University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 8

Due 11:59pm Monday, May 23, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Non-linear sigma models with more general target spaces.** [Some knowledge of differential geometry is helpful here.]

   In lecture we considered the 2d non-linear sigma model whose target space was a round 2-sphere, motivated by the low-energy physics of antiferromagnets. At weak coupling (large radius of sphere, which means large spin), we saw that the sphere wants to shrink in the IR.

   Consider now a 2d non-linear sigma model (NLSM) whose target space is a more general manifold X with Riemannian metric $ds^2 = L^2 g_{ij}(x) dx^i dx^j$. Assume that the space is *big*, in the sense that we will treat the parameter L^{-1} as a small parameter, and *smooth* in the sense that we can Taylor expand around any point.

   The NLSM is a field theory whose fields $x^i(\sigma)$ are maps from spacetime (here 2d flat space) to the *target space* X. The simplest action is

   $$S[x(\sigma)] = \int d^2\sigma L^2 g_{ij}(x) \partial_{\sigma^\mu} x^i \partial_{\sigma^\nu} x^j \eta^{\mu\nu}$$

   where $\eta^{\mu\nu}$ is the flat metric on the 2d spacetime 'worldsheet'.

   $D = 2$ is special because the free scalar field $x(\sigma)$ is dimensionless. As long as g_{ij} is nonsingular, in the limit $L \to \infty$, the local coordinate field becomes free.

   Regard $g_{ij}(x)$ as a coupling *function*. What is the leading beta function (actually beta functional) for this set of couplings?

   Hint: use the fact that the answer must be covariant under changes of coordinates on X plus dimensional analysis. Match the undetermined coefficient using our previous result for the case where $X = S^n$.

2. **Haldane phase from the path integral.**

   Consider the $D = 1 + 1$ nonlinear sigma model with target space S^2 at $\theta = 2\pi$. Recall that this describes a spin-one antiferromagnetic chain. The θ term is a total derivative in the action, so it can manifest itself when we study the path integral on a spacetime with boundary.

(a) Put this field theory on the half-line $x > 0$. Suppose that the boundary conditions respect the $\mathsf{SO}(3)$ symmetry, so that the boundary values $\vec{n}(\tau, x = 0)$ are free to fluctuate. By remembering that the θ-term is a total derivative, and considering the strong-coupling (IR) limit, $g \to \infty$, show that there is a spin-$\frac{1}{2}$ at the boundary. (Hint: Recall the coherent state path integral for a spin-$\frac{1}{2}$.)

(b) Now cut the path integral open at some fixed euclidean *time* $\tau = 0$. (Consider periodic boundary conditions in space.) Such a path integral computes the groundstate wavefunction, as a function of the boundary values of the fields, $\vec{S}(x, \tau = 0)$. Find the groundstate wavefunctional is $\Psi[\vec{n}(x, \tau = 0)]$ in the strong coupling limit $g \to \infty$ (where the gap is big).