spectrum is gapless and that each momentum state is doubly-degenerate.

iii. [Challenge problem] The previous part shows that this model produces *two* massless majorana fermions of each chirality. Find the action of the $\mathbb{Z}_2$ symmetry (2) on these fermions.

iv. [Challenge problem] Consider the effect of adding the ferromagnetic term $\sum_j \mathbf{Z}_j \mathbf{Z}_{j+1}$ on this system. Is it invariant under the symmetry?

In this problem we consider adding an extra term:

$$\mathbf{H}_2 = -J \sum_j \left(g_x \mathbf{X}_j + g_z \mathbf{Z}_j \mathbf{Z}_{j+1} + \tilde{g}_x \mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1} \right) \ .$$

When $\tilde{g}_x = -g_x$, this hamiltonian has the symmetry

$$\mathbf{S}_1 = \left(\prod_j \mathbf{X}_j \right) \left(\prod_l e^{\mathbf{i} Q_{l,l+1}} \right)$$

where $e^{\mathbf{i} Q_{l,l+1}} = \sqrt{\mathbf{Z}_l \mathbf{Z}_{l+1}} = e^{\frac{\mathbf{i}\pi}{4}(1 - \mathbf{Z}_l \mathbf{Z}_{l+1})}$.

(d) **Kitaev-honeycomb-model-like chain** [optional]

Consider

$$\mathbf{H}_{\mathrm{K}} = \sum_j \left(\mathbf{X}_{2j} \mathbf{X}_{2j+1} + \mathbf{Y}_{2j} \mathbf{Y}_{2j-1} \right)$$

where the bonds alternate between XX interactions and YY interactions. There are now two sites per unit cell, which means that the solution in terms of momentum-space fermion operators will involve two bands. Find their dispersion.

2. **Homage to Onsager.** [optional]

Show that the groundstate energy of Ising chan with $N \gg 1$ sites may be written as

$$E_0(g) = -NJ \int_0^\pi \frac{dk}{2\pi} \epsilon_k$$

where ϵ_k is the dispersion we derived for the fermions.

Show that this can be written as

$$\frac{1}{NJ} E_0(g) = -\frac{2}{\pi}(1+g) E(\pi/2, \sqrt{1-\gamma^2}), \quad \gamma = \left| \frac{1-g}{1+g} \right|$$

(notice that this expression is manifestly self-dual) where $E(\pi/2, x)$ is the elliptic integral

$$E(\pi/2, x) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - x^2 \sin^2 \theta}.$$

Expand this result in $g - g_c$.

Use the quantum-to-classical mapping to infer the critical behavior of the 2d (classical) Ising model.

3. **Heisenberg chain**

   Consider the Heisenberg hamiltonian

$$\mathbf{H} = -J \sum_j \left(\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} + v \mathbf{Z}_j \mathbf{Z}_{j+1} \right) \ .$$

   When $v = 1$ there is $\mathsf{SU}(2)$ symmetry. What are the generators?

   On the previous problem set we successfully fermionized the model with $v = 0$. Fermionize the v term.

   Take the continuum limit.

4. **T-duality: not just for the free theory.**

   Here is a path integral derivation of T-duality which is more general than just a single free boson.

   Consider the sigma model whose action is

$$S(\partial X, Y) = S(Y) + \frac{1}{4\pi\alpha'} \int d^2 z \left(\delta^{ab} G_{XX}(Y) \partial_a X \partial_b X + \left(\delta^{ab} G_{\mu X} + \epsilon^{ab} B_{\mu X} \right) \partial_a X \partial_b Y^\mu \right) \ .$$

   Here Y^μ are a bunch of coordinates on which the background fields G, B may depend in arbitrarily complicated ways. X only appears through its derivatives.

   (a) Show that by replacing $\partial_\mu X$ by $\partial_\mu X + A_\mu$ we arrive at a theory with an invariance under local shifts of $X \to X + \alpha(x)$.

   (b) Add a 2d θ term $\mathbf{i}\phi F_{\mu\nu}$, with $F = dA$ and the angle ϕ a dynamical field. Show that the path integral over ϕ undoes the previous step and returns us to the original model. Hint: use the gauge $\partial_\mu A^\mu = 0$.

   (c) Instead choose the gauge $X = 0$ and do the integral over A_μ. Identify ϕ as the T-dual variable. To get the period right, you need to think about non-perturbative parts of the gauge field path integral.

5. **T-duality as EM duality of 0-forms.**

In this problem we will contextualize the form of the T-duality map

$$\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z}) \mapsto \tilde{\phi}(z,\bar{z}) \equiv \phi_L(z) - \phi_R(\bar{z})$$

in terms of more general duality maps on form fields.

Consider a massless p-form field a in D (euclidean) dimensions, more specifically, on $\mathbb{R}^D$. We will treat it classically. Suppose its eom are

$$\mathrm{d} \star \mathrm{d}a = 0 \ .$$

[1]

This equation says $\star \mathrm{d}a$ is closed, which on $\mathbb{R}^D$ which has no nontrivial topology, this means it is exact: we can define $\star \mathrm{d}a = \mathrm{d}\tilde{a}$.

For abelian gauge theory in $D = 4$ show that this map $a \to \tilde{a}$ takes $(E, B) \to (\tilde{E}, \tilde{B}) = (B, -E)$.

Show that the map between ϕ and $\tilde{\phi}$ is of this form, if we regard ϕ as a 0-form potential.

For help see this paper by Chris Beasley.

6. $\mathsf{SU}(2)$ **current algebra from free scalar.** [bonus problem]

Consider again a compact free boson $\phi \simeq \phi + 2\pi$ in $D = 1 + 1$ with action

$$S[\phi] = \frac{R^2}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \phi \partial^\mu \phi. \tag{3}$$

[Notice that if we redefine $\tilde{\phi} \equiv R\phi$ then we absorb the coupling R from the action $S[\tilde{\phi}] = \frac{1}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}$ but now $\tilde{\phi} \simeq \tilde{\phi} + 2\pi R$ has a different period – hence the name 'radius'.[2]]

So: there is a special radius (naturally called the $\mathsf{SU}(2)$ radius) where new operators of dimension $(1, 0)$ and $(0, 1)$ appear, and which are charged under the boson

[1]By this notation, I mean the following. The exterior derivative of a p-form is a $p + 1$ form:

$$(\mathrm{d}a)_{\mu_1 \cdots \mu_{p+1}} = \left(\partial_{\mu_1} a_{\mu_2 \cdots \mu_{p+1}} \pm \text{perms} \right) \frac{1}{(p+1)!}$$

The Hodge dual of a k-form is a $d - k$ form:

$$(\star \omega_k))_{\mu_1 \cdots \mu_{d-k}} \equiv \epsilon_{\mu_1 \cdots \mu_d} (\omega_k)^{\mu_{d-k+1} \cdots \mu_d} \ .$$

[2]Relative to the notation I used in lecture, I have set $\pi T \equiv R^2$. A note for the string theorists: I am using units where $\alpha' = 2$.

number current $\partial_\pm\phi$. Their dimensions tell us that they are (chiral) currents, and their charges indicate that they combine with the obvious currents $\partial_\pm\phi$ to form the (Kac-Moody-Bardakci-Halpern) algebra $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$.

Here you will verify that the model (3) does in fact host an $SU(2)_L \times SU(2)_R$ algebra involving *winding modes* – configurations of ϕ where the field winds around its target space circle as we go around the spatial circle. We'll focus on the holomorphic (R) part, $\phi(z) \equiv \phi_R(z)$; the antiholomorphic part will be identical, with bars on everything.

Define

$$J^\pm(z) \equiv : e^{\pm i\phi(z)} :, \qquad J^3 \equiv i\partial\phi(z).$$

The dots indicate a normal ordering prescription for defining the composite operator: no wick contractions between operators within a set of dots.

(a) Show that $J^3, J^\pm$ are single-valued under $\phi \to \phi + 2\pi$.

(b) Compute the scaling dimensions of $J^3, J^\pm$. Recall that the scaling dimension Δ of a holomorphic operator in 2d CFT can be extracted from its two-point correlation function:

$$\left\langle \mathcal{O}^\dagger(z)\mathcal{O}(0)\right\rangle \sim \frac{1}{z^{2\Delta}} .$$

For free bosons, all correlation functions of composite operators may be computed using Wick's theorem and

$$\langle\phi(z)\phi(0)\rangle = -\frac{1}{R^2}\log z.$$

Find the value of R where the vertex operators $J^\pm$ have dimension 1.

(c) Defining $J^\pm \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + i\epsilon^{abc}\frac{J^c(0)}{z} + ...$$

with $k = 1$. This is the level-$k = 1$ SU(2)Kac-Moody-Bardakci-Halpern algebra.

(d) [Bonus tedium] Defining a mode expansion for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

For hints (and some applications in string theory) see problem 5 here.