University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 5 – Solutions

Due 11:59pm Monday, May 2, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

---

1. **Boson coherent states brain warmers.**

   Verify the following identities for the coherent state $|\phi\rangle = e^{\phi \mathbf{a}^\dagger}|0\rangle$ of a single mode.

   (a)
   $$\langle \phi_1 | \phi_2 \rangle = e^{\phi_1^\star \phi_2}.$$

   (b)
   $$1\!\!1 \equiv \sum_{n=0}^{\infty} |n\rangle\langle n| = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} |\phi\rangle\langle\phi|.$$

   (c)
   $$\text{tr}\cdot = \int \frac{d\phi d\phi^\star}{\pi} e^{-|\phi|^2} \langle \phi | \cdot | \phi \rangle .$$

2. **Grassmann exercises.**

   (a) A useful device is the integral representation of the grassmann delta function. Show that
   $$-\int d\bar\psi_1 e^{-\bar\psi_1(\psi_1 - \psi_2)} = \delta(\psi_1 - \psi_2)$$
   in the sense that $\int d\psi_1 \delta(\psi_1 - \psi_2) f(\psi_1) = f(\psi_2)$ for any grassmann function f. (Notice that since the grassmann delta function is not even, it matters on which side of the δ we put the function: $\int d\psi_1 f(\psi_1)\delta(\psi_1 - \psi_2) = f(-\psi_2) \neq f(\psi_2)$.)

   (b) Recall the resolution of the identity the Hilbert space of a single fermion mode in terms of fermion coherent states
   $$1\!\!1 \equiv \sum_{n=0}^{1} |n\rangle\langle n| = \int d\bar\psi d\psi \; e^{-\bar\psi\psi} |\psi\rangle\langle\bar\psi|. \tag{1}$$

   Show that $1\!\!1^2 = 1\!\!1$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was

$$\mathrm{tr}\mathbf{A} = \int d\bar{\psi}d\psi\ e^{-\bar{\psi}\psi} \langle -\bar{\psi}| \mathbf{A} |\psi\rangle \ ,$$

and the minus sign in the bra had important consequences.
(Here $\langle -\bar{\psi}| \mathbf{c}^\dagger = \langle -\bar{\psi}| (-\bar{\psi})$).
Check that using this expression you get the correct answer for

$$\mathrm{tr}(a + b\mathbf{c}^\dagger\mathbf{c})$$

where a, b are ordinary numbers.

(d) Prove the identity (1) by expanding the coherent states in the number basis.
Using $|\psi\rangle = |0\rangle + \psi|1\rangle$, $\langle -\bar{\psi}| = \langle 0| - \bar{\psi}\langle 1|$, we have

$$\begin{aligned}\int d\bar{\psi}d\psi\ e^{-\bar{\psi}\psi} |\psi\rangle\langle\bar{\psi}| &= \int d\bar{\psi}d\psi\ e^{-\bar{\psi}\psi} (|0\rangle + \psi|1\rangle)\left(\langle 0| - \bar{\psi}\langle 1|\right) \\ &= \int d\bar{\psi}d\psi\ e^{-\bar{\psi}\psi}\left(|0\rangle\langle 0| - \psi\bar{\psi}|1\rangle\langle 1|\right) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}. \qquad (2)\end{aligned}$$

3. **Fermionic coherent state exercise.**

Consider a collection of fermionic modes c_i with quadratic hamiltonian $H = \sum_{ij} h_{ij} c_i^\dagger c_j$, with $h = h^\dagger$.

(a) Compute $\mathrm{tr}e^{-\beta H}$ by changing basis to the eigenstates of h_{ij} (the single-particle hamiltonian) and performing the trace in that basis: $\mathrm{tr}... = \prod_\epsilon \sum_{n_\epsilon = c_\epsilon^\dagger c_\epsilon = 0,1} ...$.
In the eigenbasis of h_{ij},

$$H = \sum_{ij} h_{ij} c_i^\dagger c_j = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha,$$

the trace factorizes:

$$\mathrm{tr}e^{-\beta H} = \prod_\alpha \sum_{n_\alpha = c_\alpha^\dagger c_\alpha = 0,1} e^{-\beta\epsilon_\alpha n_\alpha} = \prod_\alpha \left(1 + e^{-\beta\epsilon_\alpha}\right) = \det\left(1 + e^{-\beta h}\right).$$

(b) Compute $\mathrm{tr}e^{-\beta H}$ by coherent state path integral. Compare!
[Hint: to do the Matsubara sum, it is helpful to use an integral representation such as

$$\sum_n f(\mathbf{i}\omega_n) = \frac{1}{2\pi\mathbf{i}} \oint_C \frac{\beta dz}{e^{\beta z} + 1} f(z)$$

where C is a contour that encircles all the poles of $\frac{1}{e^{\beta z}+1}$.]

In lecture we showed for a single fermionic mode how to write the thermal partition function as a grassmann path integral

$$\mathrm{tr} e^{-\beta H(c^\dagger, c)} = \int [D\psi D\bar{\psi}] e^{-\int_0^\beta d\tau \left(\bar{\psi}\partial_\tau \psi - H(\bar{\psi},\psi)\right)}$$

as long as H is normal-ordered. Here we just have many copies of that problem:

$$\mathrm{tr} e^{-\beta H(c_i^\dagger, c_j)} = \int \prod_i [D\psi_i D\bar{\psi}_i] e^{-\int_0^\beta d\tau \left(\bar{\psi}_i \partial_\tau \psi_i - h_{ij}\bar{\psi}_i \psi_j\right)}.$$

To do this integral, let's go to frequency space:

$$\psi_i(\tau) = \sum_n e^{-\omega_n \tau}\psi_{ni}, \quad \omega_n = \pi T(2n+1).$$

Further, let's change coordinates to diagonalize h, so we have

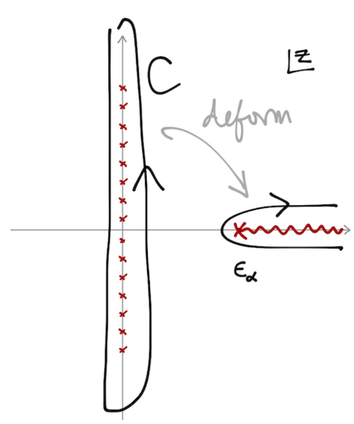
$$Z = \int \prod_{\alpha,n} d\psi_{\alpha,n} d\bar{\psi}_{\alpha,n} \prod_{\alpha,n} e^{-\bar{\psi}_{\alpha,n}(\mathbf{i}\omega_n - \epsilon_\alpha)\psi_{\alpha,n}} \tag{3}$$

$$= \prod_{\alpha,n} (\mathbf{i}\omega_n - \epsilon_\alpha) = e^{\sum_{\alpha,n} \log(\mathbf{i}\omega_n - \epsilon_\alpha)} \tag{4}$$

So

$$\log Z = \sum_{\alpha,n} \log\left(\mathbf{i}\omega_n - \epsilon_\alpha\right) \tag{5}$$

$$= \sum_\alpha \frac{1}{2\pi \mathbf{i}} \oint_C dz \frac{\beta}{e^{\beta z}+1} \log\left(z - \epsilon_\alpha\right)$$



$$= \frac{1}{2\pi \mathbf{i}} \sum_\alpha \int_{\epsilon_\alpha}^\infty dz \,\mathrm{disc}\left(\frac{\beta}{e^{\beta z}+1} \log\left(z - \epsilon_\alpha\right)\right)$$

$$= \frac{1}{2\pi \mathbf{i}} \sum_\alpha \int_{\epsilon_\alpha}^\infty dz \frac{\beta}{e^{\beta z}+1} 2\pi \mathbf{i}$$

$$= \sum_\alpha \int_{\epsilon_\alpha}^\infty dz \frac{\beta}{e^{\beta z}+1} = \sum_\alpha \log\left(1 + e^{-\beta\epsilon_\alpha}\right),$$

which gives the same answer as above.

(c) [super bonus problem] Consider the case where h_{ij} is a random matrix. What can you say about the thermodynamics?

4. **Topological terms in QM.** [from Abanov]

The purpose of this problem is to demonstrate that total derivative terms in the action (like the θ term in QCD) do affect the physics.

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B} \cdot d\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta d\tau \left(\frac{m}{2} \dot{\phi}^2 - \mathbf{i} \frac{\theta}{2\pi} \dot{\phi} \right)} .$$

Here

$$\phi \equiv \phi + 2\pi \tag{6}$$

is a coordinate on the ring. Because of the identification (2), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z} \setminus 0} \phi_\ell e^{\mathbf{i} \frac{2\pi}{\beta} \ell \tau}. \tag{7}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (3), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_n f(n) = \sum_l \hat{f}(2\pi l)$$

where $\hat{f}(p) = \int dx e^{-\mathbf{i}px} f(x)$ is the fourier transform of f.]

Applied to our case, the Poisson resummation formula says

$$\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2} t n^2 + \mathbf{i} z n} = \sqrt{\frac{2\pi}{t}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2t}(z - 2\pi \ell)^2}.$$

Using the given mode expansion and $\int_0^\beta dte^{\frac{2\pi\mathbf{i}(l-l')\tau}{\beta}} = \beta\delta_{l,l'}$ the action is

$$S[\phi] = \mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta} + \sum_{\ell\neq 0}\frac{(2\pi\ell)^2 m}{2\beta}\phi_\ell\phi_{-\ell}$$

where $\phi_\ell = \phi^\star_{-\ell}$. Thus

$$Z = \sum_{Q\in\mathbb{Z}} e^{-\mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta}}\prod_{\ell\neq 0}\int d^2\phi_\ell e^{\frac{(2\pi\ell)^2 m}{2\beta}\phi_\ell\phi^\star_\ell} \tag{8}$$

$$= \sum_{Q\in\mathbb{Z}} e^{-\mathbf{i}\theta Q + \frac{m(2\pi Q)^2}{2\beta}}\prod_{\ell\neq 0}\left(\frac{\beta}{2\pi\ell^2 m}\right) \tag{9}$$

$$\propto \sum_{n\in\mathbb{Z}} e^{-\beta\frac{1}{2m(2\pi)^2}(\theta-2\pi n)^2} = \sum_{n\in\mathbb{Z}} e^{-\beta\frac{1}{2m}\left(n-\frac{\theta}{2\pi}\right)^2} \tag{10}$$

where in the last step we used the above Poisson summation formula with $z=\theta$ and $t=\frac{m(2\pi)^2}{\beta}$.

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

After the Poisson resummation, this is manifestly the partition function of a system with energies $E_n = \frac{1}{2m}(n-\frac{\theta}{2\pi})^2$.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

Note that the action given above is the *Euclidean* action. The real time action (from which we should derive the hamiltonian) is

$$S = \int dt\left(\frac{1}{2}m\dot{\phi}^2 + \dot{\phi}\frac{\theta}{2\pi}\right).$$

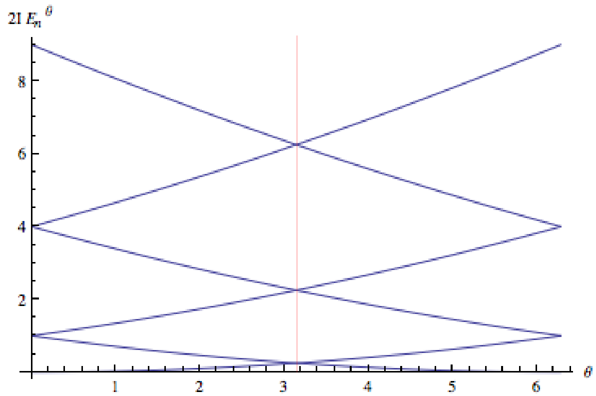
This gives $p = \frac{\partial L}{\partial\dot{\phi}} = m\dot{\phi} + \frac{\theta}{2\pi}$, and hence

$$H = \frac{\left(p-\frac{\theta}{2\pi}\right)^2}{2m}.$$

Now, since $\phi \equiv \phi + 2\pi$, its canonical momentum is quantized, $p\in\mathbb{Z}$, so

$$E_n = \frac{1}{2m}\left(n-\frac{\theta}{2\pi}\right)^2$$

as above. We find the following spectrum for various θ (I am plotting the energies of the states with wavenumbers $n\in[-3,2]$):



(In the axis label, I is the moment of inertia of the rotor.) Notice that when $\theta = \pi$, the groundstate becomes doubly degenerate.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta - \pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?

In this limit, the higher bands of energies go off to ∞, and we are left with a two-state system. X is a Zeeman field splitting the two states.