University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 7

Due 11:59pm Monday, May 16, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Kondo problem problem.** Consider a spinful Fermi liquid (treat it as free) in d dimensions coupled to a single spin s located at the origin. They are coupled by the Kondo interaction
$$H_K = J_K \mathbf{c}^\dagger \vec{\sigma} \mathbf{c} \cdot \vec{\mathbf{S}}.$$

   (a) Make a coherent state path integral representation for both the fermions and the spin. Write the action in Euclidean time.

   (b) Find the Feynman rules for perturbation theory about $J_K = 0$: the propagator for ψ (the fermion coherent state variable), the propagator for z (the spin coherent state variable, $\vec{n} = z^\dagger \vec{\sigma} z$), and the interaction vertex.

   (c) Find the one-loop beta function for the Kondo coupling.

   Two hints about how to proceed: (1) Recall from our previous discussion the methods for doing momentum integrals over functions peaked on a round Fermi surface. (2) Integrate out a shell of momentum modes with $|k| \in (\Lambda/b, \Lambda)$, where Λ is a UV cutoff (the bandwidth), and b is the RG parameter.

   (d) Solve the beta function equation for the running of the coupling, $J(b)$.

   (e) Some poetry: at finite temperature, the system explores states in a shell of width T around the Fermi surface. In your solution for $J(b)$ make the replacement $b = \Lambda/(\Lambda/b) = T/T_F$ to find $J_{\text{eff}}(T)$. Assuming that the bare J_K is antiferromagnetic, find the *Kondo temperature* T_K defined by $1 = J_{eff}(T_K)$.

   (f) What happens when the Kondo coupling becomes strong? Unlike QCD, here we can answer this question. Study the limit of the hamiltonian where J_K is the largest scale (so we may ignore the kinetic terms of the fermions at leading order) and find the groundstate.

2. **Potentials for matrix-valued fields.**

(a) By a symmetry transformation $\Sigma \to g_L \Sigma g_R^\dagger$ can we put a complex matrix Σ in the form $\Sigma = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$?

(b) Consider the $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$-symmetric potential

$$V(\Sigma) = -m^2 \text{tr}\Sigma\Sigma^\dagger + \frac{\lambda}{4}\left(\text{tr}\Sigma\Sigma^\dagger\right)^2 + g\text{tr}\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger. \tag{1}$$

Show that for any $g > 0$ this potential has a minimum at $\langle\Sigma\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find v. Show that if $g = 0$ there are other minima which are not related by rotations $\Sigma \to g_L \Sigma g_R^\dagger$.

(c) [more optional] Now consider a hermitian-matrix-valued field $\Phi = \Phi^a T^a$. Suppose T^a are generators of the adjoint of $\mathsf{SU}(5)$, so there are 24 components of Φ^a. In order for $\mathsf{SU}(5)$ grand unification to work, there must be a potential for such a Higgs field Φ that has a minimum of the form

$$\langle\Phi\rangle = v\text{diag}(2, 2, 2, -3, -3) \equiv \Phi_{3,2}$$

which breaks $\mathsf{SU}(5)$ down to $\mathsf{SU}(3)_{\text{color}} \times \mathsf{SU}(2)_{\text{weak}}$. Consider the most general quartic potential for Φ which is invariant under $\mathsf{SU}(5)$:

$$V = -m^2 \text{tr}\Phi^2 + a\text{tr}\Phi^4 + b\left(\text{tr}\Phi^2\right)^2.$$

Choose a basis where $\Phi = v\text{diag}(a_1, a_2, a_3, a_4, a_5)$, with $\sum_{i=1}^5 a_i = 0$. (Impose this last condition with a Lagrange multiplier.)

For what values of m, a, b is $\Phi_{3,2}$ an extremum?

Show that $\Phi_{3,2}$ is a minimum.

Find all possible minima of this potential.

For the minimum of the form $\langle\Phi\rangle \equiv \Phi_{4,1} = v\text{diag}(1, 1, 1, 1, -4)$, what are the masses of the massive gauge bosons, and what is the unbroken gauge group?