University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 5

Due 11:59pm Monday, May 2, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Boson coherent states brain warmers.**

   Verify the following identities for the coherent state $|\phi\rangle = e^{\phi \mathbf{a}^\dagger}|0\rangle$ of a single mode.

   (a)
   $$\langle \phi_1|\phi_2\rangle = e^{\phi_1^\star \phi_2}.$$

   (b)
   $$\mathbb{1} \equiv \sum_{n=0}^{\infty} |n\rangle\langle n| = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} |\phi\rangle\langle\phi|.$$

   (c)
   $$\text{tr}\cdot = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} \langle\phi| \cdot |\phi\rangle \,.$$

2. **Grassmann exercises.**

   (a) A useful device is the integral representation of the grassmann delta function. Show that
   $$-\int d\bar\psi_1 e^{-\bar\psi_1(\psi_1-\psi_2)} = \delta(\psi_1 - \psi_2)$$
   in the sense that $\int d\psi_1 \delta(\psi_1 - \psi_2) f(\psi_1) = f(\psi_2)$ for any grassmann function f. (Notice that since the grassmann delta function is not even, it matters on which side of the δ we put the function: $\int d\psi_1 f(\psi_1)\delta(\psi_1 - \psi_2) = f(-\psi_2) \neq f(\psi_2)$.)

   (b) Recall the resolution of the identity the Hilbert space of a single fermion mode in terms of fermion coherent states
   $$\mathbb{1} \equiv \sum_{n=0}^{1} |n\rangle\langle n| = \int d\bar\psi d\psi \; e^{-\bar\psi\psi} |\psi\rangle\langle\bar\psi|. \quad (1)$$
   Show that $\mathbb{1}^2 = \mathbb{1}$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was

$$\mathrm{tr}\mathbf{A} = \int d\bar{\psi}d\psi \; e^{-\bar{\psi}\psi} \left\langle -\bar{\psi}\right| \mathbf{A} \left|\psi\right\rangle \;,$$

and the minus sign in the bra had important consequences.
(Here $\left\langle -\bar{\psi}\right| \mathbf{c}^\dagger = \left\langle -\bar{\psi}\right| (-\bar{\psi})$).
Check that using this expression you get the correct answer for

$$\mathrm{tr}(a + b\mathbf{c}^\dagger\mathbf{c})$$

where a, b are ordinary numbers.

(d) Prove the identity (1) by expanding the coherent states in the number basis.

3. **Fermionic coherent state exercise.**

Consider a collection of fermionic modes c_i with quadratic hamiltonian $H = \sum_{ij} h_{ij} c_i^\dagger c_j$, with $h = h^\dagger$.

(a) Compute $\mathrm{tr}e^{-\beta H}$ by changing basis to the eigenstates of h_{ij} (the single-particle hamiltonian) and performing the trace in that basis: $\mathrm{tr}... = \prod_\epsilon \sum_{n_\epsilon = c_\epsilon^\dagger c_\epsilon = 0,1} \cdots$.

(b) Compute $\mathrm{tr}e^{-\beta H}$ by coherent state path integral. Compare!
[Hint: to do the Matsubara sum, it is helpful to use an integral representation such as

$$\sum_n f(\mathbf{i}\omega_n) = \frac{1}{2\pi\mathbf{i}} \oint_C \frac{\beta dz}{e^{\beta z} + 1} f(z)$$

where C is a contour that encircles all the poles of $\frac{1}{e^{\beta z}+1}$.]

(c) [super bonus problem] Consider the case where h_{ij} is a random matrix. What can you say about the thermodynamics?

4. **Topological terms in QM.**

The purpose of this problem is to demonstrate that total derivative terms in the action (like the θ term in QCD) do affect the physics.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B} \cdot \mathrm{d}\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta \mathrm{d}\tau \left(\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi} \right)} \;.$$

Here

$$\phi \equiv \phi + 2\pi \tag{2}$$

is a coordinate on the ring. Because of the identification (6), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z} \backslash 0} \phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{3}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (7), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where $\hat{f}(p) = \int dx e^{-\mathbf{i}px} f(x)$ is the fourier transform of f.]

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta - \pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?