University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2022
Assignment 3 – Solutions

Due 11:59pm Monday, April 18, 2022

Thanks in advance for following the guidelines on HW01. Please ask me by email if you have any trouble.

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1. **Right-handed neutrinos.**

[from Iain Stewart, and hep-ph/0210271]

Consider adding a right-handed singlet (under all gauge groups) neutrino N_R to the Standard Model. It may have a majorana mass M; and it may have a coupling g_ν to leptons, so that all the dimension ≤ 4 operators are

$$\mathcal{L}_N = \bar{N}_R \mathbf{i}\not{\partial} N_R - \frac{M}{2}\bar{N}_R^c N_R - \frac{M}{2}\bar{N}_R N_R^c + \left(g_\nu \bar{N}_R H_i^T L_j \epsilon^{ij} + h.c.\right)$$

where $N_R^c = C\left(\bar{N}_R\right)^T$ is the the charge conjugate field, $C = \mathbf{i}\gamma_2\gamma_0$ (in the Dirac representation), H is the Higgs doublet, L is the left-handed lepton doublet, containing ν_L and e_L. Take the mass M to be large compared to the electroweak scale. Integrate out the right-handed neutrinos at tree level. [Hint: you may find it useful to work in terms of the Majorana field

$$N \equiv N_R + N_R^c$$

which satisfies $N = N^c$.]

Show that the leading term in the expansion in $1/M$ is a dimension-5 operator made of Standard Model fields. Explain the consequences of this operator for neutrino physics, assuming a vacuum expectation value for the Higgs field.

In terms of N, the lagrangian is

$$\mathcal{L}_N = \frac{1}{2}\bar{N}\left(\mathbf{i}\not{\partial} - M\right)N + g_\nu \bar{N} H_i L_j \epsilon^{ij} + g_\nu \bar{N} H_i^\star L_j^c \epsilon^{ij}.$$

The equation of motion for N (from varying $\bar{N}$) is

$$\left(\mathbf{i}\not{\partial} - M\right)N = -g_\nu\left(H_i L_j + H_i^\star L_j^c\right)\epsilon^{ij}$$

which gives

$$\mathcal{L}_N| = -\frac{1}{2}g_\nu\left(\bar{L}_j^c H_i + \bar{L}_j H_i^\star\right)\epsilon^{ij}\frac{1}{\mathbf{i}\not{\partial} - M}g_\nu\left(H_k L_\ell + H_k^\star L_\ell^c\right)\epsilon^{k\ell}.$$

As for our discussion of W-bosons, we expand this in powers of $1/M$ to get a local effective field theory. The leading term is

$$\mathcal{O}^{(5)} = \frac{g_\nu^2}{M} \bar{L}_j^c H_i \epsilon^{ij} L_\ell H_k \epsilon^{k\ell} + h.c.$$

Plugging in $\langle H \rangle \neq 0$, this is a neutrino mass.

Place a bound on M assuming that the observed neutrinos have masses $m_\nu < 0.5$ eV.

In terms of the parameterization from lecture, $m_\nu = \frac{c_5 v^2}{2\Lambda_{\text{new}}}$. This gives $\Lambda_{\text{new}} \geq 10^{14}$GeV for $c_5 \sim 1$. We find $\Lambda_{\text{new}}/c_5 \sim M$, so $M \geq 10^{14}$GeV.

2. **Gross-Neveu model.**

Here's an example which illustrates the manipulations we did in describing the BCS phenomenon. Now that we've learned about fermionic path integrals, consider the partition function for an N-vector of fermionic spinor fields in D dimensions:

$$Z = \int [d\psi d\bar{\psi}] e^{\mathbf{i} S[\psi]}, \quad S[\vec{\psi}] = \int \mathrm{d}^D x \left(\bar{\psi}^a \mathbf{i} \partial\!\!\!/ \psi^a - \frac{g}{N} \left(\bar{\psi}^a \psi^a \right)^2 \right).$$

(a) At the free fixed point, what is the dimension of the coupling g as a function of the number of spacetime dimensions D? Show that it is classically marginal in $D = 2$, so that this action is (classically) scale invariant.

(b) We will show that this model in $D = 2$ exhibits dimensional transmutation in the form of a dynamically generated mass gap. Here are the steps: first use the Hubbard-Stratonovich trick to replace ψ^4 by $\sigma\psi^2 + \sigma^2$ in the action, where σ is a scalar field. Then integrate out the ψ fields. Find the saddle point equation for σ; argue that the saddle point dominates the integral for large N. Find a translation invariant saddle point. Plug the saddle point configuration of σ back into the action for ψ and describe the resulting dynamics.

We can decouple the quartic term by writing

$$Z = \int [D\psi \bar{\psi}] e^{\mathbf{i} S[\psi]} = \int [D\psi D\bar{\psi} D\sigma] e^{\mathbf{i} S_2[\psi] + \mathbf{i} \int d^D x \left(\sigma \bar{\psi}^a \psi^a + h.c. \right) + \mathbf{i} \int \mathrm{d}^D x \frac{N\sigma^2(x)}{2g}} . \tag{1}$$

Now the integral over ψ is gaussian:

$$\int [D\psi D\bar{\psi} D\sigma] e^{\int d^D x \bar{\psi}^a \left(\mathbf{i}\partial\!\!\!/ + \sigma \right) \psi^a} = \left(\det \left(\mathbf{i}\partial\!\!\!/ + \sigma \right) \right)^N = e^{N \operatorname{tr} \log \left(\mathbf{i}\partial\!\!\!/ + \sigma \right)}.$$

The resulting path integral is

$$Z = \int [D\sigma] e^{\mathbf{i} N S_{\text{eff}}[\sigma]}$$

with $S_{\text{eff}}[\sigma] = \int d^D x \frac{\sigma^2}{2g} + \delta S[\sigma]$ where the term generated by the fermionic fluctuations is

$$\delta S[\sigma] = \text{tr} \log \left(\not{\partial} + \sigma \right).$$

We can take care of the spin indices by noticing that

$$\text{tr}_{\text{spin}} \log \left(\not{\partial} + \sigma \right) = \frac{1}{2} \left(\text{tr}_{\text{spin}} \log \left(\not{\partial} + \sigma \right) + \text{tr}_{\text{spin}} \log \left(-\not{\partial} + \sigma \right) \right) \tag{2}$$

$$= \frac{1}{2} \text{tr}_{\text{spin}} \log \left(-\partial^2 + \sigma^2 \right) \stackrel{D=2}{=} \log \left(-\partial^2 + \sigma^2 \right) \tag{3}$$

where at the last step we used the fact that the Dirac spinor in 2D has two components.

If we assume that σ is constant in spacetime, we can do the trace in momentum space (V is the volume of spacetime):

$$\text{tr} \log \left(-\partial^2 + \sigma^2 \right) = V \int đ^D p \log \left(p^2 + \sigma^2 \right) \tag{4}$$

$$\stackrel{\text{Wick rotate}}{=} \mathbf{i} V \frac{1}{2\pi} \mathbf{i} \int_0^\Lambda p dp \log \left(p^2 + \sigma^2 \right) \tag{5}$$

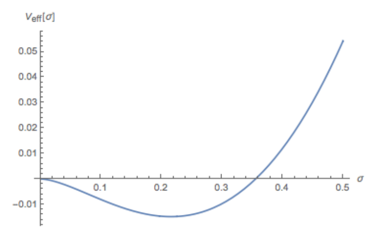
$$= \mathbf{i} \frac{V}{\pi} \left(-\sigma^2 \log \frac{\sigma^2}{\Lambda^2} + \text{UV divergent terms} \right). \tag{6}$$

I introduced a hard UV cutoff, since we have no gauge invariance to preserve. At the last step I've assumed $\sigma \ll \Lambda$. We ignore the divergent constants.

Because of the big honking factor of N in front of S_{eff}, the σ integral is dominated by its saddle point configuration, where

$$0 = \frac{\delta S_{\text{eff}}}{\delta \sigma} = V \left(\frac{\sigma}{g} + \frac{2\sigma}{\pi} \left(1 + \log \sigma / \Lambda \right) \right)$$

from which we conclude that there is a minimum for σ at $\sigma = \Lambda e^{-\frac{\pi}{g}} / \sqrt{e}$. (The figure at right is for $g = .3, \Lambda = 1000$.)



Thus, the fermions get a mass of order $\Lambda e^{-\pi/g}$, non-perturbative in g, and parametrically smaller than the cutoff.